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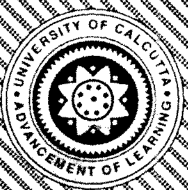

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AN INTRODUCTION TO BOOLEAN SEMIRINGS

T. K. MUKHERJEE, M.K. SEN and S. GHOSH

ABSTRACT. A commutative semiring R is called a boolean semiring if $x^2 = x$ for all $x \in R$. In this paper we study some properties of boolean semirings and characterize some special classes of boolean semirings.

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Key words and phrases : Semiring, boolean semiring, k-ideal, lattice, boolean ring.

1. INTRODUCTION

In this paper by a *semiring* we mean an algebraic structure (R, + ,.) in which (R,+) is an abelian monoid with an identity 0 (zero) and (R,.) is a semigroup, connected by the ring-like distributivity. Also $0x = 0x = 0$ for all $x \in R$. If moreover $ab = ba$ for all $a, b \in R$, then R is called commutative.

A subsemiring I of R is called an *ideal* of R if IR, RI $\subseteq$ I. An ideal I of R is called a *k-ideal* [4] if $x + y = z$, $y, z \in I$, $x \in R$ implies $x \in I$.

A commutative semiring R is called a *boolean semiring* if $x^2 = x$ for all $x \in R$. Distributive lattices with additive identity, boolean rings and any direct product of a distributive lattice with additive identity and a boolean ring are common examples of boolean semirings.

In this paper we study some properties of boolean semirings and characterize some special classes of boolean semirings.

Throughout the rest of the paper we consider R as a boolean semiring.

2. IDEALS 2R AND 3R OF R

Proposition 2.1. $2x = 4x$ for all $x \in R$.

Proof. Let $x \in R$. Then $2x = (x + x)^2 = 4x^2 = 4x$.

Definition 2.2. Let $2R = \{ 2x : x \in R \}$.

Proposition 2.3. (i) $2R$ is the set of all additive idempotents of R.
(ii) $2R$ is an ideal of R.

Proof. (i) Follows from Proposition 2.1.
(ii) Straightforward.

Boume [1] defined a congruence relation ρ_I (say) on a semiring S relative to an ideal I of S as $(a, b) \in \rho_I$ iff $a + i_1 = b + i_2$ for some $i_1, i_2 \in I$. We denote the quotient semiring S/ρ_I by S/I. $0\rho_I$, the equivalence class containing 0, is the smallest k-ideal of S, containing I and is denoted by $\bar{I}$, which equals to $\{ x \in S : x + i_1 = i_2$ for some $i_1, i_2 \in I \}$. Also $S/I = S/\bar{I}$ [4].

Proposition 2.4. $R/2R$ is a boolean ring.

Proof. Follows from the fact that, for any $x \in R$, $x\rho_{2R} + x\rho_{2R} = 2x\rho_{2R} = 0\rho_{2R}$ and $0\rho_{2R}$ is the zero of the quotient semiring $R/2R$.

For any semiring S, $Z(S) = \{ x \in S : x + z = z$ for some $z \in S \}$ is called the *zeroid* of S [2].

Proposition 2.5. $\overline{2R}$ is the zeroid of R.

Proof. Let $x \in Z(R)$. Then $x + z = z$ for some $z \in R$. Thus $x + 2z = 2z$, which implies $x \in \overline{2R}$.

Conversely, let $x \in \overline{2R}$. Then $x + 2y = 2z$ for some $y, z \in R$. Adding $2y + 2z$ to both sides we get $x + 2(y + z) = 2(y + z)$. Thus $x \in Z(R)$.

Definition 2.6. Let ρ be a binary relation on R defined by $(x, y) \in \rho$ iff $2x = 2y$, for each $x, y \in R$.

It is easy to verify that ρ is a congruence on R.

Proposition 2.7. ρ is the greatest additive idempotent separating congruence on R.

Proof. Clearly ρ separates the additive idempotents of R. Let σ be any additive idempotent separating congruence on R and $(a, b) \in \sigma$, $a, b \in R$. Now

$$2a = (a + a)\ \sigma\ (b + a)\ \sigma\ (b + b) = 2b.$$

Since $2a$ and $2b$ are additive idempotents, we have $2a = 2b$ which implies $(a, b) \in \rho$. Hence $\sigma \subseteq \rho$.

Homomorphisms of semirings are defined in a natural way. A bijective homomorphism is called an isomorphism.

Proposition 2.8. $R/\rho \cong 2R$.

Proof. Simple verification shows that the mapping $f : R/\rho \rightarrow 2R$, defined by $f(x\rho) = 2x$ is an isomorphism on R/ρ onto $2R$, as required.

Definition 2.9. Let $L_R = \{ a \in R : a + ax = a$ for all $x \in R \}$ and $B_R = \{ a \in R : a + b$

$= 0$ for some $b \in R$ }.

L_R and B_R are non-empty as they both contain 0. The following two propositions are obvious.

Proposition 2.10. (i) $L_R \subseteq 2R$.

(ii) L_R is an ideal of R.

(iii) L_R is a distributive lattice.

Proposition 2.11. (i) $B_R = \{ a \in R : 2a = 0 \}$.

(ii) B_R is a k-ideal of R.

(iii) B_R is the maximal boolean subring of R containing 0.

Let I_1 and I_2 be two ideals of a semiring S such that $I_1 \cap I_2 = \{ 0 \}$. Then $I = I_1 + I_2$ is called the internal direct sum of I_1 and I_2 if for each $i \in I$, the representation $i = i_1 + i_2$, $i_k \in I_k$, $k = 1, 2$ is unique and is denoted by $I_1 \oplus I_2$ [2].

Proposition 2.12. (i) $2\overline{R} \cap B_R = \{ 0 \} = (2\overline{R})\, B_R$.

(ii) $2\overline{R} + B_R = 2\overline{R} \oplus B_R$.

Proof. (i) Let $x \in 2\overline{R} \cap B_R$. Then $x + e = f$, where $e, f \in 2R$. Thus $x + xe = x^2 + xe = xf$. Now $xe = x(e + e) = (x + x)e = 0$, as $x \in B_R$. Similarly, $xf = 0$ and so $x = 0$.

The second part follows from the fact that $(2\overline{R})\, B_R \subseteq 2\overline{R} \cap B_R$.

(ii) Let $x \in 2\overline{R} + B_R$ and so $x = a_1 + b_1 = a_2 + b_2$ for some $a_1, a_2 \in 2\overline{R}$ and $b_1, b_2 \in B_R$. This gives $a_1 + a_1 b_1 = a_1 a_2 + a_1 b_2$ which implies $a_1 = a_1 a_2$ by (i). Similarly, $a_2 = a_1 a_2$. Thus $a_1 = a_2$. Again $a_1 = a_1 + b_1 + b_1 = a_2 + b_2 + b_1$. Then $b_2 + b_1 \in 2\overline{R} \cap B_R = \{ 0 \}$, by (i). Hence $b_1 = b_2$.

Corollary 2.13. $L_R + B_R = L_R \oplus B_R$.

Definition 2.14. Let $3R = \{ 3x : x \in R \}$.

Proposition 2.15. (i) 3R is an ideal of R.

(ii) $2R \subseteq 3R$ and $2R = 2\,(3R)$.

(iii) $B_R \subseteq 3R$.

(iv) $3a = a$ for all $a \in 3R$.

Proof. Straightforward.

Proposition 2.16. 3R is a union of mutually disjoint boolean rings.

Proof. Let us restrict ρ (cf. Definition 2.6) on 3R. Let $a \in 3R$ and $b, c \in a\rho$. Then $2b = 2c = 2a = e$ (say). Now $2(b + c) = 2b + 2c = e + e = e$ and $2bc = 2b.2c = e^2 = e$, which implies $b + c, bc \in a\rho$. Also $b + e = b + 2b = 3b = b$ and $b + b = 2b = e$. Thus $a\rho$ is a subring and hence a boolean subring of 3R, for each $a \in 3R$.

An interesting property of the ideal $3R$ is that, it is not, in general, a k-ideal (see Example 2.18); but the following proposition shows that the only k-ideal generated by $3R$ is R itself.

Proposition 2.17. $\overline{3R} = R$.

Proof. Let $x \in R$. Then $x + 2x = 3x$, which implies $x \in \overline{3R}$, as $2x, 3x \in 3R$.

We close this section with an example which shows that R is not always equal to $3R$.

Example 2.18. Let Z_0^+ be the set of non-negative integers and γ be a binary relation on Z_0^+ defined by $(a, b) \in \gamma$ iff $a = b$ or $a, b > 1$ and $a - b$ is even. Then γ is a congruence on Z_0^+ [3] and the congruence classes under γ are $\{0\}, \{1\}, \{2, 4, 6, ...\}$ and $\{3, 5, 7, ...\}$. It is easy to check that the quotient semiring $S = Z_0^+/\gamma = \{0\gamma, 1\gamma, 2\gamma, 3\gamma\}$ is a boolean semiring and $3S = \{0\gamma, 2\gamma, 3\gamma\} \neq S$.

3. CHARACTERIZATIONS OF BOOLEAN SEMIRINGS

Theorem 3.1. R is a distributive lattice iff $R = L_R$ iff $a + ax = a$ for all $a, x \in R$.

Proof. Immediate.

Theorem 3.2. R is a boolean ring iff $R = B_R$ iff $2a = 0$ for all $a \in R$ iff $2R = \{0\}$.

Proof. Immediate.

Lemma 3.3. R is a direct product of a distributive lattice and a boolean ring iff $R = L_R + B_R$.

Proof. By Corollary 2.13, it is clear that the condition is sufficient. Let $R = \pounds \times B$, where $\pounds$ is a distributive lattice and B is a boolean ring. Since R is a boolean semiring. $0 \in R$. Let $0 = (\theta_1, \theta_2)$, where $\theta_1 \in \pounds$ and $\theta_2 \in B$. It is easy to verify that θ_1 and θ_2 are the additive identities of $\pounds$ and B respectively and $L_R = \pounds \times \{\theta_2\}$ and $B_R = \{\theta_1\} \times B$.

Also since for any $x \in \pounds$ and $y \in B$, $(x, y) = (x, \theta_2) + (\theta_1, y) \in L_R + B_R$, we have $R = L_R + B_R$.

Lemma 3.4. $R = L_R + B_R$ iff for each $a \in R$, there is an element $b \in R$ such that $2b = 0$ and $a + ax = a + abx$ for all $x \in R$.

Proof. Let $R = L_R + B_R$ and $a, x \in R$. Then $a = l + b$ and $x = l' + b'$ for some $l, l' \in L_R$ and $b, b' \in B_R$. We first show that $l + lx = l$. Since L_R is a lattice, we have $l + ll' = l$ and by Proposition 2.12. (i) $lb' = 0$. Thus

$$l + lx = l + ll' + lb' = l.$$

Also $a + abx = l + b + lbx + b^2x = l + b + bx = (l + lx) + b + bx = a + ax$.

Conversely, let the condition be satisfied. Let $a \in R$. Then there is $b \in R$ such that $a + ax = a + abx$ for all $x \in R$. This implies $2a + 2ax = 2a$, as $2abx = 0$, for all $x \in R$. Then $2a \in L_R$. Also we have $2a = a + aa = a + aba = a + ab$ and $a = a + ab + ab = 2a + ab \in L_R + B_R$. Thus $R =$

$L_R + B_R$.

Theorem 3.5. A boolean semiring R is a direct product of a distributive lattice and a boolean ring iff for each $a \in R$. there is an element $b \in R$ such that $2b = 0$ and $a + ax = a + abx$ for all $x \in R$.

Proof. Follows from the above lemmas.

Example 3.6. Let B be a boolean ring and 0 be the zero of B.

Let B_n be the direct product of n copies of B for $n = 1, 2, \ldots$; and $S = \bigcup_{n=1}^{\infty} B_n$. Then S is a boolean semiring which is a union of mutually disjoint boolean rings with additions and multiplications defined as follows :

For $x \in B_m$ and $y \in B_n$ $(m \leq n)$,

$x + y = (z_1, z_2, \ldots\ldots, z_n)$ and $xy = (w_1, w_2, \ldots w_m)$,

Where

$$z_k = \begin{cases} x_k + y_k, & k \leq m \\ y_k, & k > m \end{cases} \qquad \text{and} \qquad w_k = x_k y_k,\ k \leq m.$$

A semiring S is called *separative* if $2a = a + b = 2b$. $a, b \in S$ implies $a = b$.

Theorem 3.7. The following statements are equivalent :

(i) R is a union of mutually disjoint boolean rings.

(ii) R is separative.

(iii) $a = 3a$ for all $a \in R$.

(iv) $R = 3R$.

Proof.

(i) ⇒ (ii) : Let $R = \bigcup B_\alpha$, where $\{ B_\alpha : \alpha \in \Lambda \}$ is a family of mutually disjoint boolean rings. Let $2a = a + b = 2b$. $a, b \in R$. Then $a \in B_\alpha$, $b \in B_\beta$ for some $\alpha, \beta \in \Lambda$. Since $2a = 2b$, we have $\alpha = \beta$. Thus $a = a + 0_{B_\alpha} = a + a + a = a + a + b = 0_{B_\alpha} + b = b$, as required.

(ii) ⇒ (iii) : We have for all $a \in R$, $a + a = 2a = 4a = a + 3a = 4a = 6a = 3a + 3a$ which implies $a = 3a$. as F is separative.

(iii) ⇒ (iv) : Let $a \in R$. Then $a = 3a \in 3R$. Thus $R = 3R$.

(iv) ⇒ (i) : Follows from Proposition 2.16.

Let S and S' be two semirings. A surjective homomorphism of semirings $\gamma : S \to S'$ is called a *semi-isomorphism* if kernel $(\gamma) = \{ x \in S : \gamma(x) = 0_{S'} \} = \{ 0_S \}$.

Theorem 3.8. Any boolean semiring is semi-isomorphic to a boolean semiring which is a union of mutually disjoint boolean rings.

Proof. Let R be a boolean semiring. It is easy to verify that the mapping $\gamma : R \to 3R$, which sends x to $3x$ for all $x \in R$, is a surjective homomorphism. Now, if $3x = 0$ for some $x \in R$, then $x = x + 0 = x + 3x = 4x = 6x = 3x + 3x = 0$, as required.

REFERENCES

[1] S. Bourne. The Jacobson radical of a semiring, Proc. Nat. Acad. Sci. U.S.A., **37** (1951), 163– 170.

[2] J. S. Golan, *The theory of semirings with applications in Mathematics and Theoretical Computer Science,* Longman Scientific & Technical, 1992.

[3] U. Hebisch and H. J. Weinert, *Halbringe,* B. G. Teubner Stuttgart, 1993.

[4] D. R. La Torre, On h-ideals and k-ideals in hemirings, Publ. Math. Debrecen, **12** (1965), 219-226.

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A NEW REGULAR MEASURE ON A TOPOLOGICAL SPACE-II

S. GANGULY, C.K. BASU AND S. SINHA ROY

ABSTRACT. Abstract theory of regularity of measure has been studied extensively in [2] and all its applications were with regard to Baire and Borel measure on locally compact T_2 spaces. Using abstract theory, in [3], θ-Baire measure has been introduced in locally θ-H-closed spaces which are more general than locally compact T_2 spaces. This paper introduces θ-Borel measure on locally θ-H-closed space with the main intention to offer glimpses (a natural relationship) as to how a regular θ-Baire measure can be extended uniquely to a regular θ-Borel measure.

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Key words and phrases: H-set, θopen, θclosed, H-G_δ^θ set, θ-Baire (resp. θ-Borel) measure (function), regular θ-Borel (θ-Baire) measure, Urysohn space, locally θ-H-closed, θ-CR and θ-T_2.

1. INTRODUCTION

The concept of H-sets and θ-open sets were first introduced by Velicko [5]. It can be noted that every compact set is an H-set but not conversely; so it is natural that the class $\mathcal{H}$ of all real valued continuous functions with H-set support is larger than the class of all real valued continuous function with compact support. Meanwhile, in [3], it was observed that every function in $\mathcal{H}$ is θ-Baire function i.e., measurable with respect to the σ-ring of all θ-Baire sets and a θ-Baire measure is regular on a locally θ-H-closed space. So one's objective would be to take the functions in $\mathcal{H}$ under integration theory and to find an analogue of the Riesz-Markoff representation theorem on locally θ-H-closed space. With this motivation in mind, as a primary step, it seems necessary to introduce θ-Borel measure on locally θ-H-closed space and we have obtained different equivalent criterion for a θ-Borel measure to be regular. The main aim of this paper is how a regular θ-Baire measure can be extended to a regular θ-Borel measure in one and only one way.

2. PRE REQUISITES AND DEFINITIONS

By (x, T) or simply by x we shall mean a topological space and for any subset A of X, ClA (sometimes $\bar{A}$) denotes the closure of the set A in X. A subset A of X is said to be θ-open [5] if for each $x \in A$, there exists an open set U_x such that $x \in U_x \subset \mathrm{cl}U_x \subset A$. Complement of a θ-open set is called a θ-closed set. A set A in X is said to be an H-set [5] if any system $\upsilon = \{ U_\alpha : \alpha \in \Gamma\}$ of open sets in X covering A has a finite subsystem say $\{ U_{\alpha_1} \ldots U_{\alpha_n} \}$ of υ such that $A \subset \mathrm{cl} (\bigcup_{i=1}^{n} U_{\alpha_i})$. If A = X, then X is called an H-closed space [1]. Although H-closedness is a generalization of compactness but in regular topological spaces, these two concepts coincide.

A space X is said to be θ-completely regular [4] (θ-CR, for short) if for any point p and a θ-closed set H in X with $P \notin H$, there exists a continuous function $f : X \to [0, 1]$ such that $f(p) = O$ and $f(H) = 1$. Clearly every completely regular space is θ-CR but not conversely [4]. A space X is said to be θ-T_2 [4] (resp. Urysohn [6]) if for $x \neq y$ and $x, y \in X$ there exist θ-open sets U_x and U_y (resp. open sets U_x and U_y) containing x and y respectively such that, $U_x \cap U_y = \varnothing$ (resp. $\mathrm{cl}U_x \cap \mathrm{cl}U_y = \varnothing$). Clearly every θ-$T_2$ space is Urysohn and an H-set in a Urysohn space is θ-closed [4]; further in a Urysohn space arbitrary intersection of H-sets is H-set.

A space X is said to be locally θ-H-closed [4] iff every point x of X has θ-open neighbourhood whose closure is an H-set. A subset A of X is called a G_δ^θ [3] in case there exists a sequence $\{U_n\}_{n=1}^{\infty}$ of θ-open sets of X such that $A = \bigcap_{n=1}^{\infty} U_n$. It is to be noted that every G_δ^θ set is G_δ set but not conversely [3]. A set A which is a G_δ^θ as well as an H-set is called an H-G_δ^θ set [3]. The σ-ring generated by the class of all H-G_δ^θ sets (resp. H-sets) in X is called the class of all θ-Baire (resp. θ-Borel) sets. A real valued function on X which is measurable with respect to the σ-ring of θ-Baire (resp. θ-Borel) sets is called a θ-Baire (resp. θ-Borel) function.

A θ-Baire [3] (resp. θ-Borel) measure on X is a measure μ defined on the σ-ring of all θ-Baire (resp. θ-Borel) sets such that $\mu(c) < +\infty$, for every H-G_δ^θ sets (resp. H-sets) C in X. In a locally θ-H-closed, θ-T_2 and θ-CR space we have observed in [3] that the class of all θ-Baire functions is the smallest class of all real valued functions on X which contains $\mathcal{H}$ and is closed under sequential pointwise limit. A real valued function f on X is said to have H-set support if there exists an H-set C in X such that $f = O$ on $X - C$.

3. We now recall the abstract discussion of regularity [2] relative to a system $(X\ \mathfrak{S}, \mu, \mathfrak{g}, \vartheta)$ satisfying the axioms.

Axiom 1. σ μ is a measure on X defined on a σ-ring $\mathfrak{S}$ of subsets of X.

Axiom 2. $\mathfrak{g}$ is a subclass of $\mathfrak{S}$ which is closed under countable intersections and finite unions. Further assumption is that the empty set belongs to $\mathfrak{g}$ and that $\mu\ (C) < +\infty$ for all $C \in \mathfrak{g}$

Axiom 3. υ is a subclass of $\mathfrak{S}$ which is closed under countable unions and finite intersections. Further, for each E in $\mathfrak{S}$ there is a set U in υ such that $E \subset U$.

A set E in $\mathfrak{S}$ is said to be outer regular in case $\mu(E) = \inf\{\mu(U) : E \subset U, U \in \upsilon\}$, inner rregular in case, $\mu(E) = \sup\{\mu(C) : C \subset E, C \in \mathfrak{g}\}$ and regular if it is both outer regular and inner regular. The measure μ is regular if every set E in $\mathfrak{I}$ is regular.

Throughout the paper unless otherwise stated X denotes a locally θ-H-closed, θ-CR and θ-T_2 space. It is to be noted that a locally compact T_2 space is locally θ-H-closed and an example of a space exist [3], which is locally θ-H-closed, θ-CR, θ-T_2 but is non H-closed, non-locally compact, non-regular and non-completely regular.

Now if we consider $\mathcal{g}$, the class of all H-G_δ^θ sets (resp. H-sets) in X, $\mathfrak{I}$, the σ-ring of all θ-Baire (resp. θ-Borel) sets, υ, the class of all θ-open θ-Baire (resp. θ-open θ-Borel) sets and μ any θ-Baire (resp. θ-Borel) measure on X then as an application to the abstract discussion of regulanty [2] with respect to the system (x, $\mathfrak{I}$, μ $\mathcal{g}$, υ) we get the next two theorems.

Theorem 3.1. For a θ-Baire (resp. θ-Borel) measure μ on X and a θ-Baire (resp. θ-Borel) set E, the following are true.

(i) If $\mu(E) = \infty$, then E is outer regular

(ii) If $\mu(E) = O$, then E is inner regular

(iii) Every θ-open θ-Baire (resp. θ-open θ-Borel) set is outer regular

(iv) Every H-G_δ^θ set (resp. H-set) is inner regular

(v) If $\{C_n\}$ is a sequence of H-G_δ^θ sets (resp. H-sets) then $\bigcup\limits_{n=1}^{\infty} C_n$ is inner regular.

(vi) If $\{U_n\}$ is a sequence of θ-open θ-Baire (resp. θ-open θ-Borel) sets and $\mu\ (U_1) < +\infty$ then $\bigcap\limits_{n=1}^{\infty} U_n$ is outer regular.

Theorem 3.2. (i) The union of a sequence of outer regular (resp. inner regular) θ-Baire (θ-Borel) sets is outer regular (resp. inner regular).

(ii) The intersection of a sequence of outer regular (resp. inner regular) θ-Baire (θ-Borel) sets of finite measure is outer regular (resp. inner regular).

The following theorem which was established in [3] will be frequently used.

Theorem 3.3. If $C \subset U$, where C is an H-set in X and U a θ-open set, then there exist θ-Baire sets V and D such that

(i) $C \subset V \subset D \subset U$

(ii) V is θ-open and is union of sequence of H-G_δ^θ sets.

(iii) D is an H-G_δ^θ set.

Note 3.4. We have established in [3] without using any axiom that every θ-Baire measure on X is regular. In the system (X, $\mathfrak{I}$, μ, $\mathcal{g}$, υ), where X is just a non-empty set, S.K. Berberian introduced another four axioms and by the help of these axioms (1-6) he obtained simple criterion for regularity of μ.

The other axioms are

Axiom 4. If $U \in \upsilon$ and $C \in \mathcal{G}$, then $U \backslash C \in \upsilon$. A set E in $\mathfrak{S}$ is bounded if there is a C in $\mathcal{G}$ s.t. $E \subset C$.

Axiom 5. If $C \in \mathcal{G}$. there exists a bounded U in υ s.t. $C \in U$.

Axiom 6. If $C \in \mathcal{G}$ and $U \in \upsilon$, then $C - U \in \mathcal{G}$

Axiom 7. The σ-ring generated by $\mathcal{G}$ is $\mathfrak{S}$.

To obtain simple criterion for regular θ-Borel measure on locally θ-H-closed, θ-CR and θ-T_2 space X, we at first verify the axioms (4-6).

Theorem 3.5. (i) If U is a θ-open θ-Borel set and C in an H-set then $U \backslash C$ is a θ-open θ-Borel set.

(ii) If C is an H-set then there exists a θ-bounded (a set A is said to be θ-Bounded if there exists an H-set K in X s.t. $A \subset K$) set U which is also θ-open θ-Borel set such that $C \subset U$.

(iii) If C is an H-set and U is a θ-open θ-Borel set then $C \backslash U$ is also an H-set.

Proof.

(i) Since C is an H-set in X and since X is θ-T_2 and hence Urysohn, so C is θ-closed, then $U \backslash C = U \cap (X - C)$ is θ-open. U and C being both θ-Borel sets, $U \backslash C$ is a θ-Borel set, then $U \backslash C$ is a θ-open θ-Borel set.

(ii) If C is an H-set, then by Theorem 3.3, there exist a θ-open θ-Baire set U and an H-G_δ^θ set D such that $C \subset U \subset D$, obviously U is θ-bounded since D is an H-set.

(iii) Since U is θ-open, so $X - U$ is θ-closed. Again $C - U = C \cap (X - U)$, which is intersection of an H-set and a θ-closed sets, hence it is an H-set.

As an application to the abstract discussion in [2], we get the following theorem.

Theorem 3.6. If μ is a θ-Borel measure on X, the following are equivalent

(i) μ is regular θ-Borel measure

(ii) every H-set C is outer regular

(iii) every θ-bounded θ-open θ-Borel set is inner regular.

4. In this section we shall establish that every regular θ-Baire measure can be extended uniquely to a regular θ-Borel measure.

Definition 4.1. A non-negative real valued set function m on the class of all H-sets of X is called a θ-content on X if m is monotone, additive and subadditive.

Definition 4.2. For a given θ-content m on X, the set function, $m_*(U) = \sup\{m(H) : H \subset U$, H is an H-set in X$\}$

defined for every θ-open θ-Borel set U in X, is called a inner θ-content induced by m.

Theorem 4.3. If m_* is the inner θ-content induced by a θ-content m on X, then

(i) $m_*(\varnothing) = 0$

(ii) $m_*(U) < \infty$, for every θ-bounded θ-open set U.

(iii) m_* is monotone

(iv) Countably subadditive and countably additive.

Proof. Proof of (i-iii) is trivial. We shall prove (iv).

Let U and V be θ-open θ-Borel sets and C be an H-set s.t. $C \subset U \cup v$.

Since X is Urysohn, $C \setminus U = C \cap (X-U)$ and $C - V = C \cap (X-V)$ are disjoint H-sets. As X is θ-T_2, there exist two θ-open sets A and B in X such that $C - U \subset A$, $C - V \subset B$. Defining $D = C \setminus A$ and $E = C \setminus B$, we get $C = D \cup E$ and $D \subset U$ and $E \subset V$. Clearly D and E are H-sets. As m is subadditive, it follows that $m(C) \leq m(D) + m(E) \leq m_*(U) + m_*(V)$. Hence $m_*(U \cup V) \leq m_*(U) + m_*(V)$. By induction, m_* is finitely additive. Now suppose that $\{U_k\}_{k=1}^{\infty}$ is a sequence of θ-open θ-Borel sets and $U = \bigcup_{k=1}^{\infty} U_k$. If C is any H-set s.t. $C \subset U$ then $C \subset \bigcup_{k=1}^{n} U_k$ for suitable index n and $m(C) \leq m_*(\bigcup_{k=1}^{\infty} U_k) \leq \sum_{k=1}^{n} m_*(U_k) \leq \sum_{k=1}^{\infty} m_*(U_k)$

So $m_*(U) \leq \sum_{k=1}^{\infty} m_*(U_k)$. Therefore m_* is countably subadditive. Countably additiveness of m_* can similarly be tackled.

Definition 4.4. A subset F of X is said to be σ_θ-bounded if there exist a sequence $\{H_k\}_{k=1}^{\infty}$ of H-sets in X such that $F \subset \bigcup_{k=1}^{\infty} H_k$.

The collection of all σ_θ-bounded sets of X will be denoted by H^θ. From now on θ_{Bor}^{op} (resp. θ_{Bor}) denotes the collection of all θ-open θ-Borel sets (resp. θ-Borel sets) of X.

If F is any σ_θ-bounded set i.e., $F \subset \bigcup_{i=1}^{\infty} C_i$, where C_i are H-sets, $C_i \subset X$ then by Theorem 3.3, there exists a θ-open θ-Borel set V_i s.t. $C_i \subset V_i \subset X$. Now $F \subset \bigcup_{i=1}^{\infty} C_i \subset \bigcup_{i=1}^{\infty} V_i$ and $\bigcup_{i=1}^{\infty} V_i$ is a θ-open θ-Borel set.

Thus a σ_θ-bounded set is contained in some θ-open θ-Borel set. We define the set function $m^*(B) = \inf\{m_*(U) : B \subset U, U \in \theta_{bor}^{op}\}$ for every $B \in H^0$.

Note 4.5. It can be easily shown that H^0 is the heriditary σ-ring and m^* is the outer measure on H^0 and also $m^*(B)$ is finite for every θ-bounded set B. From now on m^* is called outer measure induced by m.

Theorem 4.6. m^* extends m_* i.e., $m^*(U) = m_*(U)$ for every $U \in \theta_{Bor}^{op}$. Further if $U \in H$, where U is θ-open and H is an H-set then $m_*(U) = m^*(U) \leq m(H) \leq m^*(H)$.

Proof. Let U and $V \in \theta_{Bor}^{op}$ such that $U \subseteq V$. Then by monotonicity of m_*, $m_*(U) \leq m_*(V)$ and hence $m_*(U) \leq m^*(U)$. Obviously $m^*(U) \leq m_*(U)$. So $m^*(U) = m_*(U)$ for every $U \in \theta_{Bor}^{op}$.

For the 2nd part, as U is θ-bounded and hence θ-Borel so $U \in \theta^{op}_{Bor}$.

So $m_*(U) = m^*(U)$. Let $V \in \theta^{op}_{Bor}$ such that $H \subseteq V$. Thus $m(H) \le m_*(V)$ (by definition). Since m(H) is lower bound of $\{m_*(V) : V \in \theta^{op}_{Bor}$ and $H \subseteq V\}$. $m(H) \le m^*(H)$. Obviously as $U \subseteq H$, by definition of m_*, $m_*(U) \le m(H)$. Hence the result.

As m^* is a outer measure on the heriditary σ-ring of all σ_0-bounded sets H^θ, we now apply general theory of outer measure to the set function m^*.

Definition 4.7. A set $B \in H^\theta$ is m^*-measurable if $m^*(A) = m^*(A \cap B) + m^*(A \cap B')$, where B' is the complement of B, for every $A \in H^\theta$.

It is to be remarked that if the above relation is true for all $A \in \theta^{op}_{Bor}$ then B is m^*-measurable ; infact, if $A \in H^\theta$ then by Theorem 3.3, there exists $U \in \theta^{op}_{Bor}$ such that $A \subseteq U$ and for this U, we have $m_*(U) = m^*(U) = m^*(U \cap B) + m^*(U \cap B') \ge m^*(A \cap B) + m^*(A \cap B')$. Hence $m^*(A) \ge m^*(A \cap B) + m^*(A \cap B')$.

Theorem 4.8. Every set in θ_{Bor} is m^*-measurable and m^*/θ_{Bor} is a regular θ-Borel measure.

Proof. For the first part, because the class M of all m^*-measurable sets forms a σ-ring it suffices to show that every H-set H is m-*measurable. Let $U \in \theta^{op}_{Bor}$ then obviously $U \in H^\theta$ as well and since X is Urysohn, H' is θ-open. Hence $U \cap H'$ is θ-open and as $U \in H^\theta$, it follows $U \cap H' \in \theta^{op}_{Bor}$. Let A be an H-set such that $A \subseteq U \cap H'$, then as above $U \cap A' \in \theta^{op}_{Bor}$. If B is any H-set such that $B \subseteq U \cap A'$, then $A \cap B = \varnothing$ and hence $m^*(U) = m_*(U) \ge m(A \cup B) = m(A) + m(B)$ and this relation holds for any H-set B for which $B \subseteq U \cap A'$. So by definition of m_*, we have $m^*(U) \ge m(A) + m_*(U \cap A')$. Again as $A \subset H'$, we have $A' \supset H$ and $m^*(U \cap A') \ge m^*(U \cap H)$. So $m^*(U) \ge m(A) + m^*(U \cap H)$ [∴ $U \cap A' \in \theta^{op}_{Bor}$ and hence $m^*(U \cap A') = m_*(U \cap A')$]. Again varying A, we get $m^*(U) \ge m_*(U \cap H') + m^*(U \cap H) = m^*(U \cap H') + m^*(U \cap H)$. So H is m^*-measurable. For the 2nd part, since we have noted that m^* is finite for every θ-bounded set then m^* is finite for every H-set. Therefore m^*/θ_{Bor} is a θ-Borel measure. To show that $\mu = m/\theta_{Bor}$ is regular, it suffices to show (by Theorem 3.6) that every H-set H is outer regular.

Now, $\mu(H) = m^*(H) = \inf\{m_*(U) : H \subset U$ and $U \in \theta^{op}_{Bor}\}$. But by Theorem 4.6 $\mu(U) = m^*(U) = m_*(U)$ for every $U \in \theta^{op}_{Bor}$. So $\mu(H) = \inf\{\mu(U) : H \subset U,\ U \in \theta^{op}_{Bor}\}$. So μ is regular θ-Borel measure and is called the regular θ-Borel measure induced by m.

The last theorem asserts that to every θ-content m, the restriction of m^* to θ_{Bor} is a regular θ-Borel measure μ. It is natural to ask when μ will be the extension of m. Before further discussion we need some definitions.

Definition 4.9. For two subsets A and B of X, A is said to be $*$-contained in B denoted by $A <_* B$ if there exist a θ-open set U of X such that $A \subseteq U \subseteq B$.

Definition 4.10. A θ-content m is said to be regular if $m(H) = \inf\{m(D) : H <_* D$ and D is an H-set$\}$ for each H-set H.

Lemma 4.11. If μ_1 and μ_2 and θ-Baire (resp. θ-Borel, [illegible]ures on X, such that $\mu_1(H) = \mu_2(H)$

for every H-G_δ^θ (resp. H-set) H, then $\mu_1 = \mu_2$ i.e. $\mu_1(E) = \mu_2(E)$ for every θ-Baire (resp. θ-Borel) set E.

Proof. If R is the ring generated by the class of all H-G_δ^θ sets (resp. H-sets), it can be easily shown that $\mu_1(E) = \mu_2(E)$ for all E in R. So by Unique extension theorem, $\mu_1(E) = \mu_2(E)$ for all sets E in the σ-ring generated by R.

Theorem 4.12. If m is a regular θ-content on X, there exists a unique regular θ-Borel measure μ such that μ extends m i.e., $\mu(H) = m(H)$, for all H-set H of X.

Proof. The proof follows by three steps.

Step-1. If μ is a θ-Borel measure on X and m is the restriction of μ on the class of H-sets, then μ is a regular θ-Borel measure iff m is a regular θ-content.

Proof. Let μ be regular θ-Borel measure then the restriction of μ on the class of H-sets is obviously θ-content. We shall show that m is regular. Let H be any H-set and $\varepsilon>0$ then by regularity of μ, there exist a $U \in \theta^{op}_{Bor}$ such that $H \subset U$ and $\mu(U) \leq \mu(H) + \varepsilon$. By Theorem 3.3, there exist an H-set D and a θ-open set V such that $H \subset V \subset D \subset U$. So $H <_* D$ and $m(D) = \mu(D) \leq \mu(U) \leq \mu(H) + \varepsilon = m(H) + \varepsilon$. So m is regular. On the other hand, if m is a regular θ-content we shall show that μ is regular. Let H be any H-set and $\varepsilon>0$. As m is regular θ-content, there exists an H-set D such that $H <_* D$ such that $m(D) \leq m(H) + \varepsilon$. So there exists an θ-open set U such that $H \subset U \subset D$ and hence $U \in \theta^{op}_{Bor}$. Since $\mu(U) \leq \mu(D) = m(D) \leq m(H) + \varepsilon = \mu(H) + \varepsilon$. So μ is regular.

Step-2. If μ is a regular θ-Borel measure induced by a θ-content m, then μ extends m iff m is a regular μ-content.

Proof. Let m be a regular θ-content and let H be an H-set. For a given $\varepsilon>0$. there exissts an H-set H_1 such that $H <_* H_1$ and $m(H_1) \leq m(H) + \varepsilon$.

Let U be any θ-open set such that $H \subset U \subset H_1$ (such a set exists because $H <_* H_1$), then by the monotonicity of m^*, $m^*(H) \leq m^*(U)$. Since $U \subset H_1$, by Theorem 4.6 $m_*(U) = m^*(U) \leq m(H_1)$. Therefore

$m^*(H) \leq m^*(U) \leq m(H_1) \leq m(H) + \varepsilon$. Since ε is arbitrary, $m^*(H) \leq m(H)$. Also by Theorem 4.6, $m(H) \leq m^*(H)$. Hence $m^*(H) = m(H)$. The other part is obvious.

Step-3. If μ is any other regular θ-Borel measure which extend m then $\mu'(H) = m(H) = \mu(H)$ for all H-sets H and the uniqueness i.e., $\mu = \mu'$ follows from Lemma 4.11.

Theorem 4.13. If λ is a θ-Baire measure on X, there exists a unique regular θ-Borel measure μ on X such that $\mu(B) = \lambda(B)$ for every θ-Baire set B.

Proof. Let λ be a θ-Baire measure on X, we define the set function m on the class of all H-sets as follows:

$m(H) = \inf\{\lambda(U) : H \subset U$, U is θ-open θ-Baire set$\}$ we shall first show that m is regular θ-content on X. Let H be any H-set in X them by Theorem 3.3, there exissts θ-open θ-Baire set U and an H-G_δ^θ set D such that $H\subset U\subset D$. Therefore $0\le m(H)\le\lambda(U)\le\lambda(D)<+\infty$. Monotonicity of m follows immediately by the property of infimum.

Now if H_1 and H_2 are H-sets in X. Let U_i, i = 1, 2 be θ-open θ-Baire sets such that $H_i\subseteq U_i$ for i = 1, 2 (such set exist by Theorem 3.3). Then $U_1 \cup U_2$ is θ-open θ-Baire set containing $H_1 \cup H_2$. So,

$$m(H_1 \cup H_2) \le \lambda(U_1 \cup U_2) \le \lambda(U_1) + \lambda (U_2).$$

Clearly $m(H_1 \cup H_2) \le m(H_1) + m(H_2)$.

Let H_1 and H_2 be two disjoint H-sets in X. Since X is Urysohn H_i are θ-closed and hence by Theorem 3.3 there exists θ-open θ-Baire sets V_i, i = 1,2 such that $H_1 \subset V_1 \subset X-H_2$ and $H_2 \subset V_2 \subset X-H_1$. Obviously $V_1 \cap V_2 = \phi$. If W is any θ-open θ-Baire set containing $H_1 \cup H_2$ then $H_1\subset V_1 \subset W$ and $H_2 \subset V_2 \cap W$ and hence $\lambda(W) \ge \lambda\, [V_1\cap W) \cup (V_2\cap W)] = \lambda\, (V_1 \cap w) + \lambda\, (V_2 \cap W) \ge m\, (H_1) + m(H_2)$. Taking infimum, we have $m(H_1 \cup H_2) \ge m\, (H_1) + m(H_2)$. So by subadditivity of m we have $m(H_1 \cup H_2) = m(H_1) + m(H_2)$ i.e., m is θ-content. We shall show that m is regular θ-content. Let H be an H-set in X and $\varepsilon>0$. By definition of m there exists a θ-open θ-Baire set U such that $H\subseteq U$ and $\lambda(U) \le m(H) + \delta$. Since $H \subseteq U$, by Theorem 3.3 there exists θ-open θ-Baire set V and an H-set D such that $H\subset V\subset D\subset U$. Now $H <_* D$ and $m(D) \le \lambda\, (U) \le m\, (H) + \varepsilon$. Hence m is regular θ-content.

Moreover, we shall show that $m(D) = \lambda(D)$ for every H-G_δ^θ set D. As D is an H-G_δ^θ set, there exists a sequence $\{U_i\}$ of θ-open sets such that $D = \bigcap_{i=1} U_i$ and for each i, by Theorem 3.3, there exists θ-open θ-Baire set V_i and an H-G_δ^θ set D_i such that $D\subset V_i\subset D_i\subset U_i$. So each V_i is θ-bounded. We may assume $V_n \downarrow D$ (i.e., monotone decreasing and $D = \bigcap_{n=1}^{\infty} V_n$) and so $\lambda(V_n) \downarrow \lambda(D)$ by finitencess of λ and hence $m(D) = \lambda\, (D)$. So m is a regular θ-content and hence by Theorem 4.12, the restriction of m^* to the class of all θ-Borel sets is a regular θ-Borel measure μ that extends m. If λ' be the θ-Baire restriction of μ, then for every H-G_δ^θ set D, we have $\lambda' (D) = \mu(D) = m(D) = \lambda(D)$ i.e., $\lambda=\lambda'$, we also have $\mu(D) = \lambda(D)$ for every θ-Baire set D and that μ is unique follows from Theorem 4.12.

REFERENCES

1. P. Alexandroff and P. Urysohn, Memoire sur less espaces compacts, Verhand Lingen des Kon, Academy Amsterdum Deel XIV, No, 1 (1929).
2. S. K. Berberian, Measure and Integration, the Macmillan Company, 1967.

3. S. Ganguly, C. K. Basu and S. Sinha Roy, a new regular Measure on a topologica space-I, Far East J. Math, Sci. 4(3) 1996.

4. S. Sinha Roy and S. Bandyopadhyay, on θ-completely regular and locally θ-H-closed spaces, Bull. Cal. Math. Soc. 87, 19-28 (1995).

5. N. V. Velicko, H-closed topological spaces, Mat. Sb. 70(112)(1966), 98-112, Amer. Math. Soc, Transl. (2) 78(1969) 103-118.

6. S. Willard, General Topology, Addision-Wesley publishing Company, 1970.

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A CLASS OF ENTIRE FUNCTIONS OF BOUNDED INDEX IN SEVERAL COMPLEX VARIABLES

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ABSTRACT. In this paper we have established an inequality between maximal co-efficients in the power series expansion of a function f about two different points, where f $\in \mathcal{U}_N$, the class of all entire functions of bounded index (b.i) in several complex variables with at least one index $\leq N = (N_1,..,N_n) \in I^n$, I being the set of non-negative integers. We have also obtained a necessary and sufficient condition for a sequence of functions $f_n \in \mathcal{U}_N$ to be normal in C^n. Finally, a sufficient condition for the limit function of a sequence $f_n \in \mathcal{U}_N$ to be of bounded index has been obtained.

1. INTRODUCTION

B. Lepson [2] first introduced the concept of an entire function of bounded index in C, the complex plane, in 1968. Following him many mathematicians like Fred Gross, S.M. Shah, G.H. Fricke etc. studied further properties and characterization of such functions. It turned out that solutions of a large class of differential equations in C are entire functions of bounded index. Unfortunately, almost no work was done on the corresponding theory in several complex variables. Till date, to our knowledge, only in one paper J.G. Krishna and S.M. Shah [1] generalized this concept for analytic functions in several complex variables with respect to an arbitrary given point α in C^n. But they did not proceed further. In this paper we shall introduce the concept of bounded index for entire functions in C^n without reference to any given point. In fact, our defn. coincides with that of J.G. Krishna & S.M. Shah if the analytic function is, in particular, an entire function and the arbitrary given point, $\alpha = (1,..........,1)$.

2. NOTATIONS

C^n and R^n will denote respectively the complex and real n-space. We denote the points $(z_1..........z_n)$, $(w_1.........., w_n)$ etc. of C^n by their corresponding unsuffixed symbols z, w etc. For z, w $\in C^n$ and $\alpha \in C$ we define $\alpha z=(\alpha z_1,........,\alpha z_n)$, $z+w = (z_1+w_1,..........,z_n+w_n)$, $z.w = (z_1w_1+z_2w_2+.......+z_nw_n)$,

$|z| = (|z_1|^2 + \dots\dots + |z_n|^2)^{1/2}$. The positive hyperoctant R_+^n in R^n will be $R_+^n = \{x : x \in R^n, x_i \geq 0, i = 1, \dots\dots, n\}$ and $I_+^n = \{x : x \in I^n, x_i > 0, i = 1, \dots\dots\dots, n\}$, I being the set of non-negative integers. For $t \in R_+^n$ we set $\|t\| = t_1 + \dots\dots + t_n$ and for $m \in I^n$, $m! = m_1! \dots\dots m_n!$ $z^m = z_1^{m_1} \dots\dots z_n^{m_n}$ for $z \in C^n$ $m \in I^n$ ($z_i^{(0)} = 1$ even if $z_i = 0$). For any $k \in R$, $\bar{k} = (k, \dots\dots, k)$. In particular $\bar{0}$ and $\bar{\propto}$ will respectively denote the n tuples $(0, \dots\dots, 0)$ and $(\propto, \dots\dots, \propto)$. Also, for $x, y \in R^n$ we say that (i) $x \leq y$ iff $x_i \leq y_i$ (ii) $x < y$ iff $x \leq y$ but $x \neq y$; (iii) $x << y$ iff $x_i < y_i$, $i = 1, \dots\dots, n$. For an entire function f with domain C^n, $f^{(k)}$ will denote the partial derivative $\dfrac{\partial^{\|k\|} f}{\partial z_1^{k_1} \dots\dots \partial z_n^{k_n}}$, where $k \in I^n$ and $f(\bar{0}) = f$.

3. An entire function f in C^n has a Taylor expansion about any point $w \in C^n$ of the form :

$f(z) = \sum\limits_{n=0}^{\infty} a_m (z-w)^m$, where $a_m = \dfrac{f^{(m)}(w)}{m!}$ Since the power series of f about w is absolutely convergent everywhere in C^n, the terms $\dfrac{|f^{(m)}(w)|}{m!}$ must tend to zero. Consequently, for each $w \in C^n$ there exists

an n-tuple $(m_1, \dots\dots m_n) = m \in I^n$

$m = m(w)$, such that $\dfrac{|f^{(m)}(w)|}{m!}$ is a maximal co-efficient. Thus,

Definition 1. For each point $w \in C^n$ there exists an n–tuple $m(w) \in I^n$ s. t.

$$\max_{p \leq m} \left\{ \frac{|f^{(p)}(w)|}{p!} \right\} \geq \frac{|f^{(q)}(w)|}{q!}, \ \forall q \in I^n.$$

There may be several tuples m for which the above inequality holds. A minimal element of such tuples is called an index of f at the point w.

Definition 2. An entire function f in C^n is said to be of bounded index iff there exists an $m \in I^n$ such that $\forall\ z \in C^n$,

$$(3.1) \quad \max_{p \leq m} \left\{ \frac{|f^{(P)}(z)|}{p!} \right\} \geq \frac{|f^{(q)}(z)|}{q!}, \ \forall\ q \in I^n.$$

There may have several tuples satisfying (3.1). A minimal element of such tuples is called an index of f and is denoted by I(f). An entire function which is not of b.i is said to be of unbounded index.

Definition 3. A sequence $\{f_n\}$ of entire functions in C^n is said to be almost uniformly convergent in a domain D in C^n if it converges uniformly on every compact subset of D.

Definition 4. Normal Sequence : Let $\{f_n\}$ be a sequence of entire functions in C^n. $\{f_n\}$ is said to be normal iff there exists a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\}$ is almost uniformly convergent in C^n to an entire function g (say).

Ω_N : Ω_N will denote the class of all entire functions of bounded index in C^n with at least one index

$\le N,\ N \in I^n$.

$C_f(w)$: $C_f(w)$ will denote the maximal co-efficient of Taylor series expansion of an entire function f about the point w, $w \in C^n$, i.e.

$$C_f(w) = \max_{j \le N} \left[\frac{|f^{(j)}(w)|}{j!}\right], \text{ for } w \in C^n.$$

Theorem 1. Let $f \in \Omega_n$ Then $C_f(z) < 2^{\|N\|+n} C_f(w)$ for any two points $z, w \in C^n$ such that $|z-w| \le 1/2$.

Proof : Let one index of f be $K \le N$. Then,

$$\text{(3.2)}\quad f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(w)}{m!}(z-w)^m \text{ for all } z, w \text{ and } \frac{|f^{(q)}(w)|}{p!} \le C_f(w) = \max_{q \le k} \left\{\frac{|f^{(q)}(w)|}{q!}\right\} \forall P \in I^n$$

Since the series (3.2) is absolutely convergent and the geometric series $\sum_{m=0}^{\infty} |z_1-w_1|^{m_1} \ldots\ldots |z_n-w_n|^{m_n}$ is convergent in D, $D = \{z : |z_i-w_i| < 1,\ i=1,\ldots\ldots, n\}$ it follows

$$|f(z)| \le C_f(w)\sum_{m=0}^{\infty} |z_1-w_1|^{m_1} \ldots\ldots |z_n-w_n|^{m_n}$$

$$= \frac{C_f(w)}{\prod_{i=1}^{n} (1-|z_i-w_i|)}$$

for all z, w such that $|z-w| < 1$.

Now, $f^{(p)}(z) = \sum_{m=p}^{\infty} m_1(m_1-1) \ldots\ldots (m_1-p_1+1) \ldots\ldots m_n (m_n-1) \ldots\ldots (n_n-p_n+1)a_m(z-w)^{m-p}$.

But the series

$$\sum_{m=p}^{\infty} m_1(m_1-1) \ldots\ldots (m_1-p_1+1) \ldots\ldots m_n (m_n-1) \ldots\ldots (m_n-p_n+1)\, r^{m-p},$$

which is obtained by differentiating term by term the geometric series $\sum_{m=0}^{\infty} r^m$ p – times successively, converges to

$$= \frac{p!}{\prod_{i=1}^{n} (1-r_i)^{p_i+1}}$$

Hence

$$\frac{|f^{(p)}(z)|}{p!} \le \frac{C_f(w)}{\prod_{i=1}^{n} (1-|z_i-w_i|)^{p_i+1}}, \ \forall P \in I^n$$

and all z, w such that $|z_i - w_i| < 1$, $i = 1, \dots\dots, n$.

Therefore,

$$C_f(z) \le \frac{C_f(w)}{\prod_{i=1}^{n} (1 - |z_i - w_i|^{k_i+1}} \le 2^{\|k\|+n} C_f(w)$$

$$\le 2^{\|N\|+n} C_f(w)$$

for all z, w such that $|z_i - w_i| \le 1/2$, $(i=1, \dots\dots, n)$ and hence for all z, w such that $|z-w \le 1/2$

Theorem 2. Let $f \in \Omega_N$ and S be a bounded set in C^n.

Then there exists a positive integer t, depending only on N and S, such that the inequality (3.3) $C_f(z) \le t\, C_f(w)$ is true for any two points $z, w \in S$.

Proof. Let d be the diameter of S. Then there is a positive integer l such that $2d < l$ Let z, w, be any two points of S. Then $|z - w| \le d$.

We can choose a finite sequence $\{z^{(p)}\}_0^k$ of points with $z^{(0)}=z$ $z^{(k)}=w$ and $k \le l$ such that each $z^{(p)}$ lies on the line segment joining z and w and $|z^{(p)} - z^{(p-1)}| \le 1/2$ for all $p = 1, \dots\dots\dots, k$.

By the Theorem 1

$C_f(z) = C_f(z^{(0)}) \le 2^{\|N\| + n} C_f(z^{(1)} \le 2^2(\|N\| + n) C_f(z^{(2)}) \le \dots\dots\dots \le 2^{k(\|N\|)+n)} C_f(2^{(k)}$
$\le 2^2(\|N\| + n) C_f(w)$.

Choosing $t = 2^{l(\|N\|+n)}$ we get the theorem.

We now obtain a necessary and sufficient condition for a sequence of functions $f_n \in \Omega_N$ to be normal in C^n.

Theorem 3. Let $\{f_n\}$ be a sequence in Ω_N. The sequence $\{f_n\}$ is normal iff there is a subsequence $\{C_{f_{n_k}}\}$ of the sequence $\{C_{f_n}\}$ which is bounded at a single point in C^n.

Proof. First we let the sequence $\{f_n\}$ be normal in C^n. Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which is almost uniformly convergent in C^n and converges to an entire function g (say).

Let w be a point in C^n and $\in$ be a positive number. Now since the sequence $\{f_{n_k}^{(j)}(w)/j!\}$ converges uniformly to $g^{(j)}(w)/j!$, $j \in I^n$, we have that there is a positive integer t_j corresponding to jth partial derivative such that

(3.4) $|f^{(j)}_{t_j+p}(w)|/j! < |g^{(j)}(w)|/j! + \in$, $\forall\, p \in I$, $j \ge \bar{0}$, $j = (j_1, \dots\dots\dots, j_n)$.

Let $m = \max\limits_{0 \le j \le N} \{t_j\}$. Then

(3.5) $|f^{(j)}_{m+p}(w)|/j! < |g^{(j)}(w)|/j! + \varepsilon$, $\forall\, p \in I$, $\bar{0} \le j \le N$.

Since each $f_{m+p} \in \Omega_N$ we have, from (3.5)

$C_{f_{m+p}}(w) < C_g(w) + \varepsilon$, $\forall\, p \in I$,

which shows that the subsequence $\{C_{f_{n_k}}\}$ of the sequence $\{C_{f_n}\}$ is bounded at the point w. In fact, if $\{f_n\}$ is normal we have shown that $\{C_{f_n}\}$ is bounded at every point of C^n.

Conversely, let us suppose the subsequence $\{C_{f_{n_k}}\}$ of the sequence $\{C_{f_n}\}$ is bounded at a point $w \in C^n$.

Let D be an arbitrary compact set in C^n. We can find a bounded set G such that $w \in G$ and $D \subseteq G$. There is a positive number M such that

$C_{f_{n_k}}(w) \leq M, \forall k \in I.$

Now by the theorem 2., there is a positive number h depending on G (and hence on D) and N such that

$C_{f_{n_k}}(z) \leq h. C_{f_{n_k}}(w)$

$\leq h.M, \forall k \in I$ and $\forall z \in D.$

Since $|f_{n_k}(z)| \leq C_{f_{n_k}}(z), \forall z \in D, k \in I.$

We have,

$|f_{n_k}(z)| \leq M. h, \forall z \in D$ and $\forall k \in I.$

The above inequality shows that the sequence $\{f_{n_k}\}$ is uniformly bounded on every compact subset D of C^n.

Hence by generalized Montel's theorem the sequence $\{f_n\}$ is normal in C^n.

Theorem 4. Let $\{f_n\}$ be a sequence in Ω_N. If $\{f_n\}$ is almost uniformly convergent in C^n to an entire function g, then g is of bounded index with one index $\leq N$.

Proof : Let the sequence $\{f_n\}$ be almost uniformly convergent in C^n to an entire function g.

To prove that g is of bounded index it is enough to show that

$$M_g(z) = \max_{\bar{0} \leq j \leq N} \{\frac{|g^{(j)}(z)|}{j!}\} = C_g(z), \forall z \in C^n.$$

If possible, let $M_g(z) < C_g(z)$, for some z.

Then $C_g(z) = |g^{(s)}(z)|/s!$, for some $s > N$.

Now we choose a positive number ε such that

$2\epsilon < \delta$, where $0 < \delta = C_g(z) - M_g(z)$

Since the sequence $\{f_n^{(j)}(z)/j!\}$ converges to $g^{(j)}(z)/j!$, for all $j \in I^n$, arguing similarrly as in (3.5), there is a positive integer m such that

$|f^{(j)}_{m+k}(z)|/j! < |g^{(j)}(z)| / j! + \varepsilon,$

for all $k \in I$ and $\bar{0} \leq j \leq N$.

Since each $f_n \in \Omega$ n, we have

(3.6) $C_{f_{m+k}}(z) < M_g(z) + \epsilon, \forall k \in I.$

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Again there is a positive integer t tuch that

(3.7) $|g^{(s)}(z)|/s! < |f^{(s)}{}_{t+k}(z)|/s! + \epsilon, \ \forall\, k \in I.$

Hence if k_1 and k_2 be two integers such that $m+k_1 = t + k_2 = q$ (say), then from (3.6) and (3.7) we obtain,

$M_g(z) + \delta = C_g(z) < |f_q{}^{(s)}(z)| / s! + \epsilon$

$\leq C_{f_q}(z) + \epsilon$

$< M_g(z) + \epsilon$

which is absurd.

Therefore,

$$C_g(z) = \max_{0 \leq j \leq N} \{|g^{(j)}(z)|/j!\}, \ N \in I^n, \ \forall\, z \in C^n.$$

In other words, g is of bounded index with one index $\leq N$.

REFERENCES

[1] Krishna, J.K. : Functions of bounded indices in one and several complex variables.
Shah, S. M. Maths Essays dedicated to A.J. Macintyre.
Ohio univ Press, Athens, Ohio (1970). pp 223 - 235.

[2] Lepson. B : Differential equations of infinite order. Hyper Dirichlet series and entire functions of bounded index. Proc. Sympo. Pure Math, vol. II, Amer. Math. Soc. (1968) PP 298 - 307.

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SOME RESULTS ON SUBSERIES OF DIVERGENT SERIES

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INTRODUCTION. Let E denote the metric space whose points are the permutation of the set of all positive integers endowed with the Frechet metric

$$\rho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1+|x_i - y_i|} \quad \text{where } x=\{x_i\}_{i=1}^{\infty},\ y = \{y_i\}_{i=1}^{\infty}$$

Agnew [1] proved that the metric space (E, ρ) is of second category at each of its points. For $x \in E$ and $r > o$, we denote by S(x, r) the set of points y of E such that $\rho(x,y) < r$. Thus S(x, r) denotes the open sphere in E with x as centre and r as radius. It follows easily that if $x = \{x_n\}_{n=1}^{\infty}$ and $y=\{y_n\}_{n=1}^{\infty}$ are points of E and $y \in s\left(x, \frac{1}{2^{N+1}}\right)$, then $x_n = y_n$ for $n = 1, 2, \ldots N$ and if $x_n = y_n$ for $n = 1, 2, \ldots, N$, then $y \in s\left(x, \frac{1}{2^N}\right)$. Also it may be verified that the sequence of points $x^{(n)} = \{x_1^{(n)}, x_2^{(n)}, \ldots\}$ of E converges to a point $x^o=\{x_1^o, x_2^o, \ldots\}$ of E if and only if $x_i^{(n)} = x_i^o$ for $n>N$, $i=1, 2, \ldots$

In order to study some properties of the re-arrangements of a given conditionally convergent series $\sum_n a_n$, Agnew [1] established a mapping between the space E and all possible rearranged series of $\sum_n a_n$ by associating to the point $x = \{x_n\}_{n=1}^{\infty} \in E$ the rearranged series $\sum_{n=1}^{\infty} a_{x_n}$.

He also proved that for all $x = \{x_1, x_2, \ldots\} \in E$ with the exception of points of a set of the first category of the space E, $\limsup_{n\to\infty} \sum_{k=1}^{n} a_{x_k} = +\infty$, $\liminf_{n\to\infty} \sum_{k=1}^{n} a_{x_k} = -\infty$ ...... (1)

This result was supplemented by Sengupta [5] who established that the set over which (1) holds is a G_δ set and later on he [6] studied some properties of the rearranged series of a conditionally convergent series from the point of view of the topology in E.

For conditionally convergent series, it is known that i) $a_n \to o$ as $n \to \infty$ and ii) the series of positive terms of $\sum_{n=1}^{\infty} a_n$ and the series of absolute value of negative terms of $\sum_{n=1}^{\infty} a_n$ both diverge to $+\infty$.

Ganguli and Lahiri [3] proved that the set of points of E which corresponds to those rearrangements of a series $\sum_n a_n$ satisfying conditions (i) and (ii) for which $\limsup_{n\to\infty} \sum_{k=1}^{n} a_{x_k} = +\infty$ and $\liminf_{n\to\infty} \sum_{k=1}^{n} a_{x_k} = -\infty$ is a dense subset of E.

The purpose of the present paper is to improve the result of Ganguli and Lahiri taking weaker condition and it also sharpens the result of Sengupta [5] and an attempt has made also to study some properties of the rearranged series from the point of view of the topology on E.

Let $\sum_{n=1}^{\infty} a_n$ be a divergent series. To each rearrangement of the series $\sum_{n=1}^{\infty} a_n$ there corresponds a complex $\{x_1, x_2, \ldots\ldots\}$ and conversely, where $\{x_1, x_2, \ldots\ldots\}$ is a permutation of all positive integers. A series $\sum_{n=1}^{\infty} a_n$ with real terms is said to be of type (A) if the condition (ii) holds.

Theorem 1. Let $\sum_{n=1}^{\infty} a_n$ be any series with real terms of type (A). Then the set A $(-\infty, \infty)$ of all points $x = \{x_n\}_{n=1}^{\infty}$ of E which corresponds to those rearrangements $\sum_{n=1}^{\infty} a_{x_n}$ for which $\lim_{n\to\infty} \sup \sum_{k=1}^{n} a_{x_k} = +\infty$ and $\lim_{n\to\infty} \inf \sum_{k=1}^{n} a_{x_k} = -\infty$ is a residual G_δ set.

Proof. We can write

$$A(-\infty,\infty) = \{x \in E \mid \lim_{n\to\infty}. \text{Sup. } S_n(x) = +\infty,\ \lim_{n\to\infty}. \inf S_n(x) = -\infty\}$$

Where S_n, as usual denotes the nth partial sum.

$$\text{Therefore, } A(-\infty, \infty) = \{x \in E \mid \lim_{n\to\infty}. \text{Sup. } S_n(x) = +\infty\} \cap \{x \in E \mid \lim_{n\to\infty}. \inf. S_n(x) = -\infty\}$$
$$= A(\infty) \cap A(-\infty) \text{ (Say).}$$

We first show that the set $A(\infty)$ is a residual G_δ Set.

If $x = \{x_n\}_{n=1}^{\infty} \in A(\infty)$ then $\lim_{n\to\infty} \sup \sum_{k=1}^{n} a_{x_k} = +\infty$ i.e. $\lim_{n\to\infty} \text{Sup. } S_n(x) = +\infty$. Then corresponding to each positive integer k, there exists a positive integer N such that $S_N(x) > k$. Denoting by A_k the set of all those $x = \{x_n\}_{n=1}^{\infty} \in E$ for which there can be chosen an integer $N(x)$ such that $S_N(x) > k$. Then A_k is an open set in E. In fact, if $x = \{x_n\}_{n=1}^{\infty}$ and $y = \{y_n\}_{n=1}^{\infty}$ be any point of the open sphere $S(x, 2^{-N-1})$ then $x_i = y_i$, $i = 1, 2, \ldots\ldots, N$ and so $S_N(y) > K$ i.e., $y \in A_k$; therefore $S(x, 2^{-N-1}) \subset A_K$. Now, $A(\infty) = \bigcap_{k=1}^{\infty} A_k$ and each A_k is an open set. So $A(\infty)$ is a G_δ set in E.

We now show that $A(\infty)$ is dense in E. Let $\varepsilon > 0$ be given arbitrary number. We choose a positive integer m such that $2^{-m} < \varepsilon$.

Let $z = \{z_i\}_{i=1}^{\infty}$ be any point of E. We construct a sequence $\{x_1, x_2, \ldots, x_n, \ldots\}$ as follows:

$x_i = z_i$; $i = 1, 2, \ldots\ldots m$.

Let, $N_- = \{i \mid a_i < 0\}$.

Put $N_- - \{z_1\ z_2 \ldots\ldots z_m\} = \{l_1, l_2, \ldots\ldots\}$ where $l_k < l_{k+1}$.

If $i \in \{1, 2, \ldots\ldots, m, (m+1), \ldots\ldots, l_1-1\} \cup \{l_1+1, l_1+2, \ldots\ldots l_2-1\} \cup \ldots\ldots$

$\ldots\ldots \cup \{l_k+1, l_k+2, \ldots\ldots l_{k+1}-1\} \cup \ldots\ldots$

then $a_i > 0$.

By hypothesis of the theorem, we can choose a positive integer m_1 such that the distinct positive integers $x_{m+1}, x_{m+2} \ldots\ldots\ldots x_{m+m_1}$ all are different from x_i ($i = 1, 2 \ldots\ldots\ldots m$) but

$\{x_1, x_2 \ldots\ldots\ldots\ldots x_{m+m_1}) \subset \{1, 2 \ldots\ldots\ldots\ldots m, m+1 \ldots\ldots\ldots, l_1 - 1\}$ for which

$\sum_{i=1}^{m+m_1} a_{x_i} > a_{x_{m+m_1+1}} + 1$ where $x_{m+m_1+1} = l_1$.

Similarly, we determine the positive integer m_2 $(> m_1 + 1)$ such that the distinct positive integers $x_{m+m_1+2}, x_{m+m_1+3}, \ldots\ldots\ldots\ldots x_{m+m_2}$ are different from the integers $x_1, x_2 \ldots\ldots, x_{m+m_1+1}$ but $\{x_{m+m_1+2}, \ldots\ldots\ldots, x_{m+m_2}\} \subset \{l_1+1, \ldots\ldots\ldots, l_2-1\}$ for which $\sum_{i=1}^{m+m_2} a_{x_i} > a_{x_{m+m_2+1}} + 2$ where $x_{m+m_2+1} = l_2$

Proceeding in this way, we determine the positive integer m_n $(m_n > m_{n-1}+1)$ such that the distinct positive integers $\boxed{x_{m+m_{n-1}+2}}$ $, \ldots\ldots\ldots, x_{m+m_n}$ are different from $x_1, x_2, \ldots\ldots\ldots, x_{m+m_{n-1}} + 1$ but $\{x_{m+m_{n-1}+2}, \ldots\ldots\ldots, x_{m+m_n}\} \subset \{l_{n-1}+1, \ldots\ldots\ldots\ldots, l_n - 1\}$ for which $\sum_{i=1}^{m+m_2} a_{x_i} > a_{x_{m+m_n+1}} + n$ where $x_{m+m_n+1} = l_n$
If this process is continued, we obtain a rearrangement of the given series which diverges to $+ \propto$. Clearly $x = \{x_i\}_{i=1}^{\propto} \in E$ and also $x \in A(\propto)$.

We see that $x \in S(z, \varepsilon)$ and so $x \varepsilon A(\propto) \cap S(z, \varepsilon)$

which implies that $A(\propto)$ is dense in E. Therefore, $A(\propto)$ is a residual G_δ Set in E.

In an analogous manner, it can be proved that $A(-\propto)$ is also a residual G_δ set in E. Hence the theorem follows.

Theorem 2. Let $\sum_n a_n$ be a divergent series.
Then the set of points $x = \{x_n\}_{n=1}^{\propto} \in E$ for which the series $\sum_{n=1}^{\propto} a_{x_n}$ converges is a set of first category of Borel Multiplicative class of type two.

Proof : Let S denote the set of those points.
$x = \{x_i\}_{i=1}^{\propto}$ of E for which $\sum_{n=1}^{\propto} a_{x_n}$ converges. Then by Cauchy's criterion for convergence we can write :

$$S = \{x = \{x_i\}_{i=1}^{\propto} \in E \mid \forall_{p=1}^{\propto} \exists_{q=1}^{\propto} \forall_{m=1}^{\propto} \forall_{n=1}^{\propto} \mid a_{x_{q+m}} + \ldots\ldots + a_{x_{q+m+n}} \mid \leq \tfrac{1}{p}\}$$

Let $Sp = \{x = \{x_i\}_{i=1}^{\propto} \in E \mid \exists_{q=1}^{\propto} \forall_{m=1}^{\propto} \forall_{n=1}^{\propto} \mid a_{x_{q+m}} + \ldots\ldots + a_{x_{q+m+n}} \mid \leq \tfrac{1}{p}\}$

$$= \bigcup_{q=1}^{\propto} \bigcap_{m=1}^{\propto} \bigcap_{n=1}^{\propto} \{x = \{x_i\}_{i=1}^{\propto} \in E \mid \mid a_{x_{q+m}} + \ldots\ldots + a_{x_{q+m+n}} \mid \leq \tfrac{1}{p}\}$$

Therefore,

$$S_p = \bigcup_{q=1}^{\propto} \bigcap_{m=1}^{\propto} \bigcap_{n=1}^{\propto} S_{qmn};$$

Where

$$S_{qmn} = \{x = \{x_i\}_{i=1}^{\propto} \in E \mid \mid a_{x_{q+m}} + \ldots\ldots\ldots + a_{x_{q+m+n}} \mid \leq \tfrac{1}{p}\}.$$

This S_{qmn} is a closed set in E. In fact, if $\{x^{(k)}\}$ $(x^{(k)} = \{x_i^{(k)}\}_{i=1}^{\propto})$ is a sequence in S_{qmn} which converges to $x = \{x_n\}_{n=1}^{\propto}$ then $\lim_{k \to \propto} x_i^{(k)} = x_i$ for each $i = q + m, q + m + 1, \ldots\ldots\ldots\ldots, q + m + n$.

Therefore $x \in S_{qmn}$. Consequently each set S_{qmn} is closed. Hence S_p is an F_σ set and so $S = \bigcap_{p=1}^{\infty} S_p$ is an $F_{\sigma\delta}$ set i e., a Borel set of multiplicative class of type two.

Now to show that S is a set of first category, it is sufficient to show that the complement of the set $S_p = \bigcup_{q=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} S_{qmn}$ is dense in E as S_p being already proved an F_σ set [See [4], p-81]. For this let $z = \{z_n\}_{n=1}^{\infty} \in E$ and $S(z, \varepsilon)$ be an open sphere with z as centre and $\varepsilon(>o)$ as radius. Let K be the smallest positive integer such that $\sum_{i=k+1}^{\infty} 2^{-i} < \varepsilon$.

We define the sequence $x = \{x_n\}_{n=1}^{\infty} \in E$ as follows :

$x_i = z_i$ for $i = 1, 2 \ldots\ldots\ldots\ldots K$

If $x_k \leq k+1$ we write $x_i = i$; $i = K+1, K+2, \ldots\ldots\ldots\ldots$

If $x_k > K+1$, we put $x_t = x_k$ for $t = K+1, K+2, \ldots\ldots\ldots, l-1$,

where l is the smallest positive integer for which $l \geq x_k$; $x_i = i$ for $i = l, l+1, l+2, \ldots\ldots\ldots$

So we can find an integer N such that $x_i = i$ for $i = N, N+1 \ldots\ldots\ldots$ Obviously $x \in S(z,\varepsilon)$.

Hence for every q, there are m and n such that

$$|a_{x_{q+m}} + \ldots\ldots\ldots\ldots + a_{x_{q+m+n}}| = |a_{q+m} + \ldots\ldots\ldots + a_{q+m+n}| > \frac{1}{p}.$$

This shows that x belongs to the complement of S_p and also S_p is an F_σ Set. Therefore S_p is a set of first category in E. Since each of the sets S_p is of first category in E, therefore the set $S = \bigcap_{p=1}^{\infty} S_p$ is First category in E.

Note. It follows from the construction of the sequence $\{x_n\}_{n=1}^{\infty} \in E$ that it is possible to establish that the set $\{x \varepsilon E \mid \lim_{k\to\infty} x_k = +\infty\}$ is dense in E.

Theorem 3. Let $\sum_n a_n$ be a series of real terms of type (A) and $\lim_{n\to\infty} a_n = O$. If for each $x \in E$, except those belonging to a set of first category, the sequence $\{S_n(x)\}$ is compact, then the set of all limit points of $\{S_n(x)\}$ is precisely $<-\infty, \infty>$ (under $<-\infty, \infty>$, we understand all real numbers in $(-\infty, \infty)$ including $+\infty$ and $-\infty$) where $S_n(x) = \sum_{k=1}^{n} a_{x_k}$ $(n = 1, 2, \ldots\ldots\ldots\ldots)$

Proof. We have proved in theorem 1. that the set of all those $x \in E$ for which $\liminf_{n\to\infty} \sum_{k=1}^{n} a_{x_k} = -\infty$ and $\limsup_{n\to\infty} \sum_{k=1}^{n} a_{x_k} = +\infty$ is a residual G_δ set in E. Because of the condition $\lim_{n\to\infty} a_n = O$ we have $\lim_{n\to\infty} | S_{n+1}(x) - S_n(x) | = O$ Therefore by [2] the set of all limit points of $\{S_n(x)\}_n$ is connected. As $-\infty$ and $+\infty$ are also limit points of $\{S_n(x)\}_n$ hence $\{S_n(x)\}' = <-\infty, \infty>$ where $\{S_n(x)\}'$ stands for the set of all limit points of the sequence $\{S_n(x)\}$.

REFERENCE

1. Agnew, R.P. On rearrangement of series, Bull. Amer Math. Soc 46 (1940), 797-799.

2. Barone, H.g. Limit points of sequences and their transformation by methods of summability. Duke Math. Journal Vol.5 (1939), 740-752.

3. Ganguli, P.L. and Lahiri, B.K. Some results on certain sets of series. Czechoslovak Mathematical Journal, 18(93), 1968, Praha, 589-594.

4. Kuratowski, K. Topology, Vol.I, Academic Press, N.Y, 1964.

5. Sengupta, H.M. Rearrangements of series, Proc. Amer. Math Soc, 1(1950), 71-75

6. Sengupta, H.M. On rearrangements of series, Proc-Amer. Math. Soc 7 (1956) 347-750.

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FURTHER CHARACTERIZATIONS OF δ_p – CLOSED SPACES

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ABSTRACT : The present paper is a continuation of the study of δ_p-closedness of topological spaces, a sort of covering property strictly stronger than quasi H-closedness and independent of compactness, introduced and investigated in an earlier paper [4]. Such a space as studied earlier in terms of a generalized class of open sets termed δ-preopen sets, is characterized here via certain newly introduced concepts viz. p-continuity, and p-subclosedness and strong p-closedness of graphs of functions.

Keywords : δ_p-closed space, p-convergence, p-adherence, p-continuity, p-subclosed graph, strongly p-closed graph.

1990 Mathematics Subject Classification Code : 54C99, 54D99.

1. INTRODUCTION AND PREREQUISITES :

The concept of δ_p-closedness was introduced in [4] by using the notion of δ-preopen sets and δ-preclosure of such sets. In [4] we obtained a number of characterizations of δ_p-closed spaces in terms of nets, subnets, filters, ultrafilters and a certain type of covers termed p-covers. It was also shown that a δ_p-closed space is quasi H-closed, although the former one is independent of each of compactness and near compactness. In this paper we continue the investigation by presenting some more characterizing theorems for a δ_p-closed space in terms of certain newly introduced concepts viz. p-continuity p-subclosedness and strong p-closedness of graphs and allied ideas. Although the notions involved are not topological in the sense that δ-preopen sets do not always form a topology, the results obtained for δ_p-closed spaces resemble very much the corresponding ones for quasi H-closed or compact or nearly compact spaces.

In Section-2, we introduce the notions of p-continuous functions and p-subclosedness of graphs of functions. These concepts are characterized and studied to that extent which we require to characterize δ_p-closed spaces in terms of them. Such characterizations are emboddied in Section-3. In the last section we define strongly p-closed graphs of functions and ultimately characterize a δ_p-closed space via such a notion, the result being an analogue to the following characterization theorem for an H-closed space : A Hausdorff space Y is H-closed iff for every space in class ζ, each $g : X \to Y$ with a strongly closed graph is weakly continuous [1].

Throughout the paper, by X and Y we shall mean topological spaces. Let us recall certain definitions and results to use them frequently in the sequel. If A is a subset of X, the closure and interior of A in X are denoted by clA and intA respectively. A point $x \in X$ is called a δ-adherent point [5] of A, denoted as $x \in$ δ-clA, if (int clU) $\cap A \neq \emptyset$, for every open neighbourhood (henceforth nbd, for short) U of x. The set of all δ-adherent points of A is called the δ-closure of A [5], denoted by δ-clA. A set $A \subset X$ is said to be δ-closed in X if A = δ-clA. A set A in x is called δ-preopen if $A \subset$ intδ-clA, and the complement of a δ-preopen set is called δ-preclosed [3]. The collection of all δ-preopen sets in a space X is denoted by δ PO(X) and that of all δ-preopen sets each containing a given point x of X is denoted by δ PO(x). In this context, we note that for any topological space X, δ PO(X) is, in general, a strictly larger class than that of open sets. Arbitrary union of δ-preopen sets is δ-preopen but intersection of even two such sets may not be δ-preopen so that δ PO(X) is not, in general, a topology on X. The intersection (union) of all δ-preclosed (δ-preopen) sets each containing (contained in) a set A in X is called the δ-preclosure (resp. δ-preinterior) of A and denoted by δ-pcl A (resp. δ-p intA). A is δ-preclosed (δ-preopen) iff A = δ-pcl A (resp. A = δ-p intA). A set A in X is called a δ-prenbd of a point x of X if $x \in U \in A$ for some $U \in$ δ PO(x). A set A in X is δ-preopen iff it is a δ-prenbd of each of its points ; moreover, $x \in$ δ-pcl A iff every δ-prenbd of x intersects A [3]. A space X is said to be δ-preHausdorff [2] if for any two distinct points x and y, there exist $U \in$ δ PO(x) and $V \in$ δ PO(y) such that $U \cap V = \emptyset$, A filterbase $\mathcal{F}$ on $A \subset X$ is said to p-adhere at x, written $x \in$ p-ad $\mathcal{F}$, if for each $U \in$ δ PO(x) and each $F \in \mathcal{F}$, $F \cap$ δ-p c1U $\neq \emptyset$, and $\mathcal{F}$ is said to p-converge to x, denoted as $\mathcal{F} \xrightarrow{p} x$, if for each $U \in$ δ PO(x), there is $F \in \mathcal{F}$ such that $F \subset$δ-pcl U. A net $\{x_\alpha : \alpha \in (D, \geq)\}$ in $A \subset X$ is said to p-adhere at x (p-converge to x), written $x \in$ p-ad(x_α) (resp. $x_\alpha \xrightarrow{p} x$), if for each $U \in$ δ PO(x) and each $\alpha \in D$, there is $\beta \geq \alpha$ $(\beta \in D)$ such that $x_\beta \in$ δ-pcl U (resp. if the net is eventually in δ-pcl U, for all $U \in$ δ PO(x). For usual convergence and adherence of a fillterbase $\mathcal{F}$ we adopt the usual notations $\mathcal{F} \to x$ and $x \in$ ad$\mathcal{F}$ respectively ; similar notations for net are also adhered to.

2. p-CONTINUITY AND p-SUBCLOSED GRAPHS :

Definition 2.1. A function $f : X \to Y$ is called p-continuous if for each filterbase $\mathcal{F}$ on X, f(ad $\mathcal{F}$) $\subset$ p-adf($\mathcal{F}$).

Theorem 2.2. A function $f : X \to Y$ is p-continuous iff for each $x \in X$ and each $W \in$ δ PO(f(x)), there is an open nbd U of x such that f(U)$\subset$ δ-pcl W.

Proof. The proof is straightforward.

Theorem 2.3. For a function $f : X \to Y$ the following are equivalent :

(a) f. is p-continuous.

(b) For each $x \in X$ and each filterbase $\mathcal{F}$ on X with $\mathcal{F} \to x$, the filterbase $f(\mathcal{F}) \overset{p}{\to} f(x)$.

(c) For each $x \in X$ and each filterbase $\mathcal{F}$ on X with $\mathcal{F} \to x$, $f(x) \in$ p-adf$(\mathcal{F})$.

(d) For each $x \in X$ and every net x_α in X with $x_\alpha \to x$, $f(x_\alpha) \overset{p}{\to} f(x)$.

Proof. (a) ⇒ (b) For any $V \in \delta\, PO(f(x))$, there exists, by Theorem 2.2, an open nbd U of x such that $f(U) \subset \delta$-pcl V, and then $F \subset U$, for some $F \in \mathcal{F}$. Hence $f(F) \subset \delta$-pcl V.

(b) ⇒ (c) : Obvious.

(c) ⇒ (a) : If f were not p-continuous at some point x of X, there would exist some $V \in \delta\, PO(f(x))$ such that $[X - f^{-1}(\delta\text{-Pcl } V)] \cap U \neq \emptyset$, for all open nbds U of x. Then $\mathcal{F} = \{(X - f^{-1}(\delta\text{-Pcl } V)) \cap U : U$ is an open nbd of $x\}$

is a filterbase on X such that $\mathcal{F} \to x$, but $f(x) \notin$ p-adf$(\mathcal{F})$.

(a) ⇔ (d) : Straightforward and omitted.

Theorem 2.4. If $f : X \to Y$ is a p-continuous mapping and Y is Hausdorff (δ-preHausdorff), then the graph G(f) of f is closed (resp. δ-preclosed) in $X \times Y$.

Proof. Straightforward and omitted.

Lemma 2.5. [3] If A_i is a δ-preopen set in a topological space X_i, for i = 1,2,........, n, then $A_1 \times A_2 \times \ldots\ldots \times A_n$ is also δ-preopen in the product space $X_1 \times X_2 \times \ldots. \times X_n$.

Theorem 2.6. Suppose $f : X \to Y$ is a mapping and $g : X \to X \times Y$ be the graph mapping of f, given by $g(x) = (x, f(x))$ for $x \in X$. If g is p-continuous, then so is f.

Proof. Let $x \in X$ and V any δ-preopen set containing f(x). Then by Lemma 2.5, $X \times V \in \delta\, PO(g(x))$. Since g is p-continuous there is an open nbd U of x such that $g(U) \subset \delta\text{-pcl } (X \times V)$. It is easy to see that $\delta\text{-pcl } (X \times V) \subset X \times \delta\text{-pcl } V$, and thus we have $f(U) \subset \delta$-pcl V, proving the p-continuity of f.

Definition 2.7. A function $f : X \to Y$ is said to have p-subclosed graph if for each $x \in X$ and each filterbase $\mathcal{F}$ on $X - \{x\}$ with $\mathcal{F} \to x$, p-ad $f(\mathcal{F}) \subset \{f(x)\}$.

An equivalent form of the above definition in terms of net is given by the following theorem, the proof of which goes in the usual fashion.

Theorem 2.8. A function $f : X \to Y$ has p-subclosed graph iff for each $x \in X$ and each net $\{x_\alpha\}$ in $X-\{x\}$ with $x_\alpha \to x$, p-adf $(x_\alpha) \subset \{f(x)\}$.

Theorem 2.9. A function $f : X \to Y$ has a p-subclosed graph iff for each $(x. y) \in X \times Y - G(f)$, there exist an open nbd U of x in X and some $V \in \delta\, PO(Y)$ such that $[(U - \{x\}) \times \delta\text{-pcl } V] \cap G(f) = \emptyset$, where G(f), as usual, denotes the graph of f.

Proof. First suppose that $f : X \to Y$ has a p-subclosed graph,and $(x, y) \in X \times Y - G(f)$. Then $y \neq f(x)$. Consider $n^*_x = \{U - \{x\} : U$ is an open nbd of $x\}$. If it is a filterbase, then $n^*_x \to x$ and hence $y \notin$ p-adf(n^*_x). So there are $V \in \delta\,PO(y)$ and $U \in n^*_x$ such that δ-pcl $V \cap f(U) = \emptyset$. Then $U_0 = U \cup \{x\}$ is a nbd of x and $V \in \delta\,PO(y)$ such that $[(U_0 - \{x\}) \times \delta\text{-pcl } V] \cap G(f) = \emptyset$. If n^*_x is not a filterbase, then $U = \{x\}$ for some open nbd U of x, and the rest is obvious.

Conversely, suppose $\mathcal{F}$ is a filterbase on $X - \{x\}$ converging to $x \in X$ and the given condition holds. Suppose $y \neq f(x)$, i.e., $(x, y) \in X \times Y - G(f)$. Hence by the given condition, there are an open nbd U of x in X and a $V \in \delta\,PO(y)$ such that

$[(U - \{x\}) \times \delta\text{-pcl } V] \cap G(f) = \emptyset$, or equivalently,

$f(U - \{x\}) \cap \delta\text{-pcl } V = \emptyset$. Since $\mathcal{F} \to x$, it follows that $F \subset U - \{x\}$ for some $F \in \mathcal{F}$. Hence $f(F) \cap \delta\text{-pcl } V = \emptyset$. Then $y \notin$ p-adf$(\mathcal{F})$. Consequently, f has a p-subclosed graph.

3. CHARACTERIZATIONS OF δ_p – CLOSEDNESS VIA p-CONTINUITY AND P-SUBCLOSED GRAPHS

Definition 3.1. A space X is δ_p-closed if for every cover $\{U_\alpha : \alpha \in I\}$ of X by δ-preopen sets, there exists a finite subcollection $\{U_{\alpha_1}, U_{\alpha_2}, \ldots\ldots, U_{\alpha_n}\}$ (say) such that $X = \bigcup_{i=1}^{n} \delta\text{-pcl } U_{\alpha_i}$.

We recall here some of the characterizations of δ_p-closed spaces found in [4] to facilitate our discussion in the sequel:

Theorem 3.2. For a space X, the following are equivalent :

(a) X is a δ_p-closed space.

(b) Every filterbase on X has a p-adherent point.

(c) Every net in X has a p-adherent point.

Theorem 3.3. A space (Y, T) is δ_p-closed iff for any space X, every function $f : X \to Y$ with p-subclosed graph is p-continuous.

Proof. First let Y be δ_p-closed and suppose that $f : X \to Y$ is a function with a p-subclosed graph, X being an arbitrary topological space. Let $\mathcal{F}$ be a filterbase on X and $y \in f(\text{ad}\mathcal{F})$. Then there is an $x \in \text{ad}\mathcal{F}$ such that $y = f(x)$. Let $\mathcal{G} = \{U \cap F - \{x\} : F \in \mathcal{F}$ and $U \in \eta(x)\}$ where $\eta(x)$ denotes the system of all open nbds of x in X.

First suppose that $\mathcal{G}$ is not a filterbase. Then for some $U_1 \in \eta(x)$ and some $F_1 \in \mathcal{F}$, $U_1 \cap F_1 = \{x\}$. We assert that $x \in F$, for every $F \in \mathcal{F}$. If not, then for some $F_2 \in \mathcal{F}$, $x \notin F_2$. Choose $F_3 \in \mathcal{F}$ such that $F_3 \subset F_1 \cap F_2$. Then $U_1 \cap F_3 - \{x\} \subset U_1 \cap F_1 - \{x\} = \emptyset$, i.e., $\{x\} = U_1 \cap F_3$ (note that $F_3 \cap U_1 \neq \emptyset$), i.e. $x \in F_3 \subset F_2$ So that $x \in F_2$, a contradiction. Thus $f(x) \in f(F)$, for every $F \in \mathcal{F}$. Hence $y \in$ p-adf$(\mathcal{G})$. Next, let $\mathcal{H}$ be a filterbase on $X - \{x\}$. Clearly $\mathcal{H}$ converges to x in X. Since f has a p-subclosed graph, $f(\mathcal{H})$ has at most one p-adherent point viz. f(x) Since Y is δ_p-closed, it then follows by virtue of Theorem 3.2 that p-ad$f(\mathcal{G}) = \{f(x)\}$. Thus $\{y\}$ = p-ad$f(\mathcal{G}) \subset$ p-ad$f(\mathcal{F})$. Hence in any case, f(ad$\mathcal{F}$) $\subset$ p-ad$f(\mathcal{F})$ and consequently, f is p-continuous.

Conversely, to prove (Y, T) to be δ_p-closed under the stated condition it is to be shown, in view of Theorem 3.2, that every filterbase on Y has a p-adherent point.

If possible, let there exist a filterbase $\mathcal{F}$ on Y such that p-ad $\mathcal{F} = \emptyset$. Let us choose and fix some $y_0 \in Y$. Consider the collection.

$T^* = \{A \subset Y : y_0 \in Y-A\} \cup \{A \subset Y : y_0 \in A$ and $F \subset A$, for some $F \in \mathcal{F}\}$.

Clearly T^* is a topology on Y. Now, consider the identity function $f : (Y, T^*) \to (Y, T)$. We show that f has a p-subclosed graph. For this let $y \in Y$ and $\mathcal{B}$ be a filterbase on Y – {y} such that $\mathcal{B} \to y$ in (Y, T^*). Then by definition of T^*, $y = y_0$. For, otherewise {y} is a T^*-open nbd of y and hence the filterbase $\mathcal{B}$ on Y-{y} cannot converge to y. Also, we have $\mathcal{F} \subset \mathcal{B}$. In fact, for a given $F \in \mathcal{F}$, $\{y_0\} \cup F$ is a T^*-open nbd of y_0 and hence contains some member $B \in \mathcal{B}$ (as $\mathcal{B} \to y_0$) and then $B \subset F$. Now, in (Y, T), p-ad$f(\mathcal{B})$ = p-ad$\mathcal{F}$ (as $\mathcal{F} \subset \mathcal{B}$) = $\emptyset$. Thus f has a p-subclosed graph, and consequently by hypothesis, f is p-continuous. But $y_0 \in$ ad$\mathcal{F}$ in (Y, T^*), whereas $f(y_0) = y_0 \notin$ p-ad$f(\mathcal{F})$ $(=\emptyset)$ in (Y, T). This contradicts the p-continuity of f. The contradiction proves that (Y, T) is δ_p-closed.

Theorem 3.4. Let X be δ_p-closed. Then for all spaces Y and Z and all functions $f : Y \to X$ and $g : Z \to X$ with p-subclosed graphs, the set $\Delta(f, g) = \{(y, z) \in Y \times Z : f(y) = g(z)\}$ is closed in $Y \times Z$.

Proof. Let $(a, b) \in$ cl $\Delta(f,g)$. Then there is a filterbase $\mathcal{F}$ on $\Delta(f, g) - \{(a, b)\}$ converging to (a,b). If p_1 and p_2 denote the projection maps from $Y \times Z$ to Y and Z respectively, we have $f(p_1(F)) = g(p_2(F))$ for all $F \in \mathcal{F}$. First suppose that there is some $F_1 \in \mathcal{F}$ such that $p_1(F_1) = \{a\}$. We obtain a filterbase $\mathcal{F}^*$ on $\Delta(f,g) - \{(a,b)\}$ by replacing only those elements F of $\mathcal{F}$ which contain b as the second co-ordinate of at least one element, by an $F^* \in \mathcal{F}$ where $F^* \subset F_1 \cap F$. Clearly, $\mathcal{F}^*$ converges to (a, b). Then $p_2(\mathcal{F}^*)$ is a filterbase on Z–{b} converging to b (since projection maps are continuous). Since g has a p-subclosed graph, p-ad $(g(p_2(\mathcal{F}^*))) \subset \{g(b)\}$. In this case $a \in p_1(F)$ for all $F \in \mathcal{F}$ so that $f(a) \in$ p-ad$f(p_1(\mathcal{F}))$ = p-ad$g(p_2(\mathcal{F})$ $\subset$ p-ad$g$ $(p_2(\mathcal{F}^*))$ (since $\mathcal{F}^* \subset \mathcal{F}$) $\subset \{g(b)\}$. Thus $(a, b) \in \Delta(f,g)$. In case $P_2(F) = \{b\}$ for some $F \in \mathcal{F}$, we proceed similarly as above.

Finally suppose that $p_1(F) \neq \{a\}$ and $P_2(F) \neq \{b\}$ for all $F \in \mathcal{F}$. We replace each $F \in \mathcal{F}$ by the subset F^* obtained by deleting from F all those elements with first co-ordinate a, and thereby obtain a filterbase $\mathcal{F}^*$ on $\Delta(f,g) - \{(a,b)\}$ which clearly converges to (a,b). Obviously, $f(p_1(F)) = g(p_2(F))$ for every $F \in \mathcal{F}^*$, and $p_1(\mathcal{F}^*)$ is a filterbase on Y–{a} converging to a. In view of above, we can suppose without loss of generality that $\{b\} \neq p_2(F)$, for all $F \in \mathcal{F}^*$. Now, for each $F^* \in \mathcal{F}^*$, we consider the subset F^{**} of F^* by deleting from F^* the elements with second co-ordinate b.

Then $\mathcal{F}^{**} = \{F^{**} \subset F^* : F^* \in \mathcal{F}^*\}$ is a filterbase on $\Delta(f, g) - \{(a, b)\}$ converging to (a, b). Clearly, $f(p_1(F)) = g(p_2(F))$ for every $F \in \mathcal{F}^{**}$, and $p_1(\mathcal{F}^{**})$ and $p_2(\mathcal{F}^{**})$ are filterbases on Y – {a} and Z – {b} respectively converging to a and b respectively. Since f has p-subclosed graph, we have p-adf$(p_1(\mathcal{T}^{**}))$ $\subset \{f(a)\}$. By Theorem 3.3, g is p-continuous and hence by Theorem 2.3 we have, $g(b) \in$ p-adg$(p_2(\mathcal{T}^{**}))$ = p-adf $(p_1(\mathcal{T}^{**})) \subset \{f(a)\}$ so that $(a, b) \in \Delta(f, g)$.

Corollary 3.5. Let a space Y be δ_p-closed. Then for any space X and any function $f : X \to Y$ having p-subclosed graph, the set $\Delta(f) = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$ is closed in $X \times X$.

Theorem 3.6. A space Y is δ_p-closed if for any space X and any function $f : X \to Y$ with p-subclosed graph, the set $\Delta(f) = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$ is closed in $X \times X$.

Proof. Let the space Y be not δ_p-closed so that there is a net $S = \{S_\alpha : \alpha \in (D, \geq)\}$ in Y, which has no p-adherent point. We choose two distinct points y_1, y_2 in Y and assume without any loss of generality that S is a net in $Y - \{y_1, y_2\}$. Consider a space (X, T), where X = Y and $T = \{A \subset X : A \cap \{y_1, y_2\} = \emptyset\} \cup \{A \subset X : A \cap \{y_1, y_2\} \neq \emptyset$ and $T_\alpha \subset A$ for some $\alpha \in D\}$, where $T_\alpha = \{S_\beta : \beta > \alpha\}$. Consider the map $f : (X, T) \to Y$ given by

$$f(x) = \begin{cases} x, & \text{if } x \neq y_1, y_2 \\ y_2, & \text{if } x = y_1 \\ y_1, & \text{if } x = y_2 \end{cases}$$

In order to prove that f has a p-subclosed graph, let $R = \{R_\beta : \beta \in D_1\}$ be a net in $X - \{x\}$ converging to x ($\in X$). Then $x = y_1$ or y_2 (otherwise $\{x\}$ is a T-nbd of x and hence R cannot converge to x). For definiteness, let $x = y_1$. Suppose, if possible, f(R) p-adheres at some point $y \in Y$. Since the net f(S) = S has no p-adherent point in Y, there exist $U \in \delta PO(Y)$ with $y \in U$ and $\alpha_0 \in D$ such that for all $\alpha > \alpha_0$ ($\alpha \in D$), $S_\alpha \notin \delta$-pcl U i.e., $T_\alpha \subset Y - \delta$-pcl U. Now, $\{y_1\} \cup T_\alpha$ is a T-open nbd of y_1 in X. Then $\{R_\beta : \beta \geq \beta_0\} \subset \{y_1\} \cup T_\alpha$ (since $R \to y_1$ in X) for some $\beta_0 \in D_1$. Since $T_\beta \subset X - \{y_1, y_2\}$, no R_β can be y_2 for $\beta \geq \beta_0$. Thus $\{f(R_\beta) : \beta \geq \beta_0\} = \{R_\beta : \beta \geq \beta_0\} \subset T_\alpha \subset Y - \delta$-pcl U so that f(R) cannot p-adhere at y, a contradiction. Thus f has a p-subclosed graph. Now, clearly $(y_1, y_2) \notin \Delta(f)$. To arrive at a contradiction it suffices to show that $(y_1, y_2) \in \mathrm{cl}\Delta(f)$. In fact, let U be an open nbd of (y_1, y_2) in $X \times X$. By definition of T, there exist $\alpha, \beta \in D$ such that $(T_\alpha \cup \{y_1\}) \times (T_\beta \cup \{y_2\}) \subset U$. If $\gamma \in D$ such that $\gamma \geq \alpha, \beta$, then $(S_\gamma, S_\gamma) \in U \cap \Delta(f)$ proving that $(y_1, y_2) \in \mathrm{cl}\, \Delta(f)$.

Combining Corollary 3.5 and Theorem 3.6 we obtain:

Theorem 3.7. A topological space Y is δ_p-closed iff for any space X and any function $f : X \to Y$ with p-subclosed graph, the set $\Delta(f) = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$ is closed in $X \times X$.

Again, from Theorem 3.4 and 3.6 it follows that

Theorem 3.8. A topological space Y is δ_p-closed iff for any topological spaces X and Z, and all functions $f : X \to Y$ and $g : Z \to Y$ each having p-subclosed graph, the set

$\{(x, z) \in X \times Z : f(x) = g(z)\}$ is closed in $X \times Z$.

Theorem 3.9. A space X is δ_p-closed iff for any space Y and any functions f, g : Y → X with p-subclosed graphs, the set $\Delta^*(f, g) = \{y \in Y : f(y) = g(y)\}$ is a closed subset of Y.

Proof. Let X be δ_p-closed. Putting Y = Z in Theorem 3.4 we see that for any given space Y and any functions f, g : Y → X with p-subclosed graphs, the set $\Delta(f, g) = \{(y_1, y_2) \in Y \times Y :$

$f(y_1) = g(y_2)\}$ is closed in Y × Y. Now clearly

$\Delta^*(f, g) = p_1[\Delta(f, g) \cap \Delta_y]$, where Δ_y is the diagonal in Y × Y and $p_1 : Y \times Y \to Y$ is the first projection map. Since p_1/Δ_y is a homeomorphism, $\Delta^*(f,g)$ is closed in Y.

Conversely, suppose X is not δ_p-closed. Then for some filterbase $\mathcal{F}$ on X, p-ad $\mathcal{F} = \varnothing$. Select any two distinct points a, b ∈ X and put Y = X. Let

$T_1 = \{A \subset X : b \in Y - A \text{ or } F \subset A \text{ for some } F \in \mathcal{F}\}$.

Then (Y, T_1) is a topological space.

Consider the maps f, g : $(Y, T_1) \to X$ given by

$$f(x) = x, \text{ for } x \in Y, \text{ and } g(x) = \begin{cases} x, \text{ for } x \in Y - \{b\} \\ a, \text{ for } x = b \end{cases}$$

We show that each of f and g has a p-subclosed graph. Let y ∈ Y and $\mathcal{G}$ be any filterbase on Y − {y} such that $\mathcal{G} \to y$ in Y.

If y ≠ b, then {y} being a T_1-nbd of y, $\mathcal{G}$ cannot converge to y.

Hence y = b. Now, for each $F \in \mathcal{F}$, $F \cup \{b\} \in T_1$, so that p-adf($\mathcal{G}$) = p-ad $\mathcal{G} \subset$ p-ad $\mathcal{F} = \varnothing \subset \{f(y)\}$. Hence f has a p-subclosed graph. Similarly, g has a p-subclosed graph. Now, $\Delta^*(f, g) = Y - \{b\}$, and we see that for any open nbd V of b in Y, there is an $F \in \mathcal{F}$ such that F ⊂ V, and $V \cap \Delta^*(f, g) \supset F \cap \Delta^*(f, g) \neq \varnothing$ (as F ≠ {b}, for each $F \in \mathcal{F}$). Hence $\Delta^*(f, g)$ is not closed in Y.

4. STRONGLY p-CLOSED GRAPH AND δ_p-CLOSEDNESS :

Definition 4.1. A function f : X → Y is said to have strongly p-closed graph G(f) if whenever a net $x_\alpha \to x$ in X and $f(x_\alpha) \xrightarrow{p} y$ in Y, it follows that y = f(x).

Obviously, the above definition could equivalently be given in terms of filterbases instead of nets as enunciated below:

Theorem 4.2. A function f : X → Y has a strongly p-closed graph iff whenever a filterbase $\mathcal{F} \to x$ in X and $f(\mathcal{F}) \xrightarrow{p} y$ in Y, it follows that y = f(x).

Proof. Straightforward and omitted.

Lemma 4.3. A function f : X → Y has strongly p-closed graph G(f) iff for each point $(x, y) \in X \times Y - G(f)$, there exist an open nbd U of x in X and a δ-preopen set V in Y containing y such that $(U \times \delta\text{-pcl } V) \cap G(f) = \varnothing$.

Proof. Let the given condition hold for a function $f : X \to Y$, and let x_α be a given net in X such that $x_\alpha \to x$ and $f(x_\alpha) \xrightarrow{p} y$ in Y. It then follows that for each open nbd V of x and each δ-preopen set W in Y containing y, $(V \times \delta\text{-pcl } W) \cap G(f) \neq \emptyset$. Hence $(x, y) \in G(f)$ and so, $y = f(x)$. Hence f has a strongly p-closed graph.

Conversely, suppose that a function $f : X \to Y$ does not satisfy the stated condition of the theorem. Then for some $(a, b) \in X \times Y - G(f)$, we have $(U \times \delta\text{-pcl } V) \cap G(f) \neq \emptyset$, for every open nbd U of a and each δ-preopen set V containing b. Suppose $\mathcal{F}_1 = \{U_\alpha : \alpha \in I_1$, and U_α is an open nbd of a$\}$ and

$\mathcal{F}_2 = \{\delta\text{-pcl } V_\beta : \beta \in I_2$ and V_β is a δ-preopen nbd of b$\}$, and put

$\mathcal{F}_3 = \{W_{(\alpha, \beta)} = (U_\alpha \times \delta\text{-pcl } V_\beta) \cap G(f) : (\alpha, \beta) \in I_1 \times I_2\}$.

Then $\mathcal{F} = \{F_{(\alpha, \beta)} : (\alpha, \beta) \in I_1 \times I_2\}$, where

$F_{(\alpha, \beta)} = \{x \in U_\alpha : (x, f(x)) \in W_{(\alpha, \beta)}\}$,

is a filterbase on X such that $\mathcal{F}$ converges to the point a in X, $f(\mathcal{F}) \xrightarrow{p} b$ and $f(a) \neq b$. Thus in view of Theorem 4.2, f does not have a strongly p-closed graph.

By virtue of Theorems 2.9 and 3.3 and Lemma 4.3 it now follows that

Theorem 4.4. If Y is a δ_p-closed space, then every function f from any space X to Y with a strongly p-closed graph is p-continuous.

The next theorem shows that the converse of the above theorem also holds if the given space is δ-preHausdorff.

Theorem 4.5. A δ-preHausdorff space Y is δ_p-closed if each function f from any topological space X to Y with a strongly p-closed graph is p-continuous.

Proof. Let $x_0 \in Y$ and let (x_α, D) be a net in $Y - \{x_0\}$ with no p-adherent point in $Y - \{x_0\}$. Let $X = \{x_\alpha : \alpha \in D\} \cup \{x_0\}$ and let T be the topology on X generated by $\{\{x_\alpha\} : \alpha \in D\} \cup \{T_\mu \cup \{x_0\} : \mu \in D\}$ as basic open sets, where $T_\mu = \{x_\alpha : \alpha \geq \mu, \alpha \in D\}$. Let $i : X \to Y$ be the identity map and let $(x, y) \in X \times Y - G(i)$. If $x \neq x_0$, then $\{x\}$ is open in X. Choose, by δ-preHausdorffness of Y, a δ-preopen nbd V of y in Y with $x \notin \delta\text{-pcl } V$. Then clearly $(\{x\} \times \delta - \text{pcl } V) \cap G(i) = \emptyset$. If $x = x_0$, then $y \neq x_0$, so there is a $V \in \delta\, PO(y)$ satisfying $x_0 \notin \delta\text{-pcl } V$, as Y is δ-preHausdorff. Again, $y \neq x_0$, so y is not a p-adherent point of (x_α) in Y. Hence, there is $\mu \in D$ such that $T_\mu \cap \delta\text{-pcl } V = \emptyset$. Thus $T_\mu \cup \{x_0\}$ is an open nbd of x in X and V a δ-prenbd of y in Y such that $[(T_\mu \cup \{x_0\}) \times \delta\text{-pcl } V] \cap G(i) = \emptyset$. Hence the graph of i is strongly p-closed and by hypothesis, i is p-continuous. So, for any δ-preopen nbd V of x_0 in Y, there exists $\mu \in D$ satisfying $T_\mu \subset \delta\text{-pcl } V$. So, $x_\alpha \xrightarrow{p} x_0$ in Y. Hence Y is δ_p-closed.

The following result shows that the assumption of δ-preHausdorffness on the space Y in the above theorem is, in fact, necessary.

Theroem 4.6. If a surjection $f : X \rightarrow Y$ has strongly p-closed graph, then Y is δ-preHausdorff.

Proof. Let y and z be any two distinct points of Y. Then since f is onto, there is $x \in X$ such that $f(x) = y$. Hence $(x, z) \notin G(f)$. Since f has strongly p-closed graph, there exist an open nbd U of x and a δ-prenbd V of z such that $f(U) \cap \delta\text{-pcl } V = \emptyset$. Put

$W = Y - \delta\text{-pcl } V$, then $W \in \delta\, PO(y)$ and $W \cap V = \emptyset$.

Hence Y is δ-preHausdorff.

Combining Theorems 4.4 and 4.5 we obtain:

Theorem 4.7. A δ-preHausdorff space Y is δ_p-closed iff each function from any space X into Y with strongly p-closed graph is p-continuous.

REFERENCES

1. James E. Joseph, On H-closed spaces, Proc. Amer. Math. Soc. 55 (1976), 223-226.

2. S. Raychaudhuri Concerning δ^*-almost continuity and δ-preregularity, Bull. Cal. Math. Soc. 85 (1993), 385-392.

3. S Raychaudhuri and M. N. Mukherjee, On δ-almost continuity and δ-preopen sets, Bull. Ins. Sinica 21 (1993), 357-366.

4. S. Raychaudhuri and M. N. Mukherjee, δ_p-closedness for topological spaces, Indian Acad, Math. (To appear).

5. N. V. Velicko, H-closed topological spaces, Amer. Math. soc. Transl. 78(1968), 103-118.

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GENERALIZATION OF A THEOREM OF S. KUREPA AND ITS CATEGORY ANALOGUE

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ABSTRACT : In [4], Pal generalized a theorem of S. Kurepa. In this paper, we have generalized Kurepa's theorem from which Pal's result follows immediately as a corollary. An almost analogous result in the setting of Baire category has also been presented.

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INTRODUCTION

Throughout this paper we shall be dealing with sub-sets of an n-dimensional Euclidean space R^n where Lebesgue measure of any measurable set is denoted by m(A). Hence from now on, unless otherwise specified, the word 'set' will be used to interpret a sub-set of R^n and the terms like 'measure', 'measurability' etc. shall be used in the sense of Lebesgue.

S. Kurepa [1] proved that, if A be a set of positive measure, then for any system of p real numbers $\alpha_1, \alpha_2 .... \alpha_p$ ($\alpha_k \neq O$), there exists a sphere K_r of radius r with centre at the origin such that for any vector $x \in K_r$, there are vectors

$a_o(x), a_1(x), a_2(x), ..., a_p(x)$

in A such that :

$$x = \frac{a_1(x)-a_o(x)}{\alpha_1} = \frac{a_2(x)-a_o(x)}{\alpha_2} = \frac{a_p(x)-a_o(x)}{\alpha_p}$$

Pal [4] has given the following generalization of Kurepa's theorem.

If A and B are sets of positive Lebesgue measure, $\alpha_1, \alpha_2, \ldots\ldots, \alpha_p$ $(\alpha_k \neq 0)$ be any system of p real numbers and $\{\beta_k\}_{k=1}^{\infty}$ be any sequence of non-zero real numbers containing 1 such that $\underset{k \to \infty}{\text{Lt}} \beta_k = 1$, then there exists a positive integer N_0, spheres K_0 with the centre at the originand K_1 such that for any vector x in K_0, for a system of p vectors $z_1, z_2, \ldots\ldots, z_p$ in K_1 and for a system of p real numbers $\beta_{k_1}, \beta_{k_2} \ldots\ldots \beta_{k_p}$ of the sequence $\{\beta_k\}_{k=1}^{\infty}$ with $k_i > N_0$, there exist vectors

$$a(x, z_1, z_2, \ldots\ldots, z_p, \beta_{k_1}, \beta_{k_2}, \ldots\ldots \beta_{k_p}) \in A \text{ and}$$

$$b_i\,(x : z_1, z_2, \ldots\ldots, z_p, \beta_{k_1}, \beta_{k_2}, \ldots\ldots \beta_{k_p}) \in B \ (i = 1, 2, \ldots\ldots, p) \text{ such that:}$$

$$x = \frac{\frac{1}{\beta_{k_i}},\ b_i\,(x ; z_1, z_2, \ldots\ldots, z_p, \beta_{k_1}, \beta_{k_2}, \ldots\ldots, \beta_{k_p})}{\alpha_i} - \frac{a(x, z_1, z_2, \ldots\ldots, z_p, \beta_{k_1}, \beta_{k_2}, \ldots\ldots \beta_{k_p}) \div \frac{z_i}{\beta_{k_i}}}{\alpha_i}$$

$$(i = 1, 2, \ldots\ldots p)$$

In this paper a result has been proved which includes Kurepa's theorem as a particular case and from which Pal's generalization of Kurepa's theorem follows immediately.

A point 'a' in R^n is said to be a point of density of a measurable set A if Lt m $(A \cap C_k)$ / m(C_k) = 1 holds for all sequences $\{C_k\}_{k=1}^{\infty}$ of cubes satisfying $a \in C_k$ for all k and their diameters $\delta(C_k) - 0$ [5]. Now for any two measurable sets A and B, if we designate by $\phi(A)$ and $\phi(B)$ respectively their sets of points of densities, then on the same lines as proved for the case of linear sets [3], it can be verified that $\phi(A \cap B) = \phi\,(A) \cap \phi\,(B)$. Furthermore, if $a \in \phi\,(A)$, $b \in \phi\,(B)$, then it follows by translation invariance of Lebesgue measure that $a \in \phi(B - c)$ and hence from above $a \in \phi\,(A \cap (B - c))$. Nevertheless, we have made use of this deduction in proving our first theorem.

Theorem 1. If A and B are measurable with m(A), m(B) (> 0), $\{\alpha_k\}_{k=1}^{\infty}$ be any sequence of non-zero reals having a finite limit, then given any positive integer p, we can determine an interval I in R with positive abscissae and 1 as an interior point, spheres K_0 with centre at the origin and K_1 in R^n such that for every choice of a vector x from K_0, sequence $\{z_k\}_{k=1}^{\infty}$ of vectors from K_1 with a limit point at the centre of K_1 and a sequence $\{\beta_k\}_{k=1}^{\infty}$ of real numbers from I having 1 as a limit point, there exist subsequences $\{z_{k_i}\}_{k=1}^{\infty}$ of $\{z_k\}_{k=1}^{\infty}$ whose first p terms are $z_1, z_2, \ldots\ldots z_p$, $\{\beta_{k_i}\}_{i=1}^{\infty}$ of $\{\beta_k\}_{k=1}^{\infty}$ whose first p terms are $\beta_1, \beta_2, \ldots\ldots, \beta_p$ satisfying the following property:

there exist vectors $a(x) \in A$, $b_i(x) \in B$ $(i = 1, 2, \ldots\ldots)$ such that:

$$x = \frac{\frac{b_i(x)}{\beta_{k_i}} - \left(a(x) + \frac{z_{k_i}}{\beta_{k_i}}\right)}{\alpha_{k_i}} \ (i = 1, 2, \ldots\ldots)$$

where $\{\alpha_{k_i}\}_{i=1}^{\infty}$ is a subsequence of $\{\alpha_k\}_{k=1}^{\infty}$ with its first p terms as $\alpha_1, \alpha_2, \ldots\ldots \alpha_p$ respectively.

Lemma 1. If A be measurable and $x \in R^n$, then

$$\underset{x \to 0}{\text{Lt}}\, m(A \setminus (A+x)) = 0$$

Proof. Clearly $m(A \setminus (A+x))$

$$= \int_{R^n} x_A(z)\{x_A(z) - x_A(z-x)\}dz \quad [x_A \text{ is the characteristic function of } A]$$

$$\leq \int_{R^n} |x_A(z) - x_A(z-x)|\,dz - O \ (\text{as } x \to O) \qquad [6]$$

Lemma 2. If A is measurable and $\lambda \in R$, then

$$\underset{y \to 1}{\text{Lt}}\, m(A \setminus \lambda A) = 0$$

Proof. For linear sets, a proof has been suggested in [2]. For sets in R^n, an analogous procedure may be followed.

Lemma 3. If C_1 $(C_1^{(1)}, C_2^{(1)}, \ldots\ldots C_{2^n}^{(1)}$ and C_2 $(C_1^{(2)}, C_2^{(2)}, \ldots\ldots C_{2^n}^{(2)})$ be two cubes of equal volumes, with there corners represented by vectors $c_i^{(1)}$, $c_i^{(2)} (i = 1, 2, \ldots\ldots 2^n)$ and centres by C_1 and c_2 respectively, C^* be a proper subcube of C_1 and concentric with it and $\{\lambda_k\}_{k=1}^{\infty}$ be a bounded sequence of reals, then we can determine an interval J in R with positive abscissae and 1 as an interior point, spheres S_o with centre at the origin and S_1 in R^n such that for every choice of vectors x from S_o, z from S_1 and a real number β from J, the inclusions

$$C_1(c_1^{(1)}, c_2^{(1)}, \ldots\ldots c_{2n}^{(1)})\ C_2\Big(\frac{c_1^{(2)}}{\beta} - \frac{z}{\beta} - \lambda_k x, \frac{c_2^{(2)}}{\beta} - \frac{z}{\beta} - \lambda_k x, \ldots\ldots \frac{C_{2}^{(2)}n}{\beta} - \frac{z}{\beta} - \lambda_k x\Big)$$

$$\supseteq C^*(k = 1, 2, \ldots\ldots) \text{ holds}$$

$C_2\Big(\frac{c_1^{(2)}}{\beta} - \frac{z}{\beta} - \lambda_k x, \frac{c_2^{(2)}}{\beta} - \frac{z}{\beta} - \lambda_k x, \ldots\ldots \frac{c_2^{(2)}}{\beta} - \frac{z}{\beta} - \lambda_k x\Big)$ being the cube obtained from the cube C_2 under the transformations $Y \to \frac{y}{\beta} - \frac{z}{\beta} - \lambda_k x$ in R^n.

Proof. If we set, $\lambda = \sup\{|\lambda_k| : k = 1, 2, \ldots\ldots\}$.

$M = \sup\{\|x\| : x \in C_1(c_1^{(1)}, c_2^{(1)}, \ldots\ldots, c_{2n}^{(1)})\}$

(where $\|.\|$ represents the usual norm in R^n).

$\partial = \text{Max}\{\frac{1}{2}(\delta(c_1) - \delta(c^*)), 3M\}$, $J = (\mu, \gamma)$ where

$\mu = \text{Max}\{\frac{1}{2}, \frac{3M}{3M+\delta}\}$, $\gamma = \text{Min}\{2, \frac{3M}{3M-\delta}\}$

and $S_o = \{x \in R^n, \|x\| < \frac{\delta}{3(\lambda+1)}\}$

$S_1 = \{ z \in R^n,\ \| z - c \| < \frac{\delta\mu}{3} \}$ where

$c = c_2 - c_1$.

Then the corners $\frac{c_i^{(2)}}{\beta} - \frac{z}{\beta} - \lambda_k x$ $(i = 1, 2, \ldots\ldots, 2^n)$

of the cubes $C_2(\frac{c_1^{(2)}}{3} - \frac{z}{\beta} - \lambda_k x, \frac{c_i^{(2)}}{\beta} - \frac{z}{\beta} - \lambda_k x, \ldots\ldots \frac{c_i^{(2)}}{\beta} - \frac{z}{\beta} - \lambda_k x)$ satisfy the inequality

$\| (\frac{c_i^{(2)}}{\beta} - \frac{z}{\beta} - \lambda_k x) - c_i^{(1)} \| < \delta$ $(i = 1, 2, \ldots, 2^n)$, $(k = 1, 2, \ldots\ldots)$

whenever $x \in S_0$. $z \in S_1$ and $\beta \in J$. Hence, for every choice of a vector x from S_0, z from S_1 and a real number β from J, the inclusion

$$C_1(c_1^{(1)}, c_2^{(1)} \ldots\ldots, c_{2^n}^{(1)}) \cap C_2(\frac{c_1^{(2)}}{\beta} - \frac{z}{\beta} - \lambda_k x, \frac{c_2^{(2)}}{\beta} - \frac{z}{\beta} - \lambda_k x, \ldots \frac{c_{2^n}^{(2)}}{\beta} - \frac{z}{\beta} - \lambda_k x)$$

$\supseteq C^*$ $(k=1, 2\ldots\ldots)$ is satisfied. This completes the proof of the lemma.

Proof of the Theorem. Let $a \in \phi(a)$, $b \in \phi(B)$ and we set $E = A \cap (B - c)$ where

$c = b - a$. Then $a \in \phi(E)$ (see Introduction).

We choose $0 < \varepsilon < \dfrac{1}{2(p+1)}$. Consequently, there exists cubes

$C_1(c_1^{(1)}, c_2^{(1)}, \ldots\ldots\ldots\ldots\ldots c_{2^n}^{(1)})$ and $C_2(c_2^{(2)}, c_2^{(2)}, \ldots\ldots, c_{2^n}^{(2)})$ of equal volumes with their centres represented by vectors c_1 and c_2 respectively such that

$m(C_1 \setminus E) < \varepsilon m(C_1)$

$m(C_2 \setminus B) < \varepsilon m(C_2)$

For definitenesss, let α be a limit point of the sequence $(\alpha_k)_{k=1}^{\infty}$ in R and for vectors z and x in R^n, real numbers λ and β $(\neq 0)$, we designate by the symbol $X(\beta, \lambda, x, z)$ the set $\frac{X}{\beta} - \frac{Z}{\beta} - \lambda x$ in general. Now since the cube $\frac{C_2}{\beta} - \frac{z}{\beta} - \lambda x$ is identical with the

cube $C_2(\frac{c_1^{(2)}}{\beta} - \frac{z}{\beta} - \lambda x, \frac{c_2^{(2)}}{\beta} - \frac{z}{\beta} - \lambda x, \ldots \frac{c_{2n}^{(2)}}{\beta} - \frac{z}{\beta}\lambda x)$ by

virtue of lemma 3, we can determine an interval J in R with positive abscissae and 1 as an interior point and left hand end point greater than or equal to $\frac{1}{2}$, spheres S with centre at the origin and s_1 such the for every choice of a vector x from s_0, a system of p vectors $z_1, z_2, \ldots\ldots, z_p$ from S_1 and a system of p real numbers $\beta_1, \beta_2, \ldots\ldots \beta_p$ from J, we have $C_1 \cap C_2(\beta_k, \lambda_k, x, z_k) \supseteq C^*$ $(k=1, 2\ldots\ldots p)$ where C^* is that subcube of C_1 which is concentric with C_1 and whose volume is equal to $(1-\varepsilon)\, m(C_1)$ and

$$\lambda_k = \alpha_k - \alpha \quad (k=1, 2, \ldots\ldots p)$$

Consequently,

$$m(C_1 \cap \{\bigcap_{k=1}^{P} C_2(\beta_k, \lambda_k, x, z_k)\}) > (1-\varepsilon)m(C_1)$$

But then,

$$m(E \cap C_1) \cap \{\bigcap_{k=1}^{P} (B(\beta_k, \lambda_k, x, z_k) \cap C_2(\beta_k, \lambda_k, x, z_k))\})$$

$$> m(C_1 \cap \{\bigcap_{k=1}^{P} C_2(\beta_k, \lambda_k, x, z_k)\}) - [\sum_{k=1}^{p} m(C_2(\beta_k, \lambda_k, x, z_k) \backslash B(\beta_k, \lambda_k, x, z_k)) + m(C_1 \backslash E)]$$

$$> (1 - \varepsilon - \varepsilon - 2p)m(C_1) = \{1 - 2(p+1)\}m(C_1) > 0$$

Hence, $m(A \cap (B\text{-}c) \cap \{\bigcap_{k=1}^{P} B(\beta_k, \lambda_k, x, z_k)\}) > \{1-2(p+1)\}m(C_1) > 0$

New by elementary properties of measure theory and lemma 1 we get a sphere S_o with centre at the origin such that,

$$m((A+\alpha x) \cap (B-c) \cap \{\bigcap_{k=1}^{P} B(\beta_k, \lambda_k, x, z_k)\})$$

$$> \{\frac{1}{2} - (p+1)\}m(C_1) > 0 \text{ holds}$$

$$= \{\frac{1}{2} - (p+1)\}m(C_1) > 0 \text{ holds}$$

for every choice of a vector x from S_o, a system of p vectors $z_1, z_2, \ldots\ldots. z_p$ from S_1 and system of p real numbers $\beta_1, \beta_2, \ldots\ldots. \beta_p$ from J.

We relebel S_o by K_o, S_1 by K_1 and J by I respectively.

Now as

$$(B-c) \backslash B(\beta, \lambda, x, z)$$

$$= (B-c) \backslash (\frac{B}{\beta} - \frac{z}{\beta} - \lambda x)$$

$$\subseteq \{(B-c) \backslash (B-z)\} \cup \{(B-z) \backslash (\frac{B-z}{\beta})\} \cup \{(\frac{B-z}{\beta}) \backslash (\frac{B-z}{\beta} - \lambda x)\}$$

$$(x, z \in R^n, \lambda, \beta(\neq 0)(\in R)$$

by a successive application of lemma 1 and lemma 2, we find corresponding to any real number $\eta > 0$ a vector x in R^n, a vector z and real numbers λ and β ($\beta \neq 0$) such that

$$m(B-c) \backslash B(\beta, \lambda, x, z)) < \eta$$

A repeated application of this phenomenon yields corresponding to any given vector x in K_o, sequence $\{z_k\}_{k=1}^{\infty}$ of vectors in K_1 with a limit point at the centre of K_1 and a sequence $\{\beta_k\}_{k=1}^{\infty}$ of real numbers in I having 1 as a limit point, subsequences $\{z_{k_i}\}_{i=1}^{\infty}$ of $\{z_k\}_{k=1}^{\infty}$ whose first p terms are $z_1, z_2, \ldots\ldots z_p$, $\{\beta_{k_i}\}_{i=1}^{\infty}$ of $\{\beta_k\}_{k=1}^{\infty}$ whose first p terms as $\beta_1, \beta_2, \ldots\ldots \beta_p$ for which the inequality,

$$m((A+\alpha x) \cap \{\bigcap_{i=1}^{\infty} (B(\beta_{k_i}, \lambda_{k_i}, x, z_{k_i}))\}) > \{\tfrac{1}{2^2} - \tfrac{1}{2}(p+1)\} m(C_1) > 0$$

is satisfied, where $\lambda_{k_i} = \alpha_{k_i} - \alpha$ (i=1,2,.....) and its first p terms are $\lambda_1, \lambda_2, \ldots\ldots \lambda_p$ respectively.

$$\text{or, } m(A \cap \{\bigcap_{i=1}^{\infty} (B(\beta_{k_i}, \alpha_{k_i}, x, z_{k_i}))\}) \{\tfrac{1}{2^2} - \tfrac{1}{2}(p+1)\} \ m(C_1) > 0$$

(By translation invariance of m).

$$\text{or } m\ (A \cap \{\bigcap_{i=1}^{\infty} (\tfrac{E}{\beta_{k_i}} - \tfrac{z_i}{\beta_{k_i}} - \alpha_{k_i} x)\}) > \{\tfrac{1}{2^2} - \tfrac{1}{2}(p+1)\}.\ m(C_1) > 0$$

the remaining part is obvious. Hence the theorem follows.

Note 1.1. When A = B, we may take a = b and consequently K_1 becomes a sphere with centre at the origin. Now if we choose

(i) the sphere K as Min $\{K_o, K_1\}$,

(ii) $\{\alpha_k\}_{k=1}^{\infty}$ as the sequence $\alpha_1, \alpha_2, \ldots\ldots \alpha_p, \alpha_p, \ldots\ldots\ldots$

(iii) $\{z_k\}_{k=1}^{\infty}$ as the sequence 0, 0, 0,...... 0, 0,..........

and (iv) $\{\beta_k\}_{k=1}^{\infty}$ as the sequence 1, 1, 1,...... 1, 1,...........

then theorem 1 boils down to what has been stated as kurepa's theorem [1].

Note 1.2. Let spheres K_o, K_1 and interval I be choosen as in theorem 1. Then given any convergent sequence $\{\beta_k\}_{k=1}^{\infty}$ of real numbers with $\underset{k \to \infty}{\text{Lt}} B_k = 1$, we can find a positive integer N_o such that $\beta_k \in I$ for all $k \geq N_o$. Now if we choose $x \in K_o$, a sequence $\{z_k\}_{k=1}^{\infty}$ from K_1 such that $z_k = c$ (the centre of K_1) for k=p+1, p+2,....... and a sequence $\{\beta_{k_1}, \beta_{k_2} \ldots \beta_{k_p}, 1, 1 \ldots\ldots\}$ of real number $(k_i \geq N_o)$ then a relation of the form :

$$x = \frac{\dfrac{b_i(x)}{\beta_{k_i}} - (a(x) + \dfrac{z_{k_i}}{\beta_{k_i}})}{\alpha_i}$$

(i=1, 2,.......p) is satisfied, where the vectors $a(x) \in A$, $b_i(x) \in B$ (i=1,2,

........p). This in fact, is Pal's theorem.

We shall now state the Baire category analogue of theorem 1.

A set is said to have the property of Baire [3] if it can be expressed as the symmetric difference of an open set and a set of first category.

Theorem 2. If A and B are sets of second category with the propertyof Baire and $\{\alpha_k\}_{k=1}^{\infty}$ be any sequence of real numbers with a finite limit point then given any positive integer p, we can determine an interval I in R with positive abscissae and 1 as an interior point, spheres K_0 with centre at the origin and K_1 in R^n such that for every choice of a vector x from K_0, sequence $\{z_k\}_{k=1}^{\infty}$ of vectors from K_1 with a limit point at the centre of K_1 and a sequence $\{\beta_k\}_{k=1}^{\infty}$ of real numbers from I having 1 as a limit point, there exist subsequences $\{z_{k_i}\}_{i=1}^{\infty}$ of $\{z_k\}_{k=1}^{\infty}$ whose first p terms are $z_1, z_2, \ldots\ldots, z_p$, $\{\beta_{k_i}\}_{i=1}^{\infty}$ of $\{\beta_k\}_{k=1}^{\infty}$ whose first p terms are $\beta_1, \beta_2, \ldots\ldots \beta_p$ satisfying the following property: there exist vectors $a(x) \in A$ and $b_i(x) \in B$ $(i=1, 2, \ldots\ldots)$ such that

$$x = \frac{\dfrac{b_i(x)}{\beta_{k_i}} - \left(a(x) + \dfrac{z_{k_i}}{\beta_{k_i}}\right)}{\alpha_{k_i}} \quad (i=1, 2, \ldots\ldots)$$

where $\{\alpha_{k_i}\}_{i=1}^{\infty}$ is a subsequence of $\{\alpha_k\}_{k=1}^{\infty}$ with its first p terms as $\alpha_1, \alpha_2 \ldots\ldots \alpha_p$ respectively.

Proof of the Theorem: By the definition of a set with the property of Baire, we can write $A = (G_1 \backslash P_1) \cup Q_1$ and $B = (G_2 \backslash P_2) \cup Q_2$ where G_1 and G_2 are open and P_1, Q_1, P_2, Q_2 are first category sets. Moreover, as A and B are sets of second category, G_1 and G_2 are non-empty, hence we can fit in G_1 and G_2 cubes C_1 and C_2 of the equal volumes. Let a and b denote respectively their centres and we set $c = b - a$. Just as in theorem 1, we here also use the notational device $X(\beta, \lambda, x, z)$ to designate the set $\frac{X}{\beta} - \frac{z}{\beta} - \lambda x$ in general where x and z are vectors in R^n and $\lambda, \beta\ (\neq 0)$ are real numbers.

Let C^* be a proper subcube of C_1 concentric with C_1. Since the sequence $\{\alpha_k\}_{k=1}^{\infty}$ has a limit point in R, we can extract from it a subsequence $\{\alpha_{k_i}\}_{i=1}^{\infty}$ which is bounded in R and whose first p terms are $\alpha_1, \alpha_2, \ldots\ldots\alpha_p$. Hence by lemma 3 of theorem 1, we can determine an interval I in R with positive abscissae and 1 as an interior point, spheres K_0 with centre at the origin and K_1 such that for every choice of vector x from K_0, sequence $\{z_k\}_{k=1}^{\infty}$ of vectors from K_1 and a sequence $\{\beta_k\}_{k=1}^{\infty}$ of real numbers from I, there exist subsequences $\{z_{k_i}\}_{i=1}^{\infty}$ of $\{z_k\}_{k=1}^{\infty}$ whose first p terms are $z_1, z_2, \ldots\ldots$

z_p, $\{\beta_{k_i}\}_{i=1}^{\infty}$ of $\{\beta_k\}_{k=1}^{\infty}$ whose first p terms are $\beta_1, \beta_2, \ldots\ldots \beta_p$ for which the inclusion,

$C_1 \cap \{\bigcap_{i=1}^{\infty} C_2(\beta_{k_i}, \alpha_{k_i}, x, z_{k_i})\} \supseteq C^*$ holds.

But then,

$$A \cap \{\bigcap_{i=1}^{\infty} B(\beta_{k_i}, \alpha_{k_i}, x, z_{k_i})\} \supseteq C_1 \cap \{\bigcap_{i=1}^{\infty} C_2(\beta_{k_i}, \alpha_{k_i}, x, z_{k_i})\} \setminus \bigcup_{i=1}^{\infty} \{P_1 \cup P_2 (\beta_{k_i}, \alpha_{k_i}, x, z_{k_i})\}$$

$$\supseteq C^* \setminus \bigcup_{i=1}^{\infty} \{P_1 \cup P_2 (\beta_{k_i}, \alpha_{k_i}, x, z_{k_i})\}$$

The right hand side of the above inclusion being a set of second category it follow that $A \cap \{ \bigcap_{i=1}^{\infty} B(\beta_{k_i}, \alpha_{k_i}, x, z_{k_i})\}$ or $A \cap \bigcap_{i=1}^{\infty} \{(\frac{B}{\beta_{k_i}} - \frac{z_{ki}}{\beta_{k_i}} - \alpha_{k_i} x)\}$ is a set of second category and hence non-empty for every choice of a vector x from K_o, sequence $\{z_k\}_{k=1}^{\infty}$ of vectors from K_1 and a sequence $\{\beta_k\}_{k=1}^{\infty}$ of real number from I. The remaining part is obvious. Hence the theorem follows.

REFERENCES

[1] S. Kurepa, Note on the difference set of two measurable sets in E^n, Glas. Mat. Fiz.Astr. 15(35), (1960), 99-105.

[2] A. Majumdar, Some properties of sets with positive measures, Glas. Mat. Fiz.Astr. 16(36), 1981, 25-37.

[3] J.C. Oxtoby, Measure and category, Springer–Verlag, 1980, 18, 19.

[4] M. Pal, On an extension of a theorem of S.Kurepa, Glas. Mat. 20 (40), (1985), 327-335.

[5] H. Wisdorm, Lectures in measure and integration, Von Nostrand, 1969, 116.

[6] J. H. Williamson, Lebesgue integration, Holt. Rinehart and Winston, 1962, ,67.

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AC CONGRUENCE-FREE SEMIRINGS

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1. INTRODUCTION

A Semiring is a non-empty set R equipped with two binary operations, called addition, +, and multiplication (denoted by juxtaposition), such that R is multiplicatively a semigroup and additively a commutative semigroup and that the multiplication is distributed accross the addition both from the left and from the right. An element denoted by O is called the zero of R if a+o=a and oa = ao = o for all $a \in R$. We do not generally assume that R has a zero element but throughout this paper we assume that R has a multiplicative identity 1. A semiring R is called congruence-free if each congruence ρ on R is trivial i.e., ρ coincides with the identity relation on R or with the universal relation R x R. A k-ideal I of a semiring R is an ideal such that, if $a \in I$ and $x \in R$ and $a+x \in I$ then $x \in I$. The zeroid of a semiring R is $Z(R) = \{r \in R \mid r + a = a \text{ for some } a \in R\}$. If Z(R) = R then the semiring R is zeroic. Otherwise it is non zeroic (cf. [1]).

In a recent paper (cf.[2]), congruence free commutative semirings have been studied and categorized in three types. Our main objective in this paper is to concentrate on semirings free from proper additively cancellative congruences and analyse their interlink with condition D (which we shall shortly define).

2. CONDITION D

Definition 2.1. (cf. [4]) A semiring R is said to satisfy "condition D" if for each $a, b \in R$, $a \neq b$, there exist $r_1, r_2 \in R$ such that, $1+r_1a+r_2b=r_2a+r_1b$.

Example 2.2. Every field is a semiring with condition D.

Example 2.3. Let N be the set of all non-negative integers. We define a+b=max. {a,b} and a.b = usual product. Then N is a semiring with condition D, as can be easily verified.

Example 2.4. (cf. [3]) consider the semiring,

+	0	x	1
0	0	x	1
x	x	x	x
1	1	x	1

$\bullet$	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

Here, the set is {0, x, 1} and + is the "sup" in the chain $0 < 1 < x$ and $\bullet$k is the "inf" in the chain $0 < x < 1$. This semiring satisfies condition D. Indeed, it follows that,

for $x \neq 1$, $^{-}$ + 1.x + 1.1 = 1.1 + 1.x ;

for $x \neq 0$, $^{-}$ + 1.x + 1.0 = 1.0 + 1.x ;

and for $1 \neq 0$, $^{-}$ + 1.1 + 1.0 = 1.0 + 1.1 ;

as can be verified easily from the tables.

Following results are well known and can easily be verified.

Result 2.5 A semiring R with 0, satisfying condition D does not admit divisor of zero.

Result 2.6. (cf. [5]) A nonzeroic semiring R with condition D has no non-trivial k-ideal.

We now consider $M_n(R)$, where R is a semiring with condition D and $M_n(R)$ is the set of all n x n matrices over R. Then with respect to usual matrix addition and multiplication it is a semiring with identity and zero element.

Result 2.7. the semiring $M_n(R)$, as defined above has no non-trivial k-ideal.

Proof. We first produce the proof for $n = 2$.

Let I be a k-ideal in $M_2(R)$ such that $I \neq \{o\}$

i.e., $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in I$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ i.e. not all of a,b,c,d are simultaneously zero. Now, since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(R)$ and I is an ideal $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in I$ i.e., $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in I$. Similarly, taking $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ into consideration we can see that, $\begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$ is also in I. Now, without any loss of generality we may assume that $a \neq 0$, $d \neq 0$;

Then taking $a \neq 0$, by condition, D, we get that there exist $r_1,r_2 \in R$ such that $1+r_1 a = r_2a$. Now, $\begin{bmatrix} r_1 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(R)$ and $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in I$ implies $\begin{bmatrix} r_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r_1a & 0 \\ 0 & 0 \end{bmatrix} \in I$ and similarly $\begin{bmatrix} r_2a & 0 \\ 0 & 0 \end{bmatrix} \in I$ but $\begin{bmatrix} r_2a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1+r_1a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} r_1a & 0 \\ 0 & 0 \end{bmatrix} \in I$ implies $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in I$ (since I is a k-ideal). Similarly, taking $d \neq 0$. into consideration, we can show that $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in I$. Consequently $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in I$. Hence $I = M_2(R)$.

The proof for $n > 2$ can be done in esssentially similar manner and hence is omitted.

Definition 2.8. A k-ideal I is called minimal if for any k-ideal A with $(0) \subseteq A \subsetneq I$ implies $A = (0)$.

Definition 2.9. Let $I_1, I_2, \dots, I_n$ be k-ideals of a semiring R. We call R the internal direct sum of $I_1, I_2, \dots, I_n$ and write $R = I_1 \oplus \dots \oplus I_n$, provided that

a) $R = I_1 + I_2 + \dots + I_n$

b) $I_i \cap (I_1 + I_2 + \dots + I_{i-1} + I_{i+1} + \dots\dots + I_n) = \{0\}$, for each i.

Result 2.10. The semiring $M_n(R)$, as defined above is a direct sum of n minimal left k-ideals.

Proof. We first prove the result for $n = 2$.

Let us consirder $I = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} : a, b \in R \right\}$. Then I is a minimal left k- ideal of $M_2(R)$. In fact, that it is a left k- ideal follows obviously from the definitions. To prove the minimality we proceed as follows;

Let $A \subseteq I$ be any other left k-ideal; $A \neq (0)$ implies that there exists $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \in A$ such that $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Without any loss of generality we may assume $a \neq 0$. Then by condition D, there exist $r_1,r_2 \in R$ such that $1 + r_1a = r_2a$. Then as before in 2.7 we can show $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in A$ i.e., for any $\begin{bmatrix} p & 0 \\ q & 0 \end{bmatrix} \in I$, we see that $\begin{bmatrix} p & 0 \\ q & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} p & 0 \\ q & 0 \end{bmatrix}$ which implies $\begin{bmatrix} p & 0 \\ q & 0 \end{bmatrix} \in A$, so that $I = A$.

Similarly it can be established that , $J = \left\{ \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} : c, d \in R \right\}$ is a minimal left k-ideal of R.

It then follows obviously from the definition of direct sum that, $M_2(R) = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \right\} \oplus \left\{ \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} \right\}$ i.e. $M_2(R) = I \oplus J$.

The proof of this result for $n > 2$ being essentiallly similar is omitted.

3. AC CONGRUENCE ON SEMIRINGS

Definition 3.1. A congruence ρ defined on R is a subset of $R \times R$ such that;

i) ρ is an equivalence relation

ii) $(a,b) \in \rho$ implies $(c+a, c+b) \in \rho$, and $(ca, cb) \in \rho$ and $(ac,bc) \in \rho$ for all a, b, $c \in R$.

Definition 3.2. A congruence ρ on R is called additively cancellative (AC), if $(a+c,b+c) \in \rho$ implies $(a,b) \in \rho$ for all a. b, $c \in R$. Following result can easily be verified.

Result 3.3. A congruence ρ on a semiring R is cancellative if and only if $(a+c, b+d) \in \rho$ and $(c,d) \in \rho$ implies $(a,b) \in \rho$, a, b, c, $d \in R$.

Definition 3.4. (cf, [2]) A semiring R is Called a semifield if $(R, \bullet)$ or $(R \setminus \{0\} , \bullet)$ is a group.

Following example shows that a semifield is not necessarily congruence-free.

Example 3.5.. Let Q^0 be the semiring of all non- negative rationals with usual addition and multiplication. Then, it can be easily verified that Q^0 is a semifield. Define $\rho = \{ (a,b) \in Q^0 \times Q^0 : ab > 0 \} \cup \{ (0,0) \}$. Then ρ can be proved to be a proper congruence relation on Q^0.

We further point out that a semifield need not be free even from proper AC congruence as is evident from the following example.

Example 3.6. Consider $R = \{ (a,b) \in Q^+ \times Q : Q$ is the set of rationals and Q^+ is that of the positive rationals}. Defining $(a,b) + (c,d) = (a+c,b+d)$ and $(a,b). (c,d) = (ac, ad+bc)$ it can be verfied to be a semifield. Define a relation ρ on R as $(a,b)\ \rho\ (c,d)$ iff $a = c$ for $(a,b), (c,d) \in R$. It is a routine matter to check that ρ is a proper AC congruence on R. Indeed, let $(a,b) + (x,y)\ \rho\ (c,d) + (x,y)$ hold, for $(a,b), (c,d), (x,y) \in R$. Then, $a+x = c+x$ implies $a = c$ (as Q^+ is additively cancellative). Hence we get, $(a,b)\ \rho\ (c,d)$, showing that ρ is AC. Now $(1,2)\ \rho\ (1,3)$ implies that ρ is not the identity relation and $(3,2) \not\rho (5,1/2)$ shows that $\rho \neq R \times R$. Consequently ρ is a proper AC-congruence on R.

Definition 3.7. Closure of a congruence ρ on R is defined as, $\overline{\rho} = \{ (a,b) : (a+x,b+x) \in \rho$ for some $x \in R \}$

By definition the closure of any congruence becomes an AC congruence and it also follows trivially that $\rho \subseteq \overline{\rho}$ for any congruence ρ on R.

4. NON-TRIVIAL AC CONGRUENCE-FREE SEMIRINGS

In this section we take R to be a semiring with zero element $0 \neq 1$.

Definition 4.1. An AC congruence ρ on R is called a non-trivial AC congruence if $I_R \subsetneq \rho \subsetneq R \times R$, where I_R is the identity congruence on R.

Lemma 4.2. If an additively cancellative semiring R has no non-trivial AC congruence then it satisfies condition D.

Proof For $a,b \in R$ with $a \neq b$ we define a relation ρ on R in the following way

$\rho = \{ (x,y) : x+r_1 a+r_2 b = y+r_2a+r_1b,$ for some $r_1,r_2 \in R \}$

It is a routine matter to check that ρ , as defined above, is an equivalence relation on R. Also, ρ is an AC congruence relation on R. In fact, $(x,y) \in \rho$ implies, $x+r_1a+r_2b = y+r_2a+r_1b$ for some $r_1,r_2 \in R$ whence, $c+x+r_1a+r_2b = c+y+r_2a+r_1b$ for all $c \in R$ showing that $(c+x,c+y) \in \rho$ and $cx+cr_1a+cr_2b = cy+cr_2a+cr_1b$ i.e. $cx+s_1a+s_2b = cy+s_2a+s_1b; s_1,s_2 \in R$ $(s_1 = cr_1, s_2=cr_2)$ shows that $(cx,cy) \in \rho$, and similarly we can show $(xc,yc) \in \rho$ proving that ρ is a cogruence relation on R. Also $(x+c,y+c) \in \rho$ for any $c \in R$ implies $x+c+r_1a+r_2b = y+c+r_2a+r_1b$ for some $r_1,r_2, \in R$ implies $x+r_1a+r_2b = y+r_2a+r_1b$ (since R is additively cancellative) so that $(x,y) \in \rho$ proving there by the additive cancellativity of ρ and hence either $\rho = I_R$ or $\rho = R \times R$. We assert that $\rho \neq I_R$. To prove our assertion we are to find some $p,q \in R$ with $p \neq q$ such that $(p,q) \in \rho$. Indeed, $a \neq b$ serves our purpose with $r_1 = 0$, $r_2 = 1$. Hence $\rho = R \times R$, so that $(1,0) \in \rho$ implying that $1+r_1a+r_2b = r_2a+r_1b$ for some $r_1,r_2 \in R$ which is percisely the condition D.

Theorem 4.3. A non zeroic semiring R satisfies condition D if and only if R is non-trivial AC congruence free.

Proof. Let R be a semiring with condition D. let ρ be an AC congruence on R and let $\rho \neq I_R$. We are to prove $\rho = R \times R$.

Now, there exist $a,b \in R$ Such that $a \rho b$ and $a \neq b$ (since $\rho \neq I_R$). Then by condition D there exist some $r_1,r_2 \in R$ such that $1+r_1a+r_2b = r_2a+r_1b$. since ρ is an equivalence relation $(1+r_1a+r_2b, r_2a+r_1b) \in \rho$. Now, since $(a,b) \in \rho$ and ρ is a congruence we have $r_1a \rho r_1b$ and $r_2a \rho r_2b$ so that $(r_1a+r_2b, r_2a+r_1b) \in \rho$. Hence by Result 3.3 we get $(1,0) \in \rho$. Let us now consider $x,y \in R$. Then $(x,0) \in \rho$ and $(y, o) \in \rho$ implies $(x,y) \in \rho$. Since this holds for all $x,y \in R$. we get $\rho = R \times R$. Conversely, let us suppose that R has no non-trivial AC congruence. Now, we are to show that R satisfies condition D. We know $\bar{I}_R$ is an additively cancellative congruence. Hence either $\bar{I}_R = R \times R$ or $\bar{I}_R = I_R$. We see that if $\bar{I}_R = R \times R$ then, $(1,0) \in \bar{I}_R$ so that $(1+x,x) \in I_R$ for some $x \in R$ which implies that $1+x = x$; then for any $a \in R$ we get, $a+ax = ax$ which shows that R is zeroic, since this contradicts our hypothesis we assert that $\bar{I}_R = I_R$. This indicates that R is an additively cancellative semiring. Then by lemma 4.2 we see that R satisfies condition D.

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REFERENCES

1. Golan J.S., The Theory of semirings ; Longman Scientific and Technical, 1992, pp.43.

2. Mitchell Sidney S. and Fenoglio Paul B; Congruence-free commutative semirings, Semigroup Forum, 37 No.1 (1988) 79-91.

3. Padmanavan R. Subramanian H.; Ideals in semiring. Math. Japon. 13 (1968) 123-128.

4. Sen M.K.and P. Mukhopadhyay, Von Neumannian Regularity in Semirings, Kyungpook Math. J. 35 No.2 (1995) 249-258.

5. Ghosh, Shamik, Matrices over Semirings, Information Science, an International Journal 90 No. 1-4. (1996). 221-230.

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A NOTE ON p-RELATIONS

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ABSTRACT. The concepts of p-relations, p-transformations on a non-empty set and the weak regularity of a p-transformation have been defined. A criterion for weak regularity of a p-transformation has been established.

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1. INTRODUCTION

For a monoid M with identity 1, a non-deterministic M-automaton $\vartheta = (S,f)$ is the pair of set S and a mapping f from $S \times M$ into P(S), the set of all subsets of S, satisfying $f(s,1) = \{s\}$ and $f\,[f\,(s,m), m'] = f\,(s,mm')$ for all $s \in S$, $m,m' \in M$ [1]. Every non-deterministic M-automaton $\vartheta = (S,f)$ induces a relation between elements of S and P(S); an element $s \in S$ is related with $A \in P\,(S)$, if there exists $m \in M$ such that $f(s,m) = A$. This motivates us to introduce p-relation on a non-empty set S.

2. p-RELATIONS AND p-TRANSFORMATIONS

A binary relation ρ on a set S is a subset of the cartesian product $S \times S$. An operation o on the set ß (S) of all binary relation on S is defined for every ρ , $\sigma \in ß\,(S)$ by

$$\rho o \sigma = \{(x,y) \in S \times S \mid \text{there exists } z \in S \text{ with } (x,z) \in \rho \text{ and } (z,y) \in \sigma\}$$

With this operation o, ß(S) is a semigroup [1].

By a p-relation ρ on a non-empty set S, we mean a subset of $S \times P(S)$. We denote the set of all p-relations on S by $ß_p(S)$. For any $\rho,\sigma \in ß_p\,(S)$, we define

$\rho o \sigma = \{(x,A) \mid$ there exists non-empty $B \in P\,(S)$ and $A_b \in P(S)$ corresponding to every $b \in B$ satisfying $(x,B) \in \rho$, $(b,A_b) \in \sigma$ and $\bigcup_{b \in B} A_b = A\} \cup \{\,(x,\phi) \mid (x,\phi) \in \rho\}$

and $\rho o \varnothing = \varnothing = \varnothing o \infty$, where $\varnothing$ is the null p-relation on S.

2.1 Theorem. ($ß_p(S)$, o) is a semigroup.

Proof. For any ρ, σ, $\delta \in B_p(s)$, we take $(x,A) \in (\rho o \sigma)\, o \delta$.

If $A = \phi$, we have the following possibilities.

(i) $(x,\varphi) \in \rho o \sigma$ (ii) there exists non-empty $B \in P(S)$ such that $(x,B) \in \rho o \sigma$ and $(b,\varphi) \in \delta$ for all $b \in B$.

If $(x,\varphi) \in \rho o \sigma$ then either $(x,\varphi) \in \rho$ or there exists non-empty $B \in P(S)$ such that $(x, B) \in \rho$ and $(b,\varphi) \in \sigma$ for all $b \in B$. If $(x,\varphi) \in \rho$ then $(x,\varphi) \in \rho o (\sigma o \delta)$ from the definition of o. If there exists non-empty $B \in P(S)$ such that $(x,B) \in \rho$ and $(b,\varphi) \in \sigma$ for all $b \in B$, then $(x,B) \in \rho$ and $(b,\varphi) \in \sigma o \delta$ for all $b \in B$ and hence $(x,\varphi) \in \rho o (\sigma o \delta)$.

Now we consider the second possibility $(x,B) \in \rho o \sigma$ implies there exists non-empty $C \in P(S)$ and $B_c \in P(S)$ corresponding to every $c \in C$ such that $(x,C) \in \rho$, $(c,B_c) \in \sigma$ and $\bigcup_{c \in C} B_c = B$.

Here $(b,\varphi) \in \delta$ for all $b \in B$. So $(b,\varphi) \in \delta$ for all $b \in B_C$. $(c,B_C) \in \sigma$ and $(b,\varphi) \in \delta$ for all $b \in B_C$. imply $(c,\varphi) \in \sigma o \delta$. $(x,C) \in \rho$, $(c,\varphi) \in \sigma o \delta$ imply $(x,\varphi) \in \rho o (\sigma o \delta)$.

If $A \neq \varphi$ there exists non-empty $B \in P(S)$, $A_b \in P(S)$ corresponding to every $b \in B$ such that $(x,B) \in \rho o \sigma$, $(b,A_b) \in \delta$ and $\bigcup_{b \in B} A_b = A$. Now $(x,B) \in \rho o \sigma$ implies that there exists non-empty $C \in P(S)$, $B_c \in P(S)$ corresponding to every $c \in C$ such that $(x,C) \in \rho$, $(c,B_c) \in \sigma$ and $\bigcup_{c \in C} B_c = B$. Thus for each $c \in C$ we have $(c,B_c) \in \sigma$, $(b,A_b) \in \delta$ for all $b \in B_c$ which give $(c, \bigcup_{b \in B_c} A_b) \in \sigma o \delta$ for all $c \in C$. $(x,C) \in \rho$, $(c, \bigcup_{b \in B_c} A_b) \in \sigma o \delta$ imply $(x, \bigcup_{c \in C} (\bigcup_{b \in B_c} A_b) \in \rho o (\sigma o \delta)$.

$\bigcup_{c \in C} (\bigcup_{b \in B_c} A_b) = \{a \in A_b / b \in \bigcup_{c \in C} B_c\} = \bigcup_{b \in B} A_b = A$. So $(x,A) \in \rho o (\sigma o \delta)$

Converse part can also be proved similarly. Thus $(\mathcal{B}_p(S), 0)$ is a semigroup. ☐

The subsemigroup $\mathcal{T}_r(S)$ of $B(s)$ consisting of all binary relations f satisfying the conditions

(1) for every $x \in S$, there exists $y \in S$, such that $(x,y) \in f$,

(2) $(x,y) \in f$ and $(x,y') \in f$ imply $y = y'$, is called the semigroup of all transformations on S [1].

The subsemigroup $\mathcal{T}_p(S)$ of $\mathcal{B}_p(S)$ consisting of all p-relations f satisfying the conditions.

(1') for every $x \in S$, there exists $A \in P(S)$ such that $(x,A) \in f$

(2') $(x,A) \in f$ and $(x,B) \in f$ imply $A = B$

is called the semigroup of all p-transformations on S. An element $f \in \mathcal{T}_p(S)$ acts on the set S on the right and this action is usually indicated by writing $(x)f = A$ whenever $(x,A) \in f$. A p-transformation semigroup on a set S is a sub-semigroup of $\mathcal{T}_p(S)$. Similarly, a p-transformation monoid on S is defined as a submonoid of the monoid $\mathcal{T}_p(S)$. $\mathcal{T}_r(S)$ is isomorphic to a subsemigroup of $\mathcal{T}_p(S)$ under the isomorphism $f \to f_p$ for every $f \in \mathcal{T}_r(S)$, where $(s)f_p = \{(s)f\}$ for every $s \in S$. So $\mathcal{T}_r(S)$ can be thought of as a p-transformation semigroup on the set S.

For f, $g \in \mathcal{T}_p(S)$, we write $f \leq g$ when $(x)f \subseteq (x)g$ for all $x \in S$. An element a of a semigroup S is called regular if there exists $x \in S$ such that $axa = a$. An element $f \in \mathcal{T}_p(S)$ is called weakly regular if $fogof \leq f$ for some $g \in \mathcal{T}_p(S)$ with $(x)g \neq \varphi$ for all $x \in S$. Henceforth fg will stand for fog for any two p-transformations on S.

Example. For $S = \{1, 2, 3, 4, 5\}$, $f = \{(1, \{1\}), (2, \{1, 2\}), (3, \{1, 2, 3\}), (4, \{1, 2, 3, 4\}), (5, \{1, 2, 3, 4, 5\})\} \in \mathcal{T}_p(S)$ is weakly regular.

A p-transformation semigroup is weakly regular if each of its elements is weakly regular.

The following definitions and notions lead to a characterisation of a weakly regular element in $\mathcal{T}_p(S)$.

For any $f \in \mathcal{T}_p(S)$, we can define an equivalence relation $\equiv$ on S by stating that s, t ($s \equiv t \in S$) if and only if $(s)f = (t)f$. We call it as an f-equivalence on S and denote f-equivalence class containing $s \in S$ by f_s.

From each of the f-equivalence classes we take exactly one element and form a set T, called an f-essential subset of S. For any $s \in S$, an element $t \in T$ is denoted by t_s when $t \in f_s$.

Henceforth the letter T will stand for an f-essential subset of S.

We call the set $C_f(T) = \{A \in P(T) \mid \bigcap_{a\in A}(a)\, f \neq \varphi\}$ as the set deriving intersecting f-images.

For $s \in S$ and $A \in C_f(T)$ we call the pair (s, A) effective if $s \in \bigcap_{a\in A}(a)f$. For every $A \in C_f(T)$, there exists at least one $s \in S$ such that (s, A) is effective, but for an element $s \in S$, there may not have an element A in $C_f(T)$ so that (s, A) is effective.

For $s \in S$, $B \in C_f(T)$, the pair (s, B) is said to be an effective extension of an effective pair (s, A) if

(1) (s, B) is effective

(2) $A \subseteq B$

An effective extension (s, B) of an effective pair (s, A) is said to be maximal if $B \subseteq C \in C_f(T)$ and (s, C) is effective imply $B = C$. Maximal effective extension of an effective pair (s, A) is denoted by (s, M_A). For $A, B \in C_f(T)$ if (s, A) and (s, B) are effective then $(s, M_A) = (s, M_B)$.

An element $t \in T$ is said to be a core of an $A \in C_f(T)$ if $(t)\, f \subseteq \bigcap_{a\in A} (a)\, f$. The set of all cores of A is denoted by C_A.

For every $t \in C_A$, $A \cup \{t\} \in C_f(T)$. $A \cup C_A \in C_f(T)$ for $A \in C_f(T)$ if and only if $C_A \in C_f(T)$.

If $f \in \mathcal{T}_p(S)$ be such that $(x)\, f = (y)f$ or $(x)f \cap (y)f = \varphi$ for any $x, y \in S$, then every element A of $C_f(T)$ is singleton and so $C_A = A \neq \varphi$.

2.2 Theorem. $f \in C_f(T)$ is weakly regular if and only if $C_A \neq \varphi$ for every $A \in C_f(T)$.

Proof. Suppose the condition holds good for $f \in \mathcal{T}_p(s)$. We are to construct $g \in \mathcal{T}_p(S)$ such that $fgf \leq f$ and $(s)\, g \neq \varphi$ for all $s \in S$.

For this we fix an element x of S. Let $y \in S$ be arbitrary. If (y, A) is not an effective pair for any $A \in C_f(T)$ we write $(y)g = \{x\}$. If there exists $A \in C_f(T)$ such that (y, A) is effective we take the maximal effective extension (y, M_A) of (y, A). Clearly $M_A \in C_f(T)$. So by the condition $C_{M_A} \neq \varphi$. We write, $(y)g = \bigcup_{t \in C_{M_A}} f_t$.

Now, for any $s \in S$ if $(s) f = \varphi$, $(s)(fgf) = \varphi$ for any $g \in \mathcal{T}_p(S)$ and so $fgf \leq f$ for any $g \in \mathcal{T}_p(S)$; suppose $(s)f \neq \varphi$ for some $s \in S$ and $y \in (s)f$ be arbitrary. Then $y \in (t_s)f$ and so $y \in \bigcap_{a \in A}(a)f$ for some $A \in C_f(T)$, infact, it is true for $A = \{t_s\} \in C_f(T)$. Then (y, A) is an effective pair. So, $(y)g = \bigcup_{t \in C_{M_A}} f_t$.

Now for every $z \in (y)g$, $z \in f_t$ for some $t \in C_{M_A}$ and so $(z)f = (t) f \subseteq \bigcap_{m \in M_A} (m)f \subseteq \bigcap_{a \in A}(a)f$ [$\because A \subseteq M_A$] $\subseteq (t_s)f = (s)f$. This indicates that $(s)(fgf) \subseteq (s)f$ for any $s \in S$ i.e., $fgf \leq f$.

Conversely let $fgf \leq f$ for some $g \in \mathcal{T}_p(s)$ with $(s) g \neq \varphi$ for all $s \in S$ and $A \in C_f(T)$ be arbitrary. We take $y \in \bigcap_{a \in A}(a)f$ and we write $(y) g = B_y$

Now, $y \in (a)f$ for every $a \in A$

$\Rightarrow B_y = (y)g \subseteq (a)(fg)$ for every $a \in A$

$\Rightarrow (B_y) f \subseteq (a)(fgf)$ for every $a \in A$

$\Rightarrow (B_y) f \subseteq (a)f$ for every $a \in A$ [since $fgf \leq f$]

$\Rightarrow (B_y) f \subseteq \bigcap_{a \in A} (a)f$

$\Rightarrow (b) f \subseteq \bigcap_{a \in A} (a)f$ for every $b \in B_y$ [$B_y \neq \varphi$]

$\Rightarrow (t_b)f \subseteq \bigcap_{a \in A} (a)f$

$\Rightarrow t_b \in C_A$

$\Rightarrow C_A \neq \varphi$. □

In [1], it has been proved that the semigroup $\mathcal{T}_r(S)$ of all transformations on S is regular. Here we get the following result as a consequence of the theorem 2.2.

2.3 Corollary. Semigroup $\mathcal{T}_r(S)$ of all transformations on S is a weakly regular p-transformation semigroup.

Proof. For any $f \in \mathcal{T}_r(S)$, either $(x)f_p = (y)f_p$ or $(x)f_p \cap (y)f_p = \phi$ for any $x, y \in S$. So $C_A \neq \varphi$ for any $A \in C_{f_p}(T)$. Hence f_p is weakly regular. Thus $\mathcal{T}_r(S)$ is a weakly regular p-transformation semigroup.

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REFERENCE

1. G. Lallement –Semigroups and combinatorial applications, John wiley and sons, New York, 1979.

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CONGRUENCES AND IDEALS IN Ω-SEMIGROUP

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ABSTRACT. A semigroup (G, +) consists of a non-empty set G on which associative operation, +, is defined. Let (G, +) be a semigroup and Ω be a class of n-ary ($n \geq 1$) operators.

A semigroup (G, +) together with the operator domain Ω will be called Ω-semigroup (G,+,Ω) if the following conditions hold.

(1) $\exists$ at least one element $x \in G / ax \ldots xaw_n = a \; \forall w_n \in \Omega$ and for all $a \in G$.

(2) The following distributive law holds

For any (n-1) elements $a_1, a_2, a_3 \ldots a_{i+1} \ldots a_n$ of G and for $a, b \in G$, $w_n \in \Omega$.

$a_1, a_2 \ldots\ldots (a+b)\, a_n\, w_n$

$= a_1, a_2 \ldots\ldots a_{i-1}\, a\, a_{i+1} \ldots\ldots a_{n-1}\, a_n\, w_n$

$+ a_1, a_2 \ldots\ldots a_{i-1} b\, a_{i+1}\, a_{i+2}\, a_n\, w_n$

Our aim is to define the new class of algebra to study the common properties of a particular class of distributive lattices and rings. In this paper we have given some examples of Ω-semigroups. We have also defined ideals and congruences and established some interesting theorems.

Key words : Ω-Semigroup, Idempotent, n-ary operators, Ω-Sub Semigroup, K-ideal, congruence.

Classification : o8 A 30

§1. DEFINITION Ω-SEMIGROUP

A semigroup (G, +) consists of a non-empty set G on which associative opertion, +, is defined. Let (G, +) be a semigroup and Ω be a class of n – ary ($n \geq 1$) operators.

1.1 A semigroup (G, +) together with the operator domain Ω will be called Ω - semigroup (G,+,Ω) if the following conditions hold.

(i) $\exists$ an element $x \in G / ax \ldots\ldots xaw_n = a$, $\forall w_n \in \Omega$ and for all $a \in G$

(ii) The following distributive law holds : For any (n-1) elements a_1, a_2 .... a_{i-1} , a_{i+1} .... $a_n \in G$ and a, b ∈ G. $w_n \in \Omega$

a_1, a_2 .... a_{i-1} (a+b) a_{i+1} .... $a_n w_n = a_1, a_2$ .... a_{i-1} a a_{i+1} .... $a_n w_n + a_1 a_2$ .... a_{i-1} b a_{i+1} .... $a_n w_n$

Ex. 1.1 Every semigroup (G, +) is an example of Ω–semigroup. where Ω = ϕ.

Ex. 1.2 Every ring (R, +) with 1 is a Ω–semigroup where Ω = (.), since ax = a holds for x = 1.

Ex. 1.3 Every distributive lattice (L, V, ^) is a Ω–semigroup with Ω = { ^ } and + = V.

Ex. 1.4 Let J = positive intergers.

Define a V B max (a, b)

a ^ b = min (a, b)

a . b = product of a, b

a ✱ b = h.c.t. (a, b).

Here, + = V and Ω { ^ , . , ✱}

It is easy to see that ^ , . , ✱ distribute over ' + '.

Hence (J, V, Ω) where Ω = { ^ , . , ✱}, is a Ω - semigroup.

Propostioon 1.1. Let a Ω - semigroup G be such that G itself is a group w.r.t. the operation '+' then G becomes a Ω - group.

Indeed, θ ∈ G, $w_n \in \Omega$ is n-any operator.

$\Rightarrow \theta\theta$ .... $\theta\ (\theta + \theta)\ \theta$ .... $\theta \quad w_n$

$= \theta\theta$ .... $\theta\, w_n + \theta\theta$ .... $\theta\, w_n$

As G is a group we have by cancellation law θθ................ $\theta w_n = \theta$.

§ 2. DEFINITION Ω–SUBSEMIGROUP

2.1 Let G be a Ω –semigroup. By a Ω-subsemigroup H, we mean a non empty subset H of G, which is itself a Ω - semigroup.

2.2 Let J = set of intergers, Let S = J × J, S is a Ω - semigroup w.r.t (+, Ω ,) where Ω = { . } Let R = J × 0.

Then (R, –, .) is a Ω - subsemigroup of (S, +, .).

§ 3. DEFINITION

3.1 By an ideal of a Ω (≠ θ) semigroup G, we mean a non-empty set

I such that

1) I is itself a Ω - semigroup

2) $x_1, x_2, \ldots\ldots\ldots\ldots x_n \in G, x \in I. \Rightarrow$

$x_1, x_2 \ldots\ldots\ldots\ldots x_{i-1}\ x\ x_{i-1} \ldots\ldots x_n\ w_n \in I\ \forall\ w_1 \in \Omega.$

If $\Omega = \phi$, then I is a subsemigroup which is a very trivial case, so in case of ideal it is assumed $\Omega \neq \phi$.

Definition 3.2. K-ideal

By a k-ideal of a Ω - semigroup G we mean a non-empty set I such that

1) I is an ideal in G

2) $x + y \in I$ and either of $x, y \in I$ ==> the other also belongs to I

We now study the relations between the k-ideals in any ring with 1 or k-ideals of distributive lattices with ideals of the corresponding system.

Example 3.1. Let R be a ring w.r.t. +, and I be any ideal in R, in which the equation ax = a is solvable

Then, 1) $x, y \in I \Rightarrow x + y \in I$

2) $x, y \in I, x \in I \Rightarrow -x, \in I$ hence $y \in I$

3) $x_1, x_2 \in R$ and $x, \in I \Rightarrow x_1\ x \in I, xx_2 \in I$

4) The equation ax = a is solvable in I by hypothesis.

Thus every ideal in R in which the equation ax = a is solvable is a k-ideal in R, Conversely. let I be a k-ideal in R and let $a, b \in I$ and $r \in R$.

As $\theta \in R$, $I \neq \phi \Rightarrow \exists\ x \in I\ \theta x \in I$.

Let, $a \in I \Rightarrow -a \in I$

Indeed, $\theta \in I, a + (-a) \in I \Rightarrow -a \in I$

Now, $a, b \in I \Rightarrow a, -b \in I \Rightarrow a - b \in I$

Also $r a, a r \in I$. Thus I is an ideal in R.

Example 3.2 Let L be a distribuive lattice w.r.t. V and $\wedge$.

Let, J be an ideal in L.

Then 1) Let $x, y \in J \Rightarrow x \vee y \in J$

2) Let $x_1 \in L$ and $x \in J$ then $x_1 \wedge x \leq x \Rightarrow x_1 \wedge x \Rightarrow J$

3) Let $x \vee y \in J$, $x \in J$, then $y \leq x \vee y \Rightarrow y \in J$

4) Also $a \wedge x = a$ is solvable, in J, as every element of J is idempotent. Thus J is a k-ideal of L. Conversely, let J be a k-ideal in L.

Then 1) $x, y \in J \Rightarrow x \vee y \in J$

2) Let $x \in L$ and $a \in J$ such that $x \leq a$. Then $x \wedge a \in J$

In this case also the two notions concide.

Definition :

Let G be a Ω - semigroup. If G contains an elements an elements θ such that

$a_1\ a_2 \ldots a_{i-1}\ \theta\ a_{i+1} \ldots\ldots a_n\ w_n = \theta$ where $a_1, a_2, \ldots\ldots a_n \in G$, $\forall w_n \in \Omega$, and $n\ (W_n) = n > 1$ then θ is called an absorbing element of G. If the absorbing element θ is such that $\theta + a = a + \theta$, $\forall a \in G$ then θ is callled the absorbing element θ is such that $\theta + a = a + \theta$, Aa G is called the absorbing zero.

Remark 1. $\theta\theta \ldots\ldots\ldots \theta\ w_n = \theta$ holds for all $w_n \in \Omega$

Example 3.3.

Let E be a set and M = [θ, 1] its associated membership set. We consider the set of fuzzy subset P[E].

Let A.B ∈ P(E). we define the fuzzy suset A U B whose membersship function $^{\mu}AUB$ is as follows :

$^{\mu}AUB^{(x)} = \text{Max}\{^{\mu}A^{(x)}, {}^{\mu}B^{(x)}\}$

Then, (P (E), U) is a semigroup.

We define $\underset{\sim}{A} + \underset{\sim}{B}$, $\underset{\sim}{A}.\ \underset{\sim}{B}$ and $\underset{\sim}{A} \cap \underset{\sim}{B}$ in the usal way . We take the operator domain $\Omega = \{\dot{+}, \cap\}$. It can be easily seen that $(\underset{\sim}{P}(E))$,∪) with operator domain $\Omega = \{\dot{+}, \cap\}$ is a Ω-semigroup. Here θ, the empty set, is the absorbing zero. It can also be seen that $(\underset{\sim}{P}(E), \cap)$ is also a Ω-semigroup with the operator domain $\Omega = \{\cap\ \dot{+}, .\}$ but (P(E),) is not a Ω–semigroup with (Ω = ∪ ∩, ,).

Proposition 3.1. If G be a Ω–semigroup. then (θ) is an ideal in G.

Proof : By the definition of θ we have $\theta\theta \ldots\ldots\ldots\ldots \theta w_n = \theta$

Also, $\theta\theta \ldots\ldots\ldots \theta(\theta + \theta)\theta \ldots\ldots\ldots\ldots \theta w_n = \theta$

By distributivity.

$\theta\theta \ldots\ldots\ldots\ldots\ldots\ldots \theta w_n + \theta\theta \ldots\ldots\ldots\ldots\ldots \theta w_n = \theta$

or ,θ+θ=θ,

Let $a_1.a_2..........a_{i-1},..........a_n \in G$,

then $a_1.a_2..........a_{i-1}{}^{\theta}\, a_{i+1},..........a_n w_n = \theta$

It is not true that $x+y \in (\theta)$ and $x \in (\theta) \Rightarrow y \in (\theta)$.

Hence (θ) is not a k-ideal.

Example 3.4. Example of an absorbing element

Let X be any set.P (X) denotes the set of all subsets of X.

We define $A + B = A \cap B$

$A \times B = A \cup B$

P(X) is a Ω -semigroup w.r.t + and x Evidently $X \in P(X)$ is the absorbing element of Ω-semigroup $(P(X),+X)$.

§ 4. CONGRUENCES

Definition 4.1. Let $(S. \Omega)$ be a Ω–semigroup. An equivalence relation ρ will be called a congruence if

1) $x_1 \rho y_1 , x_2 \rho y_2 \Rightarrow x_1 + x_2 \rho y_1 + y_2$

2) $x_1 \rho y_1 , x_2 \rho y_2 .... x_n \rho y_n \Rightarrow$

$x_1 x_2 x_3 .... x_n w_n \rho y_1 y_2 .... y_n w_n$,

where $w_n \in \Omega$,and $n (w_n) = n \geq 2$.

Proposition 4.1 : If $I (\neq \phi) \subseteq C(G)$ .ideal in Ω-semigroup. G. where $C(G) = \{x \in G / x + a = a + x \; \forall a \in G\}$ then I defines a congruence in G.

Proof. Define $a \rho b$ iff $a + x = b + y$, where $x, y, \in I$ If suffices to prove only transitivity. Let $a \rho b$, $b \rho c$, hold, then $\exists$ x, y, w, v $\in$ I such that $a + x = b + y$, $b + w = c + v$.

Then $a + x + w = b + y + w$

$= b + w + y = c + v + y$ so $a \rho c$ holds.

Let $x_1 \rho y_1$ and $x_2 \rho y_2$,then $\exists$ u, v, w, t $\in$ I such that $x_1 + u = y_1 + v$, $x_2 + w = y_2 + t$.

Adding. $x_1 + u + x_2 + w = y_1 + v + y_2 + t$

$\Rightarrow x_1 + x_2 + u + w = y_1 + y_2 + v + t$

$\Rightarrow x_1 + x_2 \rho y_1 + y_2$

Let $x_1 \rho y_1, x_2 \rho y_2,.......x_n \rho y_n$

$\exists\ u_1, u_2 \ldots\ldots\ldots u_n ; v, u_2, \ldots\ldots\ldots v_n \in I$

such that $x_1 + u_1 = y_1 + v_1$

$$x_2 + u_2 = y_2 + v_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_n + u_n = y_n + v_n$$

$(x_1 + u_1)(x_2 + u_2) \ldots\ldots\ldots\ldots\ldots\ldots (x_n + u_n)\, w_n$

$= (y_1 + v_1)(y_2 + v_2) \ldots\ldots\ldots\ldots\ldots\ldots (y_2 + v_n)\, w_n$

or. $x_1 x_2 \ldots\ldots\ldots x_n w_n + \Sigma w_n (x_i, u_i) + u_1 u_2 \ldots\ldots\ldots u_n w_n,$ (where $w_n(x_1 u_1) = a_1 a_2 \ldots\ldots\ldots a_n w_n$
(all possible sum) a_i is either x_i or u_i)

$= y_1 y_2 \ldots\ldots\ldots y_r w_n + \Sigma w_n (y_1, v_1) + v_1 v_2 \ldots\ldots\ldots v_n w_n$ (where $w_n(y_1 v_1) = b_1 b_2 \ldots\ldots\ldots b_n w_n$
(all possible sum) b_i is either x_i or v_i)

$\Rightarrow x_1 x_2 \ldots\ldots x_n w_n\ \rho\ y_1 y_2 \ldots\ldots y_n w_n$

Thus I defines a congruence ρ in the Ω-semigroup G. ///. Now let us denote the equivalence class containing γ by γ/I i.e. γ/I contains all the elements s satisfying $\gamma + u = s + v$, where $u, v \in I$. Evidently $\gamma \in \gamma/I$

Note that γ/I is not the class of elements of the form. $\gamma + I$ but a set containing $\gamma + I$.

Define $\gamma/I + s/I = (\gamma + s) / I$

The operation is well defined.

Let $\gamma/I = \gamma_1/I$ and $s/I = s_1/I$ where $\gamma, \gamma_1, s, s_1 \in G$. so $\gamma + u = \gamma_1 + v$, for some $u, v \in I$ and $s + r = s_1 + t$, for some $r, t \in I$

$\Rightarrow \gamma + u + s + r = \gamma_1 + v + s_1 + t$

$\Rightarrow (\gamma + s) + u + r = (\gamma_1 + s_1) + v + t$

$\Rightarrow (\gamma + s)/I = (y_1 + s_1) / I$

Hence, $\gamma/I + s/I = (\gamma + s) / I = (\gamma_1 + s_1) / I = \gamma / I + s_1 / I$

Let $\gamma_1/I, \ldots\ldots v_n/I$ are n elements of the congruence classes, then define

$(y_1/I)\ (y_2/I)......(y_n/I)\ w_n = y_1y_2.....y_nw_n/I.w_n \in \Omega.n(w_n) = n$ and $n \geq 2$

Let $\gamma_i/I = s_i/I$

$\Rightarrow \gamma_i + u = s_i + v$, where $u, v \in I$

Now, $\gamma_1\ \gamma_2..............\gamma_i{-}1\ (\gamma_i + u)\ \upsilon_{i+1}............\gamma_n w_n$

$= \gamma_1\gamma_2.................\gamma_{i-1}(s_1+v)\ \gamma_{i+1}..........\gamma_n w_n$

or $\gamma_1\gamma_2........\gamma_{i-1}\gamma_i\gamma_{i+1}...........\gamma_n w_n$

$+\gamma_1\gamma_2..............\gamma_{1-1}\ u\gamma_{i+1}............y_n\ w_n$

$= \gamma_1\gamma_2............\gamma_{i-1}\ s_i\ \gamma_{i-1}...........\gamma_n w_n$

$+\ \gamma_1\gamma_2............\gamma_{i-1}\ v\ \gamma_{i-1}...........\gamma_n w_n$

as I is a ideal and $u, v \in I$

$\Rightarrow \gamma_1\gamma_2............\gamma_{i-1}\ \gamma_i\ \gamma_{i+1}..........\gamma_n w_n/I$

$= \gamma_1\gamma_2............\gamma_{i-1}\ s_i\ \gamma_{i+1}..........\gamma_n w_n/I$

Thus this operation is also well defined.

Let $a \in G, \exists\ x \in G\ /\ ax...........xaw_n = a$.

then $(a/I)\ (x/I)..........(x/I)\ (a/I)\ w_n = ax........x_\alpha w_n/I$

Thus the congruence classes form a Ω-semigroup G/ρ

Let $x \in I$, then $x/I = I$, Indeed $p \in I \Rightarrow x + p = p + x$, as $I \subsetneq C(G) \Rightarrow p \in x/I$ conversely, let $p \in x/I \Rightarrow \exists\ u, v, \in I\ /p + u = x + v, \Rightarrow p + u \in I$, As I is a k-ideal, and $u \in I \Rightarrow p \in I$.

Thus we see that $x/I = I$, when $x \in I$. Thus we have

$(x_1/I)\ (x_2/I).........(x_{i-1}/I)\ (x/I)\ (x_{i+1}/I).......(x_n/I)w_n$

$= x_1x_2............x_{i-1}\ x\ x_{i+1}........x_n\ w_n/I$

$= x/I$

Thus x / I is an absorbing zero of G/ρ.

Thus we have the following theorem :-

Theorem 4.1.

If G be a Ω-semigroup and if I (≠ ∅) be an ideal of G such that I $\subseteq$ C (G), where C(G) is the center of G, then G/ρ will form a Ω-semigroup where ρ is a congruence in G, defined by I.

REFERENCES

1. Birkhoff. G. Lattice, theory. Amer. Maths. Soc. colloquim publ. 1967.

2. Cohn P.M. Universal, Algebra. Harper and Row publishers 1965.

3. Gratzer, G. Universal Algerbraa. Van Nostand 1967.

4. Jonathan S. Golan

   The theory of semirings with applications in mathematics and theoretical computer science.Longman scientific and technical copublished in the United States with John wiley and Ssons Inc. New York.

5. Kurosh A.G. Lectures in General Algebra, London permagon press 1965.

6. Sen M. K. and Adhikari M. R. On maximal K-ideals of semirings. Proceeding of the American Math. Soc. VoL 118. No.3, June 1993

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NONLINEAR ANALYSIS AS AN AID TO LINEAR ANALYSIS

JYOTI DAS (NÉE CHAUDHURI)

1. INTRODUCTION

The first step of the qualitative analysis of a nonlinear differential equation (NDE) generally consists of finding a suitable linear differential equation (LDE), which is sufficiently close to the given NDE in some sense. This step is obviously taken with the presumption that the analysis of the corresponding LDE is fairly well-known. In other words, for the analysis of a NDE, we seek help of a LDE. Unfortunately, there are many LDEs, much of the analysis of which are yet to be known.

In view of this, a pertinent question is : may we seek help of NDEs in analysing LDEs? The present endeavour is to investigate the prospect of this idea of utilizing NDEs in deriving information regarding the solutions of a given LDE.

We concentrate here on the second-order linear homogeneous ordinary differential equation

$$-(p(x)\, y'(x))' + q(x)\, y(x) = 0, \quad x \,\varepsilon\, [0, \infty), \qquad \dots (1.1)$$

where (i) $p, q : [0, _\infty) \rightarrow R$ (the set of real numbers) are continuous,

(ii) $p(x) > 0$ for all $x \,\varepsilon\, [0, \infty)$

and $y'(x) \equiv \dfrac{dy}{dx}$.

It will be shown that (1.1) can be associated with a suitable Riccati's differential equation (RDE). By analysing the RDE, which is nonlinear, we derive various knowledge about the solutions of the given LDE (1.1) in regard to boundedness, monotonicity, square-integrability and oscillatory behaviour. Also, we can obtain from the RDE some results regarding the zeros of the derived functions of the solutions of the LDE (1,1). New information regarding the eigenfunctions of an eigenvalue problem associated with (1,1) can also be derived through the corresponding RDE. Finally we compare the solutions of two LDEs of the form (1.1) through the corresponding RDEs.

2. THE RICCATI'S DIFFERENTIAL EQUATION ASSOCIATED WITH (1.1)

Let $y : [\alpha, \beta] \rightarrow R$ be any solution of (1.1) such that $y(x) \neq 0$ for all $x \,\varepsilon\, [\alpha, \beta]$.

Write $z(x) = p(x)\, y'(x) / y(x)$ for all $x \in [\alpha, \beta]$. ...(2.1)

It can be easily verified that, for all $x \in [\alpha, \beta]$,

$$z'(x) = q(x) - z^2(x) / p(x). \qquad \text{... (2.2)}$$

This is the RDE associated with the given LDE (1.1).

Conversely, for any solution $z : [\alpha, \beta] \to R$ of (2.2), if we define $y : [\alpha, \beta] \to R$ by (2.1), we find that y is a solution of (1.1) with $y(x) \neq 0$ for all $x \in [\alpha, \beta]$. It is to be noted here that, (1,1) being homogeneous, if $y : [\alpha. \beta] \to R$ is a solution of (1.1), then for each real number k, $ky : [\alpha, \beta] \to R$ is also a solution of (1.1). In fact, given a solution z of (2.2), we get the set $S_z = \{ky : k \in R\text{-}\{0\}\}$ of solutions of (1.1), each member of which is linked with z by the relation (2.1). In view of this, a one-one correspondence is established between the solution-space of (2.2) and the solution-space of (1.1), grouped in the fashion mentioned above.

In the sequel, we consider a solution $\phi : [\alpha, \beta] \to R$ of (1.1) such that $\phi(x) \neq o$ for all $x \in [\alpha, \beta]$. Let $\xi : [\alpha, \beta] \to R$ be defined as follows : for all $x \in [\alpha, \beta]$,

$$\xi(x) = p(x)\, \phi'(x) / \phi(x). \qquad \text{... (2.3)}$$

Then $$\xi'(x) = q(x) - \xi^2(x) / p(x). \qquad \text{... (2.4)}$$

3. THE LDE (1.1) WITH $q : [\alpha, \beta] \to R$ OF ARBITRARY SIGN

Using (ii) of §1 in (2.4), we get, for all $x \in [\alpha, \beta]$,

$$\xi'(x) \leq q(x). \qquad \text{... (3.1)}$$

Integrating twice we get, for all $x \in [\alpha, \beta]$,

$$\ln \left| \frac{\phi(x)}{\phi(\alpha)} \right| \leq \int_\alpha^x \{ p(\alpha)\, \phi'(\alpha) / \phi(\alpha) + \int_\alpha^t q(u)\, du \} / p(t)\, dt. \qquad \text{... (3.2)}$$

Hence,

$$0 \lesseqgtr \phi(x) \lesseqgtr \phi(\alpha) \exp \{ \int_\alpha^x [p(\alpha)\, \phi'(\alpha) / \phi(\alpha) + \int_\alpha^t q(u) du] / p(t)\, dt \},$$

for all $x \in [\alpha, \beta]$ according as $\phi(x) \gtrless 0$ on $[\alpha, \beta]$.

These lead to the following results :

Theorem I. If (i) the LDE (1.1) is nonoscillatory on $[0, \infty)$,

(ii) $\int_0^\infty 1/p(x)\, dx < \infty$,

(iii) $\left| \int_0^\infty \{ \int_0^x q(t) dt \} / p(x) \right| < \infty$

then all the solutions of the LDE (1.1) are bounded on $[0, \infty]$.

Theorem II. If (i) the LDE (1.1) is nonoscillatory on $[0, \infty)$,

(ii) $pq : [0, \infty] \rightarrow R$ is monotonic decreasing,

(iii) $\exp [\{\int_0^\infty (p(a)\, q(a))^{1/2} + \int_a^x q(t)\, dt\} / p(x)\, dx] \,\varepsilon\, L^2 [a, \infty)$ for some $a > 0$,

then the LDE (1.1) is in the limit-circle case at infinity.

Remark : The results of Theorems I and II are new. These are the direct contributions of the nonlinear RDE (2.2).

4. THE LDE (1.1) WITH q NONNEGATIVE ON $[\alpha, \beta]$

Let $q(x) \geq 0$ for all $x \,\varepsilon\, [\alpha, \beta]$; let ϕ be a solution of the LDE (1.1) with $\phi(x) \neq 0$ for all $x \,\varepsilon\, [\alpha, \beta]$. Without loss of generality we may assume that $\phi(x) > 0$ on $[\alpha, \beta]$. The three cases

(1) $\phi'(\alpha) > 0$, (2) $\phi'(\alpha) < 0$, (3) $\phi'(\alpha) = 0$ are to be considered separately.

(1) $\phi'(\alpha) > 0$.

It follows from the LDE (1.1) that $\phi'(x) > 0$ for all $x \,\varepsilon\, [\alpha, \beta]$.

Therefore ϕ is strictly increasing on $[\alpha, \beta]$. Again q being nonnegative on $[\alpha, \beta]$, we have, for all $x \,\varepsilon\, [\alpha, \beta]$.

$$\xi'(x) \geq - \xi^2(x) / p(x). \qquad \text{... (4.1)}$$

Intergrating (4.1) we get, for all $x \,\varepsilon\, [\alpha, \beta]$,

$$\phi'(x) / \phi(x) \geq [\phi(\alpha) / p(\alpha)\, \phi'(\alpha) + \int_\alpha^x 1/p(t)\, dt]^{-1} / p(x). \qquad \text{... (4.2)}$$

Intergrating again we have

$$\phi(x) \geq \phi(\alpha) + p(\alpha)\, \phi'(\alpha) \int_\alpha^x 1/p(t)\, dt. \qquad \text{... (4.3)}$$

As $\phi'(x) > 0$ on $[\alpha, \beta]$, we know that $\phi(x) \geq \phi(\alpha)$ for all $x \,\varepsilon\, [\alpha, \beta]$. Note that (4.3) is a better result. If $q(x) \geq 0$ for all $x \,\varepsilon\, [\alpha, \infty)$ for some $\alpha > 0$, it follows that no solution of the LDE (1.1) can be square-integrable.

(2) $\phi'(\alpha) < 0$.

In this case we can find $\gamma \,\varepsilon\, (\alpha, \beta]$ such that, for all $x \,\varepsilon\, [\alpha, \gamma]$,

$$\phi(\alpha) / p(\alpha)\, \phi'(\alpha) + \int_\alpha^x 1/p(t)\, dt < 0.$$

Integrating (4.1) we now find that (4.2) holds on $[\alpha, \gamma]$. Integrating again we get, for all $x \,\varepsilon\, [\alpha, \gamma]$,

$$\phi(x) \geq \phi(\alpha) + p(\alpha)\, \phi'(\alpha) \int_\alpha^x 1/p(t)\, dt. \qquad \text{... (4.4)}$$

As $\phi(x) < 0$ on $[\alpha, \gamma]$, we have $\phi(\alpha) \geq \phi(x)$; so (4. 4) provides a bound on the other side.

(3) $\phi'(\alpha) = 0$.

In this case, there exists $\gamma \varepsilon (\alpha, \beta)$ such that, for all $\alpha_1 \varepsilon (\alpha, \gamma)$, either $\phi'(\alpha_1) > 0$, or $\phi'(\alpha_1) < 0$. Hence proceeding as in the cases (1) or (2), we can prove that, for all $x \ \varepsilon \ [\alpha_1, \gamma]$

$$\phi(x) \geq \phi(\alpha_1) + p(\alpha)\, \phi'(\alpha_1) \int_{\alpha_1}^{x} .\ 1/p(t)\, dt;$$

whence, making $\alpha_1 \to \alpha$ we get,

$$\phi(x) \geq \phi(\alpha) + p(\alpha)\, \phi'(\alpha) \int_{\alpha}^{x} 1/p(t)\, dt.$$

Incorporating the above discussions, we have the following results :

Theorem III **Let** (i) $q(x) \geq 0$ for all $x \ \varepsilon \ [\alpha, \beta]$,

(ii) $\phi : [\alpha, \beta] \to R$ be a solution of the LDE (1.1),

(iii) $\phi(x) > 0$ for all $x \ \varepsilon \ [\alpha, \beta]$.

Then, (a) there exists $\gamma \ \varepsilon \ [\alpha, \beta]$ such that, for all $x \ \varepsilon \ [\alpha, \gamma]$,

$$\phi'(x) / \phi(x) \geq [\phi(\alpha) / \{p(\alpha)\, \phi'(\alpha)\} + \int_{\alpha}^{x} 1/p(t)\, dt\,]^{-1}/p(x),$$

(b) [i] if $\phi'(\alpha) > 0$,

$$\phi(x) \geq \phi(\alpha) + p(\alpha)\, \phi'(\alpha) \int_{\alpha}^{x} 1/p(t)\, dt \geq \phi(\alpha),$$

for all $x \ \varepsilon \ [\alpha, \beta]$;

[ii] if $\phi'(\alpha) < 0$.

$$\phi(\alpha) \geq \phi(x) \geq \phi(\alpha) + p(\alpha)\, \phi'(\alpha) \int_{\alpha}^{x} 1/p(t)\, dt,$$

for all $x \ \varepsilon \ [\alpha, \gamma)$, where γ is the first zero of

$$\phi(\alpha) + p(\alpha)\, \phi'(\alpha) \int_{\alpha}^{x} 1\backslash p(t)\, dt \text{ on the right of } \alpha.$$

Theorem IV **If** (i) $q(x) \geq 0$ for all $x \ \varepsilon \ [\alpha, \infty)$, for some $\alpha > 0$,

then (a) all solutions of the LDE (1.1) are eventually strictly monotonic, i.e. the LDE (1.1) is non-oscillatory,

(b) no solution of the LDE (1.1) can be square-integrable on $[0. \infty)$; so the LDE

$-(p(x)\,y'(x))' + q(x)\,y(x) = \lambda\, y(x),\ x \in [0, \infty),$

λ being a complex parameter, is in the limit-point case at infinity,

(c) $\phi'(x)/\phi(x) \geq [\phi(\alpha)/\{p(\alpha)\,\phi'(\alpha)\} + \int_\alpha^x 1/p(t)\,dt\,]^{-1}/p(x),$

for all $x \in [\alpha, \infty)$, when $\phi(\alpha) > 0$ and $\phi'(\alpha) >$ or < 0.

Remark : For the results Theorem III(a) and Theorem IV(c), the help of the RDE (2.2) is essential; the other results can also be derived strainghtway from the LDE (1.1).

5. THE LDE (1.1) WITH q NONPOSITIVE ON [α, β]

Let $q(x) \leq 0$ for all $x \in [\alpha, \beta]$ and ϕ be a solution of the LDE (1.1) such that $\phi(x) \neq 0$ for all $x \in [\alpha, \beta]$. Without loss of generality we may assume that $\phi(x) > 0$ for all $x \in [\alpha, \beta]$.

As $q(x) \leq 0$ for all $x \in [\alpha, \beta]$, we have

$\xi'(x) \leq -\xi^2(x)/p(x)$ for all $x \in [\alpha, \beta]$. ... (5.1)

We are to consider separately the following three cases :

(1) $\phi'(\alpha) > 0$, (2) $\phi'(\alpha) < 0$, (3) $\phi'(\alpha) = 0$.

(1) $\phi'(\alpha) > 0$.

As $\phi'(\alpha) > 0$, ϕ is increasing in a neighbourhood of α.

If $\gamma \in (\alpha, \beta]$ is the first zero of ϕ' to the right of α, it can be proved from (5.1) that, for all $x \in [\alpha, \gamma)$,

$$\phi'(x)/\phi(x) \leq [\phi(\alpha)/p(\alpha)\,\phi'(\alpha) + \int_\alpha^x 1/p(t)\,dt]^{-1}/p(x). \qquad ... (5.2)$$

Hence, on integration, we have, for all $x \in [\alpha, \gamma)$,

$$0 < \phi(\alpha) \leq \phi(x) \leq \phi(\alpha) + p(\alpha)\,\phi'(\alpha) \int_\alpha^x 1/p(t)\,dt.$$

(2) $\phi'(\alpha) < 0$.

Here ϕ is decreasing in a neighbourhood of α.

It can be proved from (5.1) that $\phi'(x) < 0$ for all $x \in [\alpha, \beta]$.

Obviously, $\phi(x) \leq \phi(\alpha)$ for all $x \in [\alpha, \beta]$. Integrating (5.1) it can be proved that (5.2) holds on $[\alpha, \gamma)$ where $\gamma \in [\alpha, \beta]$ is such that

$$\phi(\alpha) + p(\alpha)\,\phi'(\alpha) \int_\alpha^x 1/p(t)\,dt > 0 \text{ for all } x \in [\alpha, \gamma).$$

Integrating (5.2) we then have, for all $x \in [\alpha, \gamma)$

$$0 < \phi (x) \leq \phi (\alpha) + p (\alpha) \phi' (\alpha) \int_{\alpha}^{x} 1/p (t)\, dt \leq \phi (\alpha).$$

(3) $\phi' (\alpha) = 0$.

In this case, either $\phi (x) = \phi (\alpha)$ in a neighbourhood of α, or $\phi' (\alpha_1) < 0$ for all $\alpha_1 \varepsilon (\alpha, \beta]$. Then, as in case (2), we have, for all $x \varepsilon [\alpha, \beta]$,

$$\phi (x) \leq \phi (\alpha_1) + p (\alpha_1) \phi' (\alpha_1) \int_{\alpha}^{x} 1/p (t)\, dt.$$

Hence, making $\alpha_1 \rightarrow \alpha$, we get, for all $x \varepsilon [\alpha, \beta]$,

$$\phi (x) \leq \phi (\alpha) + p (\alpha) \phi' (\alpha) \int_{\alpha_1}^{x} 1/p (t)\, dt.$$

This discussion leads to the following result :

Theorem V. **Let** (i) $q (x) \leq 0$ for all $x \leq [\alpha, \beta]$,

(ii) $\phi : [\alpha, \beta] \rightarrow R$ be a solution of the LDE (1.1),

(iii) $\phi (x) > 0$ for all $x \varepsilon [\alpha, \beta]$.

Then (a) there exists $\gamma \varepsilon (\alpha, \beta]$ such that, for all $x \varepsilon [\alpha, \gamma]$,

$$\phi' (x) / \phi (x) \leq [\phi (\alpha) / \{p (\alpha) \phi' (\alpha)\} + \int_{\alpha}^{x} 1/p (t)\, dt]^{-1}/p (x);$$

(b) [I] if $\phi' (\alpha) > 0$, for all $x \varepsilon [\alpha, \gamma]$,

$$0 < \phi (\alpha) \leq \phi (x) \leq \phi (\alpha) + p (\alpha) \phi' (\alpha) \int_{\alpha}^{x} 1/p (t)\, dt,$$

where γ is the first zero of ϕ' on the right of α;

[II] if $\phi' (\alpha) < 0$, for all $x \varepsilon [\alpha, \gamma]$

$$0 < \phi (x) \leq \phi (\alpha) + p (\alpha) \phi' (\alpha) \int_{\alpha}^{x} 1/p (t)\, dt,$$

where γ is the first zero of

$$\phi (\alpha) + p (\alpha) \phi' (\alpha) \int_{\alpha}^{x} 1/p (t)\, dt$$

on the right of α.

6. BEHAVIOUR OF ANY SOLUTION OF THE LDE (1.1) IN THE VICINITY OF A ZERO OF IT.

Let $\delta \varepsilon [\alpha, \beta]$ be a zero of a solution $\phi : [\alpha, \beta] \rightarrow R$ of the LDE (1.1). Then we note that there exists $\eta > 0$ such that

$$\phi (x) \phi' (x) > 0 \text{ for all } x \varepsilon (\delta, \delta + \eta],$$
$$< 0 \text{ for all } x \varepsilon [\delta - \eta, \delta).$$

Clearly, the investigations of the behaviour of a solution ϕ of (1.1) on the left, and on the right of a zero δ of it are to be carried out separately. Further the analysis depends heavily on whether

(1) $q(x) \geq 0$, or (2) $q(x) \leq 0$,

on $[\delta - \eta, \delta)$ or $(\delta, \delta + \eta]$, as the case may be.

As for example, let us suppose that $q(x) \geq 0$ for all $x \varepsilon [\delta - \eta, \delta)$. Using (2.4). we can deduce on integrating twice that, for all $x \varepsilon (\delta - \eta, \delta)$,

$$\text{either } \phi(x) \geq -p(\delta)\,\phi'(\delta) \int_x^{\delta} 1/p(t)\,dt \geq 0,$$

$$\text{or } \quad \phi(x) \leq -p(\delta)\,\phi'(\delta) \int_x^{\delta} 1/p(t)\,dt \leq 0.$$

Similarly for the other three cases. The results obtained are presented in the following theorem :

Theorem VI. Let (i) $\phi : [\alpha, \beta] \rightarrow R$ be a solution of the LDE (1.1),

(ii) $\delta \varepsilon (\alpha, \beta)$ be such that $\phi(\delta) = 0$,

(iii) $\eta > 0$ be such that $[\delta - \eta, \delta + \eta] \subset (\alpha, \beta)$ and
$\phi(x)\,\phi'(x) > 0$ for all $x \varepsilon (\delta, \delta + \eta]$,
$\phi(x)\,\phi' x < 0$ for all $x \varepsilon [\delta - \eta, \delta)$.

Then the following will be true :

(a) (1) If $q(x) \geq 0$ on $[\delta, \delta + \eta]$, then, for all $x \varepsilon [\delta, \delta + \eta]$

$$\text{either } \phi(x) \geq p(\delta)\,\phi'(\delta) \int_{\delta}^{x} 1/p(t)\,dt \geq 0,$$

$$\text{or } \phi(x) \leq p(\delta)\,\phi'(\delta) \int_{\delta}^{x} 1/p(t)\,dt \leq 0.$$

(2) If $q(x) \leq 0$ on $[\delta, \delta + \eta]$, then, for all $x \varepsilon [\delta, \delta + \eta]$,

$$\text{either} \qquad 0 \leq \phi(x) \leq p(\delta)\,\phi'(\delta) \int_{\delta}^{x} 1/p(t)\,dt,$$

$$\text{or} \qquad 0 \geq \phi(x) \geq p(\delta)\,\phi'(\delta) \int_{\delta}^{x} 1/p(t)\,dt.$$

(b) (1) If $q(x) \geq 0$ on $[\delta - \eta, \delta]$, then, for all $x \varepsilon [\delta - \eta, \delta]$,

$$\text{either} \qquad \phi(x) \geq -p(\delta)\,\phi'(\delta) \int_x^{\delta} 1/p(t)\,dt \geq 0,$$

$$\text{or} \qquad \phi(x) \leq -p(\delta)\,\phi'(\delta) \int_x^{\delta} 1/p(t)\,dt \leq 0.$$

(2) If $q(x) \leq o$ on $[\delta - \eta, \delta]$, then, for all $x \varepsilon [\delta - \eta, \delta]$,

either $$0 \leq \phi(x) \leq -p(\delta)\, \phi'(\delta) \int_x^{\delta} 1/p(t)\, dt,$$

or $$0 \geq \phi(x) \geq -p(\delta)\, \phi'(\delta) \int_x^{\delta} 1/p(t)\, dt.$$

If we normalize the solution ϕ by stipulating $\phi(\delta) = 0$, $\phi'(\delta) = 1$, and if $\psi : (\delta - \eta, \delta + \eta) \to R$ be defined by

$$\psi(x) = p(\delta) \int_{\delta}^{x} 1/p(t)\, dt \int_{\delta}^{x} \text{ for all } x \in (\delta - \eta, \delta + \eta),$$

then the curve $y = \psi(x)$ serves as a guiding curve to the solution-curve $y = \phi(x)$, both on the left, as well as, on the right of a zero of it.

7. THE DERIVED FUNCTION OF ANY SOLUTION OF THE LDE (1.1)

In this sction, we investigate the zeros of the derived function ϕ' of any solution ϕ of the LDE (1.1). Some of the results are deduced directly from the LDE (1.1), while some insights are achiveved if the associated RDE (2.2) is taken into consideration. The following result can be deduced directly from the LDE (1.1).

Theorem VII. If (i) ϕ is any nontrivial solution of the LDE (1.1),

(ii) α, β ($\alpha < \beta$) are two consecutive zeros of ϕ,

(iii) $q(x) \neq 0$ for all $x \in [\alpha, \beta]$,

then ϕ' vanishes exactly once in (α, β).

If we use the RDE (2.2), we may allow q to change sign in between two consecutive zeros of ϕ as revealed below :

Theorem VIII. If(i) ϕ is any nontrivial solution of the LDE (1.1),

(ii) α, β ($\alpha < \beta$) are two consecutive zeros of ϕ,

(iii) $p\, q : [\alpha, \beta] \to R$ is strictly monotonic,

then ϕ' vanishes exactly once in (α, β).

Corollary. If pq is strictly monotonic on $[\alpha, \beta]$, then any nontrivial solution ϕ of (1.1) has exactly one extremum between any two consecutive zeros of it lying in $[\alpha, \beta]$.

Remark. The stipulation "pq is strictly monotonic on $[\alpha, \beta]$" does not exclude the possibility of q changing sign. However, the sign of q does govern the behaviour of a solution of (1.1), which has been depicted in the following theorem :

Theorem IX. Let (i) $\phi : [\alpha, \beta] \to R$ be a nontrivial solution of (1.1),

(ii) $c, d \in [\alpha, \beta]$, ($c<d$) be two consecutive zeros of ϕ',

(iii) $\phi(x) > 0$ for all $x \in [c, d]$.

Then (a) there will be odd number of zeros of q in (c, d);

(b) if $q(c) = 0 = \phi'(c)$, then

$\phi''(c) = 0$ and $q'(c)\,\phi'''(c) < 0$;

(c) there exists $\varepsilon > 0$ such that

$q(x)\,\phi''(x) > 0$ for all $x \,\varepsilon\, (c, c+\varepsilon) \cup (d-\varepsilon, d)$.

Proof. (a) From the LDE (1.1), we have

$$p(c)\,p(d)\,\phi''(c)\,\phi''(d) = q(c)\,q(d)\,\phi(c)\,\phi(d).$$

c, d being two consecutive zeros of ϕ', we have $\phi''(c)\,\phi''(d) < 0$. Hence $q(c)\,q(d) < 0$. Consequently q vanishes odd number of times in (c,d).

(b) If $q(c) = 0 = \phi'(c)$, then from the LDE (1.1) we have

$\phi''(c) = 0$ and $q'(c)\,\phi'''(c) < 0$;

(c) Let u be the first zero of q on the right of c. The cases

(1) $\phi''(c) > 0$, (2) $\phi''(c) < 0$

are to be considered separately.

(1) As $\phi''(c) > 0$, there exists $\gamma \,\varepsilon\, (c, u)$ such that $\phi''(x) > 0$ for all $x \,\varepsilon\, (c, \gamma)$. Then, for all $x \,\varepsilon\, (c, \gamma]$, $\phi'(x) > 0$.

Now, if $q(x) < 0$ on $[c, \gamma]$, we have from (2.4)

$\xi'(x) < -\xi^2(x) / p(x)$.

Then, for $c < t < x < \gamma$,

$$\frac{1}{\xi(x)} - \frac{1}{\xi(t)} > \int_t^x \frac{du}{p(u)}.$$

Making $t \to c + 0$ in the above inequality we arrive at a contradiction since $\xi(t) \to +\infty$ as $t \to c + o$. Hence $q(x) \not< 0$. on $[c, \gamma]$.

The other cases can be dealt with similarly.

8. THE ASSOCIATED EIGENVALUE PROBLEM

In this section, we consider the eigenvalue problem (EVP π) consisting of the LDE

$$-(p(x)\,y'(x))' + q(x)\,y(x) = \lambda\, y(x), \quad x \,\varepsilon\, [a, b], \qquad \ldots (8.1)$$

where (i) $p, q : [a, b] \to R$ are continuous,

(ii) $p(x) > 0$ for all $x \,\varepsilon\, [a, b]$,

(iii) λ is a parameter,

associated with the following boundary conditions :

$$a_1 y(a) + a_2 y'(a) = 0, \qquad \dots (8.2)$$

$$b_1 y(b) + b_2 y'(b) = 0, \qquad \dots (8.3)$$

where a_1, a_2, b_1, b_2 are prescribed real numbers.

It is wellknown that there exists an increasing sequence $\{\lambda_n\}_n$ of real numbers, such that (8.1), with $\lambda = \lambda_n$, has a nontrivial solution ψ_n, that satisfies (8.2)-(8.3). Each such λ_n is called an eigenvalue and ψ_n, an eigenfunction corresponding to the eigenvalue λ_n, of the EVP π.

Usually investigations are carried out to determine the orders of λ_n and ψ_n as $n \to \infty$. While we know that the eigenvalues $\{\lambda_n\}_n$ obey the inequality $\lambda_n < \lambda_{n+1}$ for all positive integers n, surprisingly no such comparison result is known for the eigenfunctions $\{\psi_n\}_n$. The following theorem reveals an attempt in this direction. It is to be noticed that the role of the corresponding RDE

$$z'(x) = q(x) - \lambda - z^2(x) / p(x), \quad x \in [\alpha, \beta], \qquad \dots (8.4)$$

is crucial in deriving this result.

Theorem X. Let (i) ψ_n be the nth eigenfunction of the EVP π,

(ii) $a' \in (a, b]$ be such that none of ψ_n, ψ'_n, ψ_{n+1}, ψ'_{n+1}, vanishes on $(a, a']$.

Then

(a) $\psi'_n(x) / \psi_n(x) > \psi'_{n+1}(x) / \psi_{n+1}(x)$, for all $x \in (a, a']$; (8.5)

(b) $|\psi_n(x) / \psi_n(\alpha)| \geq |\psi_{n+1}(x) / \psi_{n+1}(\alpha)|$ (8.6)

for all $x \in [\alpha, a']$, for any $\alpha \in (a, a']$.

Proof. Let $\xi_n(x) = p(x)\, \psi'_n(x) / \psi_n(x)$, $n \in N$, $x \in (a, a')$. (8.7)

We consider the two cases (I) $a_2 \neq 0$ and (II) $a_2 = 0$ separetely.

(I) $a_2 \neq 0$.

As both ψ_n and ψ_{n+1} satisfy (8.2), we have $\xi_n(a) = \xi_{n+1}(a)$. (8.8)

Then the fact that $\lambda_n < \lambda_{n+1}$ implies, for all $x \in [a, a']$,

$$\xi_n(x) > \xi_{n+1}(x).$$

Hence, $\psi'_n(x) / \psi_n(x) > \psi'_{n+1}(x) / \psi_{n+1}(x)$, for all $x \in [a, a']$. (8.9)

(II) $a_2 = 0$. We write $\eta_n(x) = \dfrac{\psi_n(x)}{p(x)\psi'_n(x)}$, $(n \in N)$

Then $\eta_n(a) = 0 = \eta_{n+1}(a)$.

and $\eta^1_n(x) = 1/p(x) - (q(x) - \lambda_n)\, \eta^2_n(x)$.

As before it follows that

$$\eta_n(x) \leq \eta_{n+1}(x) \text{ for all } x \in [a, a^1].$$

Hence (8.9) follows.

Thus (a) is proved whether or not $a_2 = 0$.

Then (b) follows on integration of (a) on $[\alpha, x]$ for any $\alpha \in (a, a^1]$ and any $x \in [c, a^1]$.

9. A COMPARISON RESULT

In the Sturm comparison theorem, attention is paid to the number of zeros of the solution of two linear second-order ordinary differential equations. In the following we derive a comparison result between the said solutions, valid at each point of a certain interval.

Theorem XI. If (i) $L_i[y] = -(p_i y')' + q_i y$, $i = 1, 2$,

(ii) $p_i, q_i : I \to R$: continuous, $i = 1,2$, $I \subset R$,

(iii) $p_i(x) > 0$, for all $x \in I$,

(iv) $\phi_1 : I \to R$: nontrivial solution of $L_1[y] = o$,

(v) $a \in : I$ is such that $\phi_1(a) \neq 0$,

(vi) $\phi_2 : I \to R$: a nontrivial solution of $L_2[y] = 0$, such that $p_1(a)\, \phi'_1(\alpha) / \phi_1(a) = p_2(a)\, \phi'_2(a) / \phi_2(a)$,

(vii) $b \in I$ is such that $\phi_i(x) \neq 0$ for all $x \in [a, b] \subset I$, $(i = 1, 2)$,

(viii) $p_1(x) \leq p_2(x)$, $q_1(x) \leq q_2(x)$ for all $x \in [a, b]$,

then

(a) $p_1(x)\, \phi'_1(x) / \phi_1(x) \leq p_2(x)\, \phi'_2(x)/\phi_2(x)$ for all $x \in [a, b]$

(b) $|\phi_1(x) / \phi_1(a)| \leq |\phi_2(x) / \phi_2(a)|$ for all $x \in [a, b]$, provided $p_1(x) = p_2(x)$ for all $x \in [a, b]$.

The proof follows using the same technique as in Theorem X.

10. EXAMPLES

Example 1. Taking $p(x) = 1$, $q(x) = -1$, $\phi(x) = \sin x$, $\delta = 0$ in Theorem VI (a) (2), we have $0 \leq \sin x \leq x$ for all $x \in [0, \pi/2]$.

Taking $\phi(x) = \cos x$, $\delta = \pi/2$ there, we get

(i) $0 \geq \cos x \geq \pi/2 - x$ for all $x \in [\pi/2, \pi]$

and (ii) $0 \leq \cos x \leq \pi/2 - x$ for all $x \in [0, \pi/2]$.

Example 2. Taking p (x) = 1, q (x) = 0, $y'(0) = 0 = y'(\pi)$ in the EVP (π), we get

$|\cos nx| > |\cos(n+1)x|$, ($n \varepsilon N$) for all $x \varepsilon [0, a')$, where $a' > 0$ is such that none of cos nx, sin nx, cos (n + 1) x, sin (n + 1) x vanishes on (0, a').

Similarly we can prove that $|\sin nx| \geq |\sin(n+1)x|$, ($n \varepsilon N$).

11. CONCLUSION

The investigations elaborated above do indicate that sometimes the study of nonlinear differential equations proves to be quite fruitful in respect to the study of linear equations, although the customary procedure is exactly in the opposite direction. There are scopes of extending the idea to higher order differential equations and to some qualitative analysis not touched upon in the present endeavour.

REFERENCES

1. M.S.P. Eastham Theory of Ordinary Differential Equations, Van Nostrand Reinhold, London, 1970.
2. E.L. Ince Ordinary Differential Equations, Dover Publications, New York, 1956.

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SEMI-SYMMETRIC NOŃ-METRIC CONNECTION IN AN SP-SASAKIAN MANIFOLD

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Abstract : In the present paper we have studied some properties of the curvature tensor and the Weyl conformal curvature tensor of a semi-symmetric non-metric connection in an SP-Sasakian manifold.

1. INTRODUCTION

Let M^n be an n-demensional C^∞-manifold. If there exists in M^n a tensor field ϕ of type (1,1), a vector field ξ and a l-form η satisfying.

(1.1) $\bar{\bar{X}} = X - \eta(X)\xi$, $\bar{X}$ $\underline{\underline{\mathrm{def}}}\, \phi(X)$, $\eta\xi = 1$,

then M^n is called an almost paracontact manifold. Let g be the Riemannian metric on M satisfying

(1.2) $\eta(X) = g(X, \xi)$,

(1.3) $\eta(\phi X) = o$, $\phi(\xi) = o$, rank $(\phi) = n-1$,

(1.4) $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y)$

The set (ϕ, ξ, η, g) satisfying (1.1), (1.2), (1.3), and (1.4) is called an almost paracontact Riemannian structure and the manifold with such a structure is called an almost paracontact Riemannian manifold [1].

If we define $'\phi(X, Y) = g(\phi(X), Y)$, then in addition to the above relations the followings are satisfied,

(1.5) $'\phi(X,Y) = {}'\phi(Y, X)$ and $'\phi(\bar{X}, \bar{Y}) = {}'\phi(X, Y)$

Now, let M^n be an n-dimensional Riemannian manifold with a positive definite metric g admitting a 1-form η which satisfies the conditions

(1.6) $(\nabla_x\eta)(Y) - (\nabla_y\eta)(X) = o$

and,

(1.7) $(\nabla_x\nabla_y\eta)(Z) = -g(X,Z)\,\eta(Y) - g(X,Y)\,\eta(Z) + 2\eta(X)\,\eta(Y)\,\eta(Z)$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. If, moreoever, we put

(1.8) $g(X,\xi) = \eta(X),\ \nabla_X\xi = \phi X,$

then it can be easily verified that the manifold in consideration becomes an almost paracontact Riemannian manifold. Such a manifold is called a para-Sasakian manifold or briefly a P-Sasakian manifold [2].

Let us consider an n-dimensional differentiable manifold M^n with a Riemannian metric g and a 1-from η satisfying

(1.9) $(\nabla_x\eta)(Y) = -g(X,Y) + \eta(X)\,\eta(Y).$

Putting $\eta(X) = g(X,\xi)$ and $(\nabla_x\eta)(Y) = {}'\phi(X,Y)$, it is easily verified that the manifold in consideration is a P-Sasakian manifold. Such a manifold is called an SP-Sasakian manifold [2].

So, in an SP-Sasakian manifold, we get

(1.10) ${}'\phi(X,Y) = -g(X.Y) + \eta(X)\,\eta(Y)$

A semi-symmetric non-metric connection $\overset{*}{\nabla}$ in an almost paracontact metric manifold can be defined by [3]

(1.11) $\overset{*}{\nabla}_x Y = \nabla_x Y + \eta(Y)X$

where ∇ is a Riemannian connection with respect to the metric g.

2. CURVATURE TENSOR

Let $\overset{*}{R}$ and R be the curvature tensors of the connections $\overset{*}{\nabla}$ and ∇ respectively in an SP-Sasakian manifold. Then

(2.1) $\overset{*}{R}(X,Y)Z = \overset{*}{\nabla}_x\overset{*}{\nabla}_y Z - \overset{*}{\nabla}_y\overset{*}{\nabla}_x Z - \overset{*}{\nabla}_{[x,y]}Z$

and

(2.2) $R(X,Y)Z = \nabla_x\nabla_y Z - \nabla_y\nabla_x Z - \nabla_{[x,y]}Z$

Using (1.11) and (2.2) in (2.1) we get

(2.3) $\overset{*}{R}(X,Y)Z = R(X,Y)Z + l(X,Z)Y - l(Y,Z)X$

where l is a tensor of type (0,2) defined by

(2.4) $l(X,Y) = (\nabla_x\eta)(Y) - \eta(X)\,\eta(Y) = (\overset{*}{\nabla}_x\eta)(Y)$

From (2.3) we get

(2.5) $\overset{*}{S}(Y,Z) = S(Y,Z) - (n-1)\, l(Y,Z)$

where $\overset{*}{S}$ and S denote the Ricci tensors of $\overset{*}{\nabla}$ and ∇ respectively.

Again, from (2.5) we get

(2.6) $\overset{*}{r} = r-(n-1)\, a$

where a is the trace of l.

Let us consider M^n to be SP-Sasakian. Then from (1.9) and (1.10) we get

(2.7) $(\nabla_x \eta)(Y) = \phi(X,Y) = -g(X,Y) + \eta(X)\eta(Y)$

which gives

(2.8) $(\nabla_x \eta)(\xi) = o$

and

(2.9) $\nabla_x \xi = \phi(X)$

From (2.4) and (2.7) we have

(2.10) $l(X,Y) = -g(X,Y)$

Equation (2.3) can now be written as

(2.11) $'\overset{*}{R}(X,Y,Z,W) = {'R}(X,Y,Z,W) + l(X,Z)\, g(Y,W) - l(Y,Z)\, g(X,W)$

where $'\overset{*}{R}(X,Y,Z,W) = g(\overset{*}{R}(X,Y)Z,W)$ and $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

Using (2.10) in (2.11) we get

(2.12) $'\overset{*}{R}(X,Y,Z,W) = {'R}(X,Y,Z,W) - g(X,Z)\, g(Y,W) + g(Y,Z)\, g(X,W)$

Theorem : 2.1. In an SP-Sasakian manifold with semi-symmetric non-metric connection $\overset{*}{\nabla}$ we have

(a) $\overset{*}{R}(X,Y)Z + \overset{*}{R}(Y,Z)X + \overset{*}{R}(Z,X)Y = o$

(b) $'\overset{*}{R}(X,Y,Z,W) + {'\overset{*}{R}}(X,Y,W,Z) = o$

(c) $'\overset{*}{R}(X,Y,Z,W) - {'\overset{*}{R}}(Z,W,X,Y) = o$

Proof- Using (2.12) and the first Bianchi identity with respect to the Riemanian connection ∇, we have

$$'\overset{*}{R}(X,Y,Z,W) + '\overset{*}{R}(Y,Z,X,W) + '\overset{*}{R}(Z,X,Y,W) = 0$$

and hence

$$(2.13)\quad \overset{*}{R}(X,Y)Z + \overset{*}{R}(Y,Z)X + \overset{*}{R}(Z,X)Y = 0$$

From (2.12), we have

$$(2.14)\quad '\overset{*}{R}(X,Y,Z,W) + '\overset{*}{R}(X,Y,W,Z) = 0$$

and

$$(2.15)\quad '\overset{*}{R}(X,Y,Z,W) - '\overset{*}{R}(Z,W,X,Y) = 0$$

This completes the proof.

Remarks- The above properties of the curvature tensor of $\overset{*}{\nabla}$ does not hold, in general, in a Riemannian manifold admitting a semisymmetric non-metric connection.

Theorem 2.2 In an SP-Sasakian manifold the Ricci tensor of a semi-symmetric non-metric connection $\overset{*}{\nabla}$ is symmetric.

Proof. From (2.5) and (2.10) we get

$$(2.16)\quad \overset{*}{S}(Y,Z) = S(Y,Z) + (n-1)\, g(Y,Z)$$

From (2.16) we have

$$(2.17)\quad \overset{*}{S}(Y,Z) = \overset{*}{S}(Z,Y)$$

Hence the Ricci tensor $\overset{*}{S}$ with respect to the semi-symmetric non-metric connection $\overset{*}{\nabla}$ is symmetric.

This completes the proof of theorem.

Remarks : The Ricci tensor of $\overset{*}{\nabla}$ in a Riemannian manifold admitting a semi-symmetric non-metric connection is not, in general, symmetric.

Theorem 2.3. If an SP-Sasakian manifold admits a semi-symmetric non-metric connection whose curvature tensor $'\overset{*}{R}$ vanishes, then the manifold is of constant curvature - 1.

Proof. Putting $'\overset{*}{R} = 0$ in (2.12) we get

(2.18) $'R(X,Y,Z,W) = (-1)\,[g(Y,Z)\,g(X,W)-g(X,Z)\,g(Y,W)]$

The equation (2.18) shows that the manifold is of constant curvature- 1.

Corollary 2.1. If an SP-Sasakian manifold admits a semi-symmetric non-metric connection whose curvature tensor $'\overset{*}{R}$ vanishes then the manifold is an Einstein manifold.

Theorem 2.4. In an SP-Sasakian manifold the conformal curvature tensor of a semi-symmetric non-metric connection $\overset{*}{\nabla}$ is equal to the conformal curvature tensor of the manifold.

Proof. Let $\overset{*}{C}$ and C denote the conformal curvature tensors of $\overset{*}{\nabla}$ and ∇ respectively. Then we have

$$(2.19)\quad '\overset{*}{C}(X,Y,Z,W) = '\overset{*}{R}(X,Y,Z,W) - \frac{1}{n-2}\,[\overset{*}{S}(Y,Z)\,g(X,W) - \overset{*}{S}(X,Z)\,g(Y,W) + \overset{*}{S}(X,W)\,g(Y,Z)$$

$$- \overset{*}{S}(Y,W)\,g(X,Z)] + \frac{\overset{*}{r}}{(n-1)(n-2)}\,[g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W)]$$

From (2.5) and (2.10) we get

$$(2.20)\quad \overset{*}{S}(X,Y) = S(X,Y) + (n-1)\,g(X,Y)$$

Also from (2.6) and (2.10) we get

$$(2.21)\quad \overset{*}{r} = r+n(n-1)$$

Using (2.12), (2.20) and (2.21) we get

$$(2,22)\quad '\overset{*}{C}(X,Y,Z,W) = 'C(X,Y,Z,W)$$

This proves the theorem.

Theorem 2.5. If an SP-Sasakian manifold admits a semisymmetric non-metric connection $\overset{*}{\nabla}$ whose Ricci tensor vanishes, then the curvature tensor of $\overset{*}{\nabla}$ is equal to the conformal curvature tensor of the manifold.

Proof. Since $\overset{*}{S} = o$, (2.19) gives

$$(2,23)\quad '\overset{*}{C}(X,Y,Z,W) = '\overset{*}{R}(X,Y,Z,W)$$

From (2,22) and (2,23) we get

(2,24) $'\overset{*}{R}(X,Y,Z,W) = {}'C(X,Y,Z,W)$

This completes the proof of the theorem.

If in particular $'\overset{*}{R} = o$, then from (2,24) we have

(2.25) $'C(X,Y,Z,W) = o$ and hence we have the following

Corollary 2.2. If an SP-Sasakian manifold admits a semi-symmetric non-metric connection $\overset{*}{\nabla}$ whose curvature tensor vanishes, then the manifold is conformally flat.

From Theorem (2 5) we have

Theorem 2.7. If an SP-Sasakian manifold with vanishing Ricci tenor with respect to the semi-symmetric non-metric connection $\overset{*}{\nabla}$ is conformally flat, then the curvature tensor with respect to the semi-symmetric non-metric connection $\overset{*}{\nabla}$ vanishes.

Hence from Theorem (2.6) and (2.7) we have

Theorem 2.8. If an SP-Sasakian manifold with vanishing Ricci tensor with respect to the semi-symmetric non-metric connection $\overset{*}{\nabla}$ exists, then the manifold is conformally flat if the curvature tensor with respect to the semi-symmetric non-metric connection $\overset{*}{\nabla}$ vanishes.

REFERENCES

1. I. SATO, On a structure similar to almost contact structure I, Tensor N.S. 30 (1976, 219-224.
2. T. ADATI and T. MIYAZAWA, Some properties of P-Sasakian manifold. TRU Maths. 13(1) (1977), 33-42.
3. N. S. AGASHE and M.R. CHAFLE A semi-symmetric non-metric connection on a Riemannian manifold, Indian J. Pure appl. Math. 23(6) (1992) 399-409.

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ON GROUP-PSEUDO-IDEALS OF SEMIGROUPS

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Abstract : The aim of this paper is to characterise those congruences δ on a semigroup S for which S/δ is a group of exponent two. For this purpose we define group-pseudo-ideal in a semigroup. We show that the congruences of the above type on a semigroup can be completely characterised by the group-pseudo-ideals.

1. INTRODUCTION

A nonempty subset I of a semigroup S is called a left pseudo-ideal if i) $ab \in I$ for all a, $b \in I$ and ii) $x^2 a \in I$ for all $x \in S$ and $a \in I$. The right pseudo-ideal and the two sided pseudo-ideal or simply a pseudo-ideal are defined analogously [6].

In [2] - [6] we studied different properties of pseudo-ideals in semigroups and groups.

In this paper we define group-pseudo-ideal in semigroup and study some properties of it. We obtain that if I is a group-pseudo-ideal of a semigroup S then there exists a congruence δ_I on S such that S/δ_I is a group of exponent two and conversely if δ is a congruence on a semigroup S such that S/δ is a group of exponent two then there exists a group-peseudo-ideal I in S such that $\delta = \delta_I$. Lastly we obtain a necessary and sufficient condition for the existence of a group-pseudo-ideal in a commutative semigroup S.

2. GROUP-PSEUDO-IDEAL

Definition 2.1. A pseudo-ideal I (two sided) of a semigroup S is called a group-pseudo-ideal if $x^2 a \in I$ for x, $a \in S$ implies that $a \in I$.

Example 2.2. Let $S = \{x : x \in J \text{ or } x = a\sqrt{2} \text{ where } a \in J$, the ring of all integers$\}$. Then S is a noncommutative semigroup with respect to the multiplication defined by $xy = |x|y$ where x, $y \in S$. Let $I = \{x : x \in J\}$. Then I is a group-pseudo-ideal of S.

We are familiar with the Rees-congruence that corresponds very closely to an ideal of a semigroup.

If I is an ideal of a semigroup S then $\delta_I = (I \times I) \cup 1_S$ is a congruence on S which is the Rees-congruence.

From the above example it follows that corresponding to a pseudo-ideal which is not an ideal of a semigroup there does not exist Rees-congruence since $(2,3) \in I$ and so $2\ \delta_I\ 3$ but $2\sqrt{2}\ \delta_I\ 3\sqrt{2}$ does not hold.

Again every pseudo-ideal in a group is a normal subgroup [6] and so there exists a congruence corresponding to a pseudo-ideal in a group. The group-pseudo-ideal in a semigroup possesses some properties which the pseudo-ideal in a group possesses. So we want to investigate whether there exists a congruence corresponding to a group-pseudo-ideal of a semigroup.

Let H be a nonempty subset of a semigroup S. Then we have a congruence $\wp_H = \{(a,b) \in S \times S \mid H \,..\, a = H \,..\, b\}$ where $H \,..\, a = \{(x,y) \in S \times S \mid xay \in H\}$ on S[1]. We shall now show that the congruence $\wp_H$ on a semigroup S where H is a group-pseudo-ideal of S is a congruence for which $S/\wp_H$ is a group of exponent two and any congruence δ on S for which S/δ is a group of exponent two is of the form $\wp_H$ for some group-pseudo-ideal H of S.

To show this we require the following lemmas.

Lemma 2.3. If I is a group-pseudo-ideal of a semigroup S then $xIx \subseteq I$ for all $x \in S$.

Proof. Let $a \in I$ and $x \in S$. Then $a(ax)^2 \in I$ i.e. $a^2(xax) \in I$ which implies that $xax \in I$ Hence $xIx \subseteq I$.

Lemma 2.4. If I is a group-pseudo-ideal of a semigroup S then $ax^2 \in I$ for $a, x \in S$ implies that $a \in I$.

Proof. From lemma 2.3, it follows that $ax^2 \in I$ implies that $a(ax^2)a \in I$ i.e. $a^2x^2a \in I$ which implies that $a \in I$.

Lemma 2.5. If I is a group-pseudo-ideal of a semigroup S then $\bar{S} \subseteq I$ where $\bar{S} = \{x^2 \mid x \in S\}$.

Proof. Let $x \in S$. Then for any $a \in I$, $a^2x^2 \in I$ which implies that $x^2 \in I$. Hence $\bar{S} \subseteq I$.

Lemma. 2.6. If I is a group-pseudo-ideal of a semigroup S then $ab \in I$ if and only if $ba \in I$ for $a, b \in S$.

Proof. Let $ab \in I$. Then by lemma 2.3, it follows that $a(ab)a \in I$ i.e. $a^2(ba) \in I$ which implies that $ba \in I$.

Lemma 2.7. Let I be a group-pseudo-ideal of a semigroup S; then $ab \in I$ if and only if $axbx \in I$ for all $x \in S$ where a, $b \in S$.

Proof. Suppose $ab \in I$ and $x \in S$. Then by lemmas 2.5 and 2.3 it follows that $bxaxab = b(xa)^2b \in I$. Hence $bxax(ab)^2 \in I$. This implies that $bxax \in I$ i.e. $axbx \in I$.

Conversely if $axbx \in I$ for all $x \in S$ then in particular $a^2ba \in I$ which implies that $ba \in I$ i.e. $ab \in I$.

Definition 2.8. Let I be a group-pseudo-ideal of a semigroup S. We define a relation δ_I on S as follows: $a\ \delta_I\ b$ if and only if $ab \in I$.

Lemma 2.9. Let I be a group-pseudo-ideal of a semigroup S. Then $\delta_I = \wp_I$.

Proof. Let $a\ \delta_I b$. Then $ab \in I$. Let $(x,y) \in I$.: a. Then $xay \in I$ which implies that $yxa \in I$. So $yxa^2b \in I$ i.e., $byxa^2 \in I$. This implies that $byx \in I$ i.e. $xby \in I$. Hence $(x,y) \in I$ .. b. So I .. $a \subseteq I$ .. b. Similarly I .. $b \subseteq I$ .. a. So I .. a = I .. b i.e. $a \wp_I b$. Again if $a \wp_I b$ then I . a = I .. b. From lemma 2.5 it follows that $a^2b^2 \in I$ i.e. $a(a)b^2 \in I$. So $(a,b^2) \in I$ .. a = I .. b i.e. $abb^2 \in I$ which implies that $ab \in I$. So $a\ \delta_I b$. Hence $\delta_I = \wp_I$.

We now recall the following definitions.

Definition 2.10. ([1]). Let S be a semigroup and U a subset of S. U is said to be left (right) unitary (in S) if $u \in U$ and $ux \in U$ [$xu \in U$] imply that $x \in U$ when $x \in S$. A subset U of S which is both left and right unitary is said to be unitary.

Definition 2.11. ([1]). Let H be a subset of a semigroup S. Then the biresidue W of H is the set W = {$a \in S$ | H .. a = □}.

Definition 2.12. ([1]) Let H be a subset of a semigroup S. The set H is said to be bistrong (in S) if, for any $a, b \in S$, (H .. a) $\cap$ (H .. b) $\neq$ □ implies that H .. a = H .. b.

Lemma 2.13. (i) Let I be a pseudo-ideal of a semigroup S. If I is left unitary or right unitary then I is a group-pseudo-ideal of S.

Conversely every group-pseudo-ideal of S is unitary.

(ii) If I is a group-pseudo-ideal of a semigroup S then I is bistrong and the biresidue of I is empty.

Proof. Let I be a pseudo-ideal of S and also I be left unitary. Let $x^2a \in I$ for $x, a \in S$. Let $a_1 \in I$. Then $a_1x^2 \in I$. Since I is left unitary it follows that $x^2 \in I$. Again $x^2a \in I$. So $a \in I$. Hence I is a group-pseudo-ideal of S.

Conversely let I be a group-pseudo-ideal of S and $a \in I$ and $xa \in I$. Then $xa^2 \in I$ which implies that $x \in I$. So I is right unitary. Similarly we can show that I is left unitary. Hence I is unitary.

(ii) Let (I .. a) $\cap$ (I .. b) $\neq$ □ and (p,q) $\in$ (I .. a) $\cap$ (I .. b). Then paq, $pbq \in I$.

Now $(x,y) \in I$ .. a $\Leftrightarrow$ $xay \in I$ $\Leftrightarrow$ $yxa \in I$ (lemma 2.6) $\Leftrightarrow$ $yxa^2qp \in I$ (since $paq \in I \Rightarrow aqp \in I$) $\Leftrightarrow$ $qpyxa^2 \in I$ $\Leftrightarrow$ $qpyx \in I$ $\Leftrightarrow$ $qpyxbqp \in I$ (since $pbq \in I$) $\Leftrightarrow$ $yxb(qp)^2 \in I$ $\Leftrightarrow$ $yxb \in I$ $\Leftrightarrow$ $xby \in I$ $\Leftrightarrow$ $(x,y) \in I$. b. It is to be noted here that to obtain $\Leftarrow$ we use the unitarity of I. Thus I .. a = I .. b. Hence I is bistrong.

Lastly, for any a, b in S, $a.a.b^2 = a^2b^2 \in I$, so $(a, b^2) \in I$..a i,e I .. a $\neq$ □. Hence biresidue of I is empty.

Theorem 2.14. Let I be a group-pseudo-ideal of a semigroup S. Then there exists a congruence δ_I on S such that S/δ_I is a group of exponent two.

Proof. The Proof of the existence of a congruence δ_I in S for which S/δ_I is a group follows from lemma 2.9 and lemma 2.13 of this paper and the theorem 10.34 of [1]. Also it follows from the same theorem that I is a congruence class of δ_I and for $a \in I$, $a\delta_I = I$ is the identity element of S/δ_I. Now for every $r\delta_I \in S/\delta_I$, $(r\delta_I)^2 = r^2\delta_I = a\delta_I$ since $r^2 \in I$. Hence S/δ_I is of exponent two.

Theorem 2.15. Let δ be a congruence on a semigroup S such that S/δ is a group of exponent two. Then there exists a group-pseudo-ideal I in S such that $\delta = \delta_I$.

Proof. Suppose $a\delta$ is the identity element of S/δ. Let $I = \{x \in S | x\delta = a\delta\}$. Then I is a group-pseudo-ideal of S. We shall now show that $\delta = \delta_I$. Let $c\,\delta\,d$ where $c, d \in S$. Then $cd\,\delta\,d^2$; but $d^2\,\delta\,a$ where $a \in I$. So $cd\,\delta\,a$. Hence $cd \in I$ i.e. $c\,\delta_I\,d$. Conversely if $c\,\delta_I\,d$ then $cd \in I$. So $(cd)\,\delta = a\,\delta$ i.e. $cd\,\delta\,a$. This implies that $cd^2\,\delta\,ad$ i.e. $(c\delta)(d\delta)^2 = (a\delta)(d\delta)$ which implies that $c\delta = d\delta$ i.e. $c\,\delta\,d$. Hence $\delta = \delta_I$.

Theorem 2.16. Let S be a semigroup. There exists a one-to-one correspondence between the set of all group-pseudo-ideals of S and the set of all congruences δ on S for which S/δ is a group of exponent two.

Theorem 2.17. Let G be a nontrivial group of exponent two. If G is a homomorphic image of a semigroup S then S has a proper group-pseudo-ideal.

Proof. Let f be the homomorphism of S onto G. Let $I = \{a \in S | f(a) = e$, the identity element of G$\}$. Then I is a proper group-pseudo-ideal of S.

Combining Theorem 2.14 and Theorem 2.17 we get the following theorem.

Theorem 2.18. A semigroup S has a proper group-pseudo-ideal if and only if there exists an epimorphism from S onto a nontrivial group of exponent two.

In [7] Sen defined generalised left semi (g. l. s) ideal of a semigroup as follows.

A nonempty subset I of a semigroup S is called a g. l. s (g. r. s) ideal if $x^2 I \subseteq I$ (resp. $Ix^2 \subseteq I$) for all $x \in S$. I is called a g.s. ideal if it is a g. l. s.. ideal as well as a g r. s. ideal of S.

Theorem 2.19. A pseudo-ideal I of a semigroup S is a group-pseudo-ideal if and only if $S \backslash I$ is a g. s. ideal of S.

Theorem 2.20. A semigroup S is a group if and only if every one sided pseudo ideal of S is a two sided pseudo-ideal and every pseudo-ideal is a group-pseudo-ideal.

Remark. We can show by examples that both the conditions stated in Theorem 2.20 are necessary.

Before proving the next theorem we consider the following example.

Example 2.21. Consider the residue class ring J/(3) modulo 3. J/(3) is a multiplicative commutative semigroup. $\overline{J/_{(3)}} = \{x^2 | x \in J/(3)\} = \{(0),(1)\}$ where (a) denotes the residue class containing the integer a. {0} and $\overline{J/_{(3)}}$ are the only two proper pseudo-ideals of J/(3) and none of them is a group-pseudo-ideal.

Theorem 2.22. (Existence). Let S be a commutative semigroup. S contains a proper group-pseudo-ideal if and only if there exists an element $a \in S$ such that $\overline{S}a \cap \overline{S} = \square$

Proof. Suppose S has a proper group-pseudo-ideal I. Then by lemma 2.5, $\overline{S} \subseteq I$. Since I is proper, $S \setminus I \neq \square$ Let $a \in S/I$. Then $\overline{S}a \subseteq S \setminus I$ (lemma 2.19). Hence $\overline{S}a \cap \overline{S} = \square$

Conversely let there exist an element a in S such that $\overline{S}a \cap \overline{S} = \square$. Since S is commutative, $\overline{S}$ is a pseudo-ideal of S. Also $\overline{S}$ is proper. Let $\mathfrak{I} = \{I | I$ is a proper pseudo-ideal of S with $\overline{S} \subseteq I$ and $\overline{S}a \cap I = \square\}$. $\mathfrak{I} \neq \square$ since $\overline{S} \in \mathfrak{I}$. Applying Zorn's lemma we can find a maximal element P in $\mathfrak{I}$. We shall show that P is a proper group-pseudo-ideal of S. If possible, let $x^2b \in P$ but $b \notin P$. Set $A = P \cup Pb \cup \{b\}$. A is a pseudo-ideal of S with $\overline{S} \subseteq A$ and $P \subsetneq A$. So $\overline{S}a \cap A \neq \square$. Let $c \in \overline{S}a \cap A$. Then $x^2c \in \overline{S}a$ and also $x^2c \in x^2(P \cup Pb \cup \{b\}) \subseteq P$. So $\overline{S}a \cap P \neq \square$, a contradiction. Thus S contains a proper group-pseudo-ideal P.

REFERENCES

1. A. H. Clifford and G. B. Preston, The algebraic theory of semigroups. Vol (2) (1967) American Math. Society, Providence Rhode Island, U.S.A.
2. T. K. Dutta, Pseudo-ideals in some finite groups and in finitely generated abelian groups, Bull. Cal. Math. Soc. 75(1983), 331.
3. T. K. Dutta, On pseudo-ideal series of groups, Bull. Cal. Math. Soc. 75(1983), 191.
4. T. K. Dutta, On normal pseudo-ideals in semigroups, Jour. Pure Maths. 5(1985-86),
5. M. K. Sen and T. K. Dutta, On pseudo-ideals of groups, 'AL-I-CUZA'. Tomul XXV. (1979), 269.
6. M. K. Sen, On pseudo-ideals of semigroups, Bull. Cal. Math. Soc. 67(1975), 109.
7. M. K. Sen, On the generalised semi ideals of a semigroup, Glasnik matematicki 10 (30) (1975),

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A B*-ALGEBRA AND A BANACH LATTICE OF A CLASS OF ENTIRE FUNCTIONS IN SEVERAL COMPLEX VARIABLES

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ABSTRACT. We consider a family Ω of functions $f(z) = \sum_{m=0}^{\infty} a_m z^m$ in n complex variables for which the sequence $\{m!\,|a_m|\}$ is bounded. We show that the members of Ω are entire functions in C^n. Introducing usual addition, a special multiplication called star-multiplication and a norm function in Ω we show that Ω is a commutative B*-algebra with identity. We then restrict ourselves to the subfamily F of Ω for which the coefficients a_m are real. Defining a partial order relation in F we show it to be an Archimedian function ring and also a Banach lattice. We also study a few properties of two important sublattices of F.

1. NOTATIONS

We denote complex and real n-space by C^n and R^n respectively while I^n will be the Cartesian product of n copies of I, the set of non-negative integers. We indicate the points $(z_1,\ldots,z_n)$, $(m_1,\ldots,m_n)$ etc. of C^n or R^n by their corresponding unsuffixed symbols z,m etc. For $z \in C^n$, $|z| = \{|z_1|^2+\ldots+|z_n|^2\}^{1/2}$. The positive hyperoctant R_+^n in R^n will be $R_+^n = \{x : x \in R^n, x_i \geq 0, i = 1,\ldots,n\}$. For $t \in R_+^n$ we set $||t|| = t_1+\ldots+t_n$ and for $m \in I^n$, $m! = m_1!\ldots m_n!$.

For $z \in C^n$, $m \in I^n$, $z^m = z_1^{m_1}\ldots z_n^{m_n}$ ($z_i^0 = 1$ even if $z_i = 0$).

2. Let Ω be the family of functions in C^n represented by a multiple power series of the form

(2.1) $f(z) = \sum_{m=0}^{\infty} a_m z^m$, $a_m \in C$, where

(2.2) $\{m! \, |a_m|\}$ is bounded.

The condition (2.2) implies $\lim_{||m|| \to \infty} |a_m|^{\frac{1}{||m||}} = 0$ and hence f is an entire function. [2] Thus Ω represents a family of entire functions in C^n. Throughout the paper f will be represented by (2.1). Let $f,g \in \Omega$, where $g(z) = \sum_{m=0}^{\infty} b_m z^m$. We define addition, scalar multiplication and star-multiplication in Ω as follows : $(f+g)(z) = \sum_{m=0}^{\infty} (a_m+b_m)z^m$, $\lambda f(z) = \sum_{m=0}^{\infty} \lambda a_m z^m$, $\lambda \in C$ and $(f_* g)(z) = \sum_{m=0}^{\infty} m! a_m b_m z^m$.

The function $e(z) = \exp(z_1+\ldots+z_n) = \sum_{m=0}^{\infty} \frac{1}{m!} z^m$ is the identity element w.r.t. star-multiplication while the function $0(z) = \sum_{m=0}^{\infty} 0_m z^m$ where $0_m = 0$ for all $m \in I^n$ is the null element in Ω. It is easy to see that Ω is a complex linear space. We define the norm of $f \in \Omega$ by $||f|| = \sup_m \{m! \, |a_m|\}$.

Theorem 1. Ω is a complex Banach space.

Proof. We need to prove the completeness only. Let the sequence $\{f_p\}$, where $f_p(z) = \sum_{m=0}^{\infty} a_m^{(p)} z^m \in \Omega$ for $p = 1,2,\ldots$, be a Cauchy sequence. Then for every $\epsilon > 0$ there exists an integer $N > 0$ such that $||f_p - f_q|| < \epsilon$ for $p,q \geq N$. That is, $\sup_m \{ m! \, |a_m^{(p)} - a_m^{(q)}| \} < \epsilon$ for $p,q \geq N$. This implies

(2.3) $m! \, |a_m^{(p)} - a_m^{(q)}| < \epsilon$ for all $m \in I^n$ and hence

(2.4) $|a_m^{(p)} - a_m^{(q)}| < \epsilon$ for $p,q \geq N$ and for all $m \in I^n$.

We fix m and consider the sequence $\{a_m^{(1)}, a_m^{(2)}, \ldots\}$. Due to (2.4) this sequence will converge to a limit, say, a_m. We now form the multiple power series $f(z) = \sum_{m=0}^{\infty} a_m z^m$. From (2.3) we have,

$$m!\,|a_m^{(p)} - a_m| < \epsilon \quad \text{for } p \geq k \text{ and for all } m. \tag{2.5}$$

Hence $m!\,|a_m| < m!\,|a_m^{(k)}| + \epsilon$ for all m. But $\sum_{m=0}^{\infty} a_m^{(k)} z^m \in \Omega$ implies $\{m!\,|a_m^{(k)}|\}$ is bounded and hence $f \in \Omega$.

From (2.5) we have $\sup_m\{m!\,|a_m^{(p)} - a_m|\} < \epsilon$ for $p \geq k$; that is, $\|f_p - f\| < \epsilon$ for $p \geq k$. Thus, $f_p \to f \in \Omega$ as $p \to \infty$. Hence Ω is complete and so a complex Banach space.

Theorem 2. Ω is a commutative B*-algebra with identity.

Proof. It is easy to see that Ω is a commutative ring with the identity element e(z). Also, for any $f,g \in \Omega$,

$$\begin{aligned} \|f * g\| &= \sup_m\{(m!)^2 |a_m b_m|\} \leq \sup_m\{m!\,|a_m|\}.\sup_m\{m!\,|b_m|\} \\ &= \|f\|\,\|g\|. \end{aligned}$$

We now define a mapping ϕ in Ω by $\phi : \Omega \to \Omega$ such that $\phi(f) = f^*$ where $f^*(z) = \sum_{m=0}^{\infty} \overline{a_m} z^m$. Clearly ϕ is an involution in Ω. Again for any $f \in \Omega$, $\|f^* \star f\| = \sup_m\{(m!)^2 |a_m|^2\} = \|f\|^2$. Hence Ω is a commutative B*-algebra with identity.

3. In this section we restrict ourselves to the family $F \subset \Omega$ of entire functions with real coefficients. We first show that

Proposition. F is a lattice.

Proof. We introduce a partial order relation in F as follows :

Let $f,g \in F$ where $g(z) = \sum_{m=0}^{\infty} b_m z^m$. Then $f \leq g$ iff $a_m \leq b_m$ for all $m \in I^n$.

Let $p(z) = \sum_{m=0}^{\infty} c_m z^m$ and $q(z) = \sum_{m=0}^{\infty} d_m z^m$ where $c_m = \min(a_m, b_m)$ and $d_m = \max(a_m, b_m)$ for all m. Clearly the sequence $\{m!\,|c_m|\}$ and $\{m!\,|d_m|\}$ are bounded and hence $p,q \in F$. Since $c_m \leq a_m$ and $c_m \leq b_m$ for all $m \in I^n$, it follows that p is a lower bound of f and g and evidently it is the g.l.b of f and g. Similarly, q is the l.u.b of f and g. That is, $f \wedge g = p$ and $f \vee g = q$. Hence F is a lattice.

We observe that $f+g = (f \wedge g) + (f \vee g)$ and $f * g = (f \wedge g)*(f \vee g)$. It is easy to show that F is a commutative ring in which negative of f is $-f(z) = \sum_{m=0}^{\infty} (-a_m)z^m$.

Lemma 1. F is an Archimedian *l*-group.

Proof. In F, $f \leq g \Rightarrow u+f+v \leq u+g+v$ for all $u,v \in F$. Hence every group translation in F is isotone. Also, if $f \leq g$, then $j.a_m \leq b_m$ for every integer j implies $a_m = 0$ for all $m \in I^n$. Hence, $f << g \Rightarrow f = 0$ and so F is an Archimedian *l*-group.

Theorem 3. F is an Archimedian function ring.

Proof. Evidently, $f \geq g \Rightarrow h + f \geq h + g$ for all $h \in F$ and $f \geq 0, g \geq 0 \Rightarrow f*g \geq 0$ in F. Hence by lemma 1, F is an Archimedean *l*-ring. Again,

let $f \wedge g = 0$ and $h \geq 0$ where $h(z) = \sum_{m=0}^{\infty} c_m z^m$. Then, $\min(a_m, b_m) = 0$ and $c_m \geq 0$ for all m. It follows $a_m \geq 0$, $b_m \geq 0$ and at least one of a_m, b_m is 0 for all m. Hence $\min(m!c_m a_m, b_m) = 0$ for all m which implies $(h*f) \wedge g = (f*h) \wedge g = 0$. Thus,

$f \wedge g = 0$ and $h \geq 0 \Rightarrow (h*f) \wedge g = (f*h) \wedge g = 0$. Hence F is an Archimedean function ring.

Lemma 2. F is a vector lattice.

Proof. We restrict the scalars λ in the field of real numbers. Clearly, F is a real linear space. Also, $f \leq g \Rightarrow f + h \leq g + h$ for all $h \in F$ and $f \leq g \Rightarrow \lambda f \leq \lambda g$ for all $\lambda \geq 0$.

Hence F is a vector lattice.

Theorem 4. F is a Banach lattice.

Proof. We can show, as in theorem 1, that F is a real Banach space with $||f|| = \sup_m \{ m! |a_m| \}$. For any $f \in F$ we define the absolute value by $|f| = f \vee (-f)$. Obviously, $|f|(z) = \sum_{m=0}^{\infty} |a_m| z^m$. Also, $|f| \leq |g| \Rightarrow |a_m| \leq |b_m|$ for all $m \Rightarrow \sup_m \{ m! |a_m| \} \leq \sup_m \{ m! |b_m| \}$; that is, $|f| \leq |g| \Rightarrow ||f|| \leq ||g||$.

Hence by lemma 2, F is a Banach lattice.

4. We now consider two important sub-lattices of F. Let Γ be the subset of F consisting of the functions in the form (2.1) where $a_m \in R_+$. Then for $f, g \in \Gamma$, $f \wedge g, f \vee g \in \Gamma$. Thus Γ is a sub-lattice of F and has the least element $0(z)$. Again, let Δ be the subset of F consisting of the functions with non-positive coefficients. Then Δ is also a sub-lattice of F possessing the greatest element $0(z)$. If $f \in \Delta$, $g \in F$, then $g \leq f \Rightarrow g \in \Delta$ and $f,h \in \Delta \Rightarrow f \vee h \in \Delta$. Hence Δ is an ideal of the lattice F. In fact, it is the principal ideal generated by $0(z)$.

Theorem 5. Γ is a commutative *l*-semigroup.

Proof. It is easy to show that closure, associative and commutative laws for star-multiplication hold in Γ. We need to prove only

(4.1) $f*(g \vee h) = (f*g) \vee (f*h)$ and

(4.2) $(f \vee g) * h = (f*h) \vee (g*h)$ for all $f,g,h \in \Gamma$.

Let $f*(g \vee h) = \sum_{m=0}^{\infty} d_m z^m$ and $(f*g) \vee ((f*h) = \sum_{m=0}^{\infty} t_m z^m$

where $g(z) = \sum_{m=0}^{\infty} b_m z^m$ and $h(z) = \sum_{m=0}^{\infty} c_m z^m$. Then,

$t_m = \max(m!a_m b_m, m!a_m c_m) = m!a_m\{\max(b_m, c_m)\}$

$= d_m$ for all $m \in I^n$ which proves (4.1). The proof of (4.2) is similar.

Hence Γ is a commutative *l*-semigroup.

Lemma 3. In Δ, the following hold :

(4.3) $f + g = f \wedge g + f \vee g$

(4.4) $(f \vee g) \vee (f \vee h) = f \vee (g \vee h)$

(4.5) $(f \overset{\vee}{\underset{\wedge}{}} g) + h = (f+h) \overset{\vee}{\underset{\wedge}{}} (g+h)$

Proof. The proof is straightforward and hence omitted.

Theorem 6. Δ is a distributive lattice.

Proof. The null function $0(z)$ is the universal upper bound in Δ. It is known [1,P,34] the Δ will be a distributive lattice if the following four conditions are satisfied.

For $f, g, h \in \Delta$,

(4.6) $f \wedge f = f$

(4.7) $f \vee 0 = 0 \vee f = 0$

(4.8) $f \wedge 0 = 0 \wedge f = f$

(4.9) $f \wedge (g \vee h) = (f \wedge g) \vee (f \wedge h)$ and

$(g \vee h) \wedge f = (g \wedge f) \vee (h \wedge f)$.

To verify (4.6), (4.7) and (4.8) is a routine affair. We prove 1st part of (4.9). Clearly, $f \wedge (g \vee h) \geq f \wedge g, f \wedge h$. Hence $f \wedge (g \vee h)$ is an upper bound of $f \wedge g$ and $f \wedge h$. Let p be an arbitrary upper bound of $f \wedge g$ and $f \wedge h$. Then, $p \geq f \wedge g = f + g - f \vee g$; that is, $f + g \leq p + f \vee g$. Similarly, $f + h \leq p + f \vee h$. Hence by lemma 3,

$p + (f \vee g) \vee (f \vee h) = (p + f \vee g) \vee (p + f \vee h) \geq (f + g) \vee (f + h) = f + (g \vee h)$. Thus, $p \geq f + (g \vee h) - f \vee (g \vee h) = f \wedge (g \vee h)$. Hence, $f \wedge (g \vee h)$ is the l.u.b of $f \wedge g$ and $f \wedge h$; that is, $f \wedge (g \vee h) = (f \wedge g) \vee (f \wedge h)$.

The 2nd part of (4.9) follows from the first part due to the commutativity of the meet operation. Hence Δ is a distributive lattice.

REFERENCES

1. G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloquium Publications, 25 (1967).
2. N. C. Mazumdar, On the space of integral functions of several complex variables, I.J.M.M. 12 (No. 1) (1974), 70–81.

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A NOTE ON CONTRACTIFIABILITY OF SELF-MAPPINGS

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INTRODUCTION

The notion of contractifiability of a self-mapping in a metric space was first introduced by Meyers in connection with the investigation in the converse problem of Banach contraction mapping principle. Some of his most significant results are stated in the following section. During a similar investigation in [11] an operation (*) on non-negative real numbers proved to be very fruitful. This paper deals with the interrelation between contractifiability and the operation (*). Although not dealt with here, the interrelation bears some consequences to the converse problem of Banach Contraction principle.

SECTION 1

Definition 1 Let (X,d_1) and (X,d_2) be two metric spaces. If the identity mapping $I : (X,d_1) \to (x,d_2)$ is a homeomorphism, then d_1, d_2, are said to be equivalent on X.

One can easily observe that following statements are equivalent.

(1) The metrics d_1, d_2 are equivalent on X,

(2) the sequence $\{x_n\}$ converges to x in (X,d_1) if and only if it converges to x in (X,d_2).

We write $d_1 \simeq d_2$ to indicate that d_1 and d_2 are equivalent.

Definition 2 Two metrics d_1, d_2 on the set X are said to be uniformly equivalent if and only if the identity mapping $I : (X, d_1) \to (X, d_2)$ and the inverse mapping $I^{-1} : (X, d_2) \to (X, d_1)$ are uniformly continuous.

We write $d_1 \cong d_2$ twhen d_1 and d_2 are uniformly equivalent.

Definition 3 A continuous self-mapping T is said to be contractifiable in a metric space (X, d) if and only if there exists an equivalent metric d_1 such that T is a contraction in (X, d_1).

One significant result on contractifiably was established by Meyers [8]. This is stated below.

The following results are equivalent.

a) T is contractifiable in (X, d),

b) there exists $w \in X$ such that

i) $Tw = w$,

ii) $T^n x \to w$, for all $x \in X$,

iii) $T^n U \to w$ for some neighbourhood U of w.

Leader has offered a set of simplified conditions in this respect [6] viz.

(L_i)$T^n x \to w$, for all $x \in X$,

(L_{ii})$T^n x \to w$, uniformly, for all x in some neighbourhood of w.

SECTION 2

In this section a binary operation is defined on which rests the main results of this paper.

Definition 4 * is a binary operation on non-negative real numbers satisfying the following conditions

i) $u*0 = k_1 u$, $k_1 > 0$,

ii) $0 * v = k_2 v$, $k_2 > 0$,

iii) $u \leq u_1 - u_2$, $v \leq v_1 + v_2$ imply $u * v \leq u_1 * v_1 + u_2 * v_2$ for all u, u_1, u_2, v, v_1, v_2 .

Remark 1 $0 * 0 = 0$.

Examples.

1. $u * v = k_1 u + k_2 v$, $k_1 > 0$, $k_2 > 0$.
2. $u * v = \max(u,v)$.
3. $u * v = (k_1 u^p + k_2 v^p)^{1/p}$, $k_1>0$, $k_2>0$, p being any integer > 1.
4. $u * v = (u^2+uv+v^2)^{1/2}$.

The following properties of the * operation are interesting and useful.

$P_1 : (u_1+u_2) * v_1 \leq u_1 * v_1 + u_2 * v_1$.

$P_2 : u * (v_1+v_2) \leq u * v_1 + u * v_2$.

$P_3 : u \leq u_1, v \leq v_1$ imply $u * v \leq u_1 * v_1$.

$P_4 : k_1 u, k_2 u \leq u * v \leq k_1 u + k_2 v$.

P_5 : * is neither commutative nor associative generally.

P_6 : For $k_1+k_2 \ge 1$, $k_1^n u \le ((\dots((u * u) * u)\dots) * u) \le (k_1+k_2)^n u$ [operation being n times].

P_7 : For $k_1+k_2 \ge 1$, $k_2^n u \le (u * (u * \dots (u * (u * u))\dots)) \le (k_1+k_2)^n u$ [operation being n times].

P_8 : For $k_1+k_2 \le 1$, $u \ge u * u \ge ((u * u) * u) \ge (((u * u) * u) * u) \ge \dots$

P_9 : For $k_1+k_2 \le 1$, $u \ge u * u \ge (u * (u * u)) \ge (u * (u * (u * u))) \ge \dots$

P_{10} : For $k_1 \ge 1$, $u \le u * u \le ((u * u) * u) \le (((u * u) * u) * u) \le \dots$

P_{11} : For $k_2 \ge 1$, $u \le u * u \le (u * (u * u)) \le (u * (u * (u * u))) \le \dots$

Definition 5. Let d_1 and d_2 be two metrics defined on the same set X. We define a function d by

$$\begin{aligned} d(x,y) &= d_1 * d_2(x,y) \\ &= d_1(x,y) * d_2(x,y) \end{aligned}$$

Theorem 1. d is a metric on X.

The proof follows immediately from the definition.

But the converse is not true i.e. under some operation $\square$, if $d_1(x,y) \square d_2(x,y)$ is a metric for two metrics d_1 and d_2 then $\square$ is not necessarily a *-operation.

Example d_1 and d_2 are metrics on the real line defined by

$$d_1(x, y) = \begin{cases} 1 & \text{for } x \neq y \\ 0 & \text{for } x = y \end{cases}$$

and $d_2(x, y) = |x-y|$.

Let $d(x, y) = d_1(x, y) \square d_2(x, y) = \min(d_1(x, y), d_2(x, y))$.

Then d(x, y) is a metric.

But $u \square 0 = \min(u,0) = 0 \neq k_1 u$ for any $k_1 > 0$, and this implies that $\square$ is not a *-operation.

Theorem 2. The topology on a set X induced by the metric $d_1 * d_2$ is finer than the topologies induced by the metrics d_1 and d_2.

Proof. Let τ_1, τ_2 and τ_3 be the topologies induced by the metrics d_1, d_2 and $d_1 * d_2$ respectively.

Let $S_r^{d_1}(x)$ be an open sphere in (X, d_1).

Now by the property P_4

$k_1 d_1(w,x) \le d_1(w,x) * d_2(w,x) = d_1 * d_2(w,x)$.

So, $d_1 * d_2(w,x) \le k_1 r_1$ implies that $d_1(w,x) \le r_1$.

Thus $\quad S^{d_1 * d_2}_{k_1 r_1}(w) \subseteq S^{d_1}_{r_1}(w)$.

Similarly from $k_2 d_2(w,x) \le d_1(w,x) * d_2(w,x) = d_1 * d_2(w,x)$, it can be estqablished that

$$S^{d_1 * d_2}_{k_2 r_2}(w) \subseteq S^{d_2}_{r_2}(w),$$

where $S^{d_2}_{r_2}(w)$ is an open sphere in (X, d_2).

Hence τ_3 is finer than both τ_1 and τ_2.

Theorem 3. If $d_1 \simeq d_2$ then each of d_1, d_2 is equivalent to $d_1 * d_2$.

Proof. Let $\{x_n\}$ be any sequence which converges to x with respect to d_1 i.e., $\lim\limits_{n\to\infty} d_1(x_n, x) = 0$.

Since $d_1 \simeq d_2$, $\lim d_2(x_n, x) = 0$

Then $\lim\limits_{n\to\infty} (k_1 d_1(x_n, x) + k_2 d_2(x_n, x)) = 0$.

i.e., $\lim\limits_{n\to\infty} d_1 * d_2(x_n, x) = 0$ since $d_1 * d_2(x_n, x) \le k_1 d_1(x_n, x) + k_2 d_2(x_n, x)$.

Conversely, let a sequence $\{y_n\}$ converge to y with respect to $d_1 * d_2$.
i.e., $\lim\limits_{n\to\infty} d_1 * d_2(y_n, y) = 0$.

So, $k_1 d_1(y_n, y) \le d_1 * d_2(y_n, y)$ implies that $\lim\limits_{n\to\infty} d_1(y_n, y) = 0$.

Hence $d_1 \simeq d_2 * d_2$ and so $d_2 \simeq d_1 * d_2$.

Corollary 3.1. Putting $d_1\ d_2 = d$, we get that $d \simeq d * d$.

Definition 6. A word of metrics d_1 and d_2 on the same set X is defined recursively by

i) d_i ; $i=1,2$, is a word.

ii) If d and d' are words, then $d * d'$ is a word.

Definition 7. The length of a word is the number of *'s in the word.

Theorem 4. If $d_1 \simeq d_2$, then any two words are equivalent to each other.

The proof follows from theorem 3.

Towards the converse, we state the following theorem the proof being omitted.

Theorem 5. Let $d_1 \simeq d_1 * d_2$ and $d_1 * d_2 \le d_2$, then $d_1 \simeq d_2$

Theorem 6. If $d_1 \simeq d_1'$, and $d_2 \simeq d_2'$ then $d_1 * d_2 \simeq d_1' * d_2'$.

Proof. Let $\{x_n\}$ be a sequence which converges to x with respect to $d_1 * d_2$, i.e. $\lim\limits_{n\to\infty} d_1 * d_2(x_n, x) = 0$.

Hence $\lim\limits_{n\to\infty} d_1(x_n, x) = 0$, $\lim\limits_{n\to\infty} d_2(x_n, x) = 0$ as $k_1 d_1, k_2 d_2 \le d_1 * d_2$.

Since $d_1 \simeq d_1'$, $d_2 \simeq d_2'$, $\lim\limits_{n\to\infty} d_1'(x_n, x) = 0$, $\lim\limits_{n\to\infty} d_2'(x_n, x) = 0$.

Since $d_1' * d_2' \le k_1 d_1' + k_2 d_2'$, $\lim\limits_{n\to\infty} d_1' * d_2'(x_n, x) = 0$,

i.e. the sequence $\{x_n\}$ which converges to x with respect to $d_1 * d_2$ also converges to x with respect to $d_1' * d_2'$.

Hence $d_1{*}\,d_2 \cong d_1'{*}\,d_2'$.

Theorem 7. If $d_1 \cong d_2$, then they are uniformly equivalent to $d_1 * d_2$.

Proof. We take any arbitrary $\varepsilon > 0$ and $0 < \delta_1 < \varepsilon$ so that $\varepsilon - \delta_1 > 0$.

Since d_1 is uniformly equivalent to d_2.

$I : (X, d_1) \to (X, d_2)$ is uniformly continuous. For $\frac{\delta_1}{k_2}$, there exists δ_2 such that for all $x,y \in X$, $d_1(x, y) < \delta_2$ implies $d_2(x, y) < \frac{\delta_1}{k_2}$ i.e., $k_2 d_2(x, y) < \delta_1$.

Let $\delta = \min\left(\delta_2, \frac{(\varepsilon - \delta_1)}{k_1}\right)$.

Then $d_1(x, y) < \delta$ implies that $k_2 d_2(x, y) < \delta_1$

and $d_1(x, y) \leq \delta$, implies that $d_1(x, y) < \frac{(\varepsilon - \delta_1)}{k_1}$, i.e. $k_1 d_1(x, y) < \varepsilon - \delta_1$.

Therefore, if $d_1(x, y) < \delta$ then $k_1 d_1(x, y) + k_2 d_2(x, y) < \varepsilon - \delta_1 + \delta_1 = \varepsilon$.

Hence by P_4, $d_1 * d_2(x, y) < \varepsilon$.

Hence $I : (X, d_1) \to (X, d_1 * d_2)$ is uniformly continuous.

Again, given $\varepsilon > 0$, we take $0<\delta<k_1\varepsilon$.

Then $d_1 * d_2(x, y) < \delta < k_1 \varepsilon$ implies that $k_1 d_1(x, y) < k_1 \varepsilon$ i.e. $d_1(x, y) < \varepsilon$.

So, $I^{-1} : (X, d_1 * d_2) \to (X, d_1)$ is uniformly continuous.

Hence $d_1 \cong d_1 * d_2$.

The following theorems can be proved in a straight forward way.

Theorem 8. If $d_1 \cong d_2$ then any two words are uniformly equivalent to each other.

Theorem 9. If $d_2 \leq d_1$, then $d_1 \cong d_1 * d_2$ and $d_1 \cong d_2 * d_1$.

SECTION 3

In this section some results on contractifiability are established.

Theorem 10. Let a self mapping T be a contractifiable in both the metric spaces (X, d_1) and (X, d_2). Then T is so in $(X, d_1 * d_2)$.

Proof. We first prove that if T is continuous in (X, d_1) and $(X. d_2)$, then T is continuous in $(X, d_1 * d_2)$.

Let x be an arbitrary point in X.

Then given $\varepsilon > 0$, there are δ_1, δ_2 such that

if $d_1(x, y) < \delta_1$ then $d_1(Tx, Ty) < \frac{\varepsilon}{2k_1}$

and if $d_2(x, y) < \delta_2$ then $d_2(Tx, Ty) < \frac{\varepsilon}{2k_2}$, for all $y \in X$.

Then $d_1 * d_2(Tx, Ty) \leq k_1 d_1(Tx, Ty) + k_2 d_2(Tx, Ty) < \varepsilon$.

Now by P_4. $d_1 * d_2(x, y) < k_1\delta_1$ implies that $d_1(x, y) < \delta_1$

and $d_1 * d_2(x, y) < k_2\delta_2$ implies that $d_2(x, y) < \delta_2$.

So, when $d_1 * d_2(x, y) < \min(k_1\delta_1, k_2\delta_2)$, we have both $d_1(x, y) < \delta_1$

and $d_2(x, y) < \delta_2$

and then $d_1 * d_2(Tx, Ty) < \varepsilon$.

Hence T is continuous in $(X, d_1 * d_2)$.

Given that T is contractifiable in (X, d_1) and (X, d_2),

we have

i) $Tw = w$,

ii) $T^n x \xrightarrow{d_1} w$,

iii) $T^n x \xrightarrow{d_2} w$, since w is the unique fixed point.

iv) $T^n x \xrightarrow{d_1} w$ uniformly in some neighbourhood U of w

and v) $T^n x \xrightarrow{d_2} w$ uniformly in some neighbourbood V of w.

From (ii) and (iii), given $\frac{\varepsilon}{2k_1} > 0$ and $\frac{\varepsilon}{2k_2} > 0$, there exist N_1, N_2 such that

$$d_1(T^n x, w) < \frac{\varepsilon}{2k_1} \text{ for } n \geq N_1$$

$$\text{and } d_2(T^n x, w) < \frac{\varepsilon}{2k_2} \text{ for } n \geq N_2.$$

Let us take $N = \max(N_1, N_2)$.

Then, $d_1 * d_2(T^n x, w) < \varepsilon$ for $n \geq N$

Hence $T^n x \xrightarrow{d_1 * d_2} w$. So (L_1) is satisfied for $d_1 * d_2$.

Finally, we know that $d_1 * d_2$ generates a topology finer than the topologies generated by d_1 and d_2 (section II, Theorem 2).

Now as U and V are neighbourhoods of w in (X, d_1) and (X, d_2), there are open spheres $S_{r_1}^{d_1}(w)$ and $S_{r_2}^{d_2}(w)$ such that

$$S_{r_1}^{d_1}(w) \subseteq U \text{ and } S_{r_2}^{d_2}(w) \subseteq V.$$

As $d_1 * d_2$ generates a finer topology, there is an open spheres $S_r^{d_1 * d_2}(w)$ such that

$$S_r^{d_1 * d_2}(w) \subseteq S_{r_1}^{d_1}(w) \subseteq U$$

$$\text{and } S_r^{d_1 * d_2}(w) \subseteq S_{r_2}^{d_2}(w) \subseteq V.$$

So, $S_r^{d_1 * d_2}(w) \subseteq U \cap V$.

This means that $U \cap V$ is a neighbourhood of w in $(X, d_1 * d_2)$.

Let $x \in X$.

Then from (iv) and (v) we obtain for ε (> 0), there exists $n \geq N$ such that for all x in $U \cap V$.

$$d_1(T^n x, w) < \frac{\varepsilon}{2k_1}$$

$$\text{and} \quad d_2(T^n x, w) < \frac{\varepsilon}{2k_2}$$

Then $d_1 * d_2\,(T^n x, w) < \varepsilon$ for $n \geq N$ and for all x in $U \cap V$.

Thus the condition (L_{ii}) is satisfied for $d_1 * d_2$.

Hence T is contractifiable in $(X, d_1 * d_2)$.

Corollary 1. Under the conditions of the theorem, T is contractifiable in (X, D), where D is any word of d_1, d_2.

Corollary 2. Let $d_1 = d_2 = d$.

We get :

let T be contractifiable in (X, d), then T is so in $(X, d * d)$ and in (X, D'), where D' is any word of d.

Theorem 11. Let a self-mappint T be contractifiable in (X, d_2).

If $d_1 \leq d_2$, then T is contractifiable in $(X, d_1 * d_2)$.

Proof. Given that T is contractifiable in (X, d_2) we have

(i) T is continuous in (X, d_2),

(ii) $Tw = w$ for some $w \in X$.

(iii) $T^n x \xrightarrow{d_2} w$ for all $x \in X$.

and (iv) $T^n x \xrightarrow{d_2} w$ uniformly in some neighbourhood V of w.

Also it is given that $d_1 \leq d_2$.

From (i), given $\varepsilon > 0$ there exists δ such that $d_2(x, y) < \delta$. This implies that $d_2(Tx, Ty) < \frac{\varepsilon}{k_1 + k_2}$ for some arbitrary but fixed x.

Let $0 < \delta_1 < k_2\,\delta$.

Then $d_1 * d_2\,(x, y) < \delta_1$ implies that $k_2 d_2\,(x, y) < \delta_1 < k_2\,\delta$.

So $d_2 (x, y) < \delta$ and hence $d_2 (Tx, Ty) < \frac{\varepsilon}{k_1 + k_2}$.

Also from (v), $d_1 * d_2 (x, y) < \delta_1$ implies that $d_1 (Tx, Ty) < \frac{\varepsilon}{k_1 + k_2}$.

So,
if $d_1 * d_2 (x, y) < \delta_1$ then

$$d_1 * d_2 (Tx, Ty) \leq k_1 d_1 (Tx, Ty) + k_2 d_2 (Tx, Ty) \leq k_1 \frac{\varepsilon}{k_1 + k_2} + k_2 \frac{\varepsilon}{k_1 + k_2} = \varepsilon .$$

Hence T is continuous in $(X, d_1 * d_2)$. From (iii), given $\varepsilon > 0$, there exists N such that $d_2 (T^n x, w) < \frac{\varepsilon}{k_1 + k_2}$ for some $n \geq N$ and for arbitrary but fixed x in X.

Since $d_1 \leq d_2$

$$d_1 (T^n x, w) \leq d_2 (T^n x, w) < \frac{\varepsilon}{k_1 + k_2} \text{ for } n > N.$$

Then $d_1 * d_2 (T^n x, w) \leq k_1 d_1 (T^n x, w) + k_2 d_2 (T^n x, w) < k_1 \frac{\varepsilon}{k_1 + k_2} + k_2 \frac{\varepsilon}{k_1 + k_2} = \varepsilon$ for $n \geq N$.

Hence $T^n x \xrightarrow{d_1 * d_2} w$.

From (iv), since V is a neighbourhood of w, there exists an open spheres $S^{d_2}_{r_2} (w)$ such that

$$S^{d_2}_{r_2} (w) \subseteq V.$$

Since the topology of $d_1 * d_2$ is finer than d_2, we can get an open spheres $S^{d_1 * d_2}_{r} (w)$ such that

$$S^{d_1 * d_2}_{r} (w) \subseteq S^{d_2}_{r_2} (w) \subseteq V.$$

Hence V is also a neighbourhood of w in $(X, d_1 * d_2)$.

From (iv), given ε (> 0) there exists N such that $d_2 (T^n x, w) < \frac{\varepsilon}{k_1 + k_2}$, for all $n \geq N$ and for all x in V.

Since $d_1 \leq d_2$,

$$d_1 (T^n x, w) < \frac{\varepsilon}{k_1 + k_2}, \text{ for } n \geq N \text{ and for all } x \text{ in } V.$$

Hence $k_1 d_1 (T^n x, w) + k_2 d_2 (T^n x, w) < \varepsilon$, for $n \geq N$, for all x in V.

This implies $d_1 * d_2 (T^n x, w) < \varepsilon$, for $n \geq N$, for all x in V.

i.e. $T^n x \xrightarrow{d_1 * d_2} w$ uniformly in V.

Hence the theorem is proved.

Theorem 12. Let a self-mappint T be contractifiable in $(X, d_1 * d_2)$ and let $d_2 \leq d_1$. Then T is contractifiable in (X, d_1).

Proof. Since $d_2 \leq d_1$, by theorem 9, $d_1 \cong d_1 * d_2$. Since continuity is a topological property, T is continuous in (X, d_1).

Now T is contractifiable in $(X, d_1 * d_2)$ implies

(i) $Tx = w$ for some w in X,

(ii) $T^n x \xrightarrow{d_1 * d_2} w$

and (iii) $T^n x \xrightarrow{d_1 * d_2} w$ uniformly in some neighbourhood V of w.

Let x be an arbitrary but fixed point in X.

Then from (ii), given ε (> 0) there exists N such that $d_1 * d_2 (T^n x, w) < k_1 \varepsilon$ for $n \geq N$.

So, by P_4, $d_1 (T^n x, w) < \varepsilon$ for $n \geq N$, i.e., condition (L_i) holds for the metric d_1.

Again by (iii), $S_r^{d_1 * d_2}(w) \subseteq V$, for some r.

Now consider $S_{r_1}^{d_1}(w)$, where $r_1 = r/2k$, $k = \max (k_1, k_2)$.

Then for $x \in S_{r_1}^{d_1}(w)$, we get

$d_1(x, w) < r/2k$ and since $d_2 \leq d_1$, $d_2(x, w) < r/2k$.

So, $k_1 d_1(x, w) + k_2 d_2(x, w) < r$.

Hence $x \in S_r^{d_1 * d_2}(w)$, by P_4 again.

This implies that $S_{r_1}^{d_1}(w) \subseteq S_r^{d_1 * d_2}(w) \subseteq V$.

Hence V is a neighbourhood of w with respect to d_1.

Now from (iii), we get that given ε (> 0) there exists N such that for all $x \in V$,

$d_1 * d_2(T^n x, w) < k_1 \varepsilon$, when $n \geq N$.

So, $k_1 d_1(T^n x, w) < k_1 \varepsilon$, when $n \geq N$.

This implies that $d_1(T^n x, w) < \varepsilon$, when $n \geq N$ and for all $x \in V$.

Thus (L_{ii}) is established for d_1.

Hence the theorem is proved.

REFERENCES

1. Anglov, V. G., *A Converse to a Contraction Mapping Theorem in Uniform Spaces,* 89K : 54101. Non-linear Analysis.

2. C. Bessaga, *On the Converse of Banach's Fixed point Principle,* colloq. Math 7 (1959) P. 41-43.
3. Valle, I Istvatesca, *Fixed Point Theory, an Introduction,* D. Riedel Publishing Company, 1981.
4. L. Janos, *A Converse of Banach's Contraction Principle,* Proc. Amer. Math. Soc 18 (1967).
5. K. D. Joshi, *Introduction to General Topology.* Wiley Eastern Limited, 1983.
6. S. Leader, A Topological Characterisation of Banach's Contraction, Pacific J. Math 69 (1977) p. 461-466.
7. P. R. Meyers, *A Converse of Banach's Contraction Theorems,* J. Res. Bur. Sutdents 71B (1967), p. 73-76.
8. P. R. Meyers, *Contractifiable Semi groups,* J. Res. Bur. Students 74B (1970B), p. 315-332.
9. P. R. Meyers, *On Contractifiable Self-mapping Non-linear Analysis,* World Scientific Publishing Co. Singapore, 1987, p. 407-432.
10. C. G. Pits, *Introduction to metric spaces.* Oliver and Boyd, 1972.
11. A. Sen, Ph. D. thesis, University of Calcutta.

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LOCALLY HAUSDORFF SPACES

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Abstract : In this paper the notion of "LOCALLY HAUSDORFF SPACES" has been introduced. A topological space X is called Locally Hausdorff, if each point has a neighbourhood, which is a Hausdorff subspace of X. These spaces are seen to lie between T_1 and T_2 spaces. A space is seen to be locally Hausdorff, if and only if, the diagonal $A(X) = \{(x, x) : x \in X\}$, happens to be a locally closed subset of the product space XxX. Locally Hausdorff spaces are seen to be only finitely productive (i.e. Tychonoff productive) and arbitrarily box productive. Finally a characterisation of the epimorphisms in the category LHAUS of all locally Hausdorff spaces and continuous functions between them has been furnished.

1. INTRODUCTION

This paper is an outcome of an attempt to localise the celebrated Hausdorff separation axiom. The class of spaces so obtained in the process are seen to lie strictly between T_1 and T_2 spaces. In §1, we have shown that locally Hausdorff spaces stand independent of a few other well known separation axioms between T_1 and T_2. In §2, a characterisation of a locally Hausdorff space X via the diagonal of the product space XxX is given; in the same section an extrinsic characterisation, also of such spaces in terms of the kernels of continuous maps into these spaces is furnished. In § 3, it is shown that local hausdorffness of a topological space is finitely Tychonoff productive, but arbitrarily box productive topological property. In § 4, the problem when the one point compactification of a Hausdorff space is locally Hausdorff is considered; this leads to a method of constructing non-Hausdorff locally Hausdorff spaces. In the final section of the paper we have given a characterisation of the class of epimorphisms in the category LHAUS of locally Hausdorff spaces with continuous functions between them.

Definition 1.1. A topological space X is said to be locally Hausdorff (hence forth abbreviated to LH), if each point on it has a Hausdorff neighbourhood (hence forth abbreviated to nbd).

It is plain that every subspace of an LH space is LH, every Hausdorff space is LH, and every LH space is a T_1 space.

The following is a list of some familiar separation axioms that lie between T_1 and T_2 spaces. [See [4], [6], [7], [9].]

Definition 1.2. A topological space X is said to be

(i) KC space, if every compact subset is closed;

(ii) US space, if every convergent sequence has exactly one limit point;

(iii) T_{AB} space, if it is a T_1 space and for every pair of disjoint compact sub sets K and L of X, there exists an open set containing one of them and disjoint with the other.

It is easy to prove that a KC space is a T_{AB} space, which in turn is a US space. The following examples entail the independence of each of these three separation axioms with local Hausdorfness, which in the hierarchy of separation axioms lie strictly between T_1 and T_2 spaces.

Example 1.3. (a) Any infinite set with topology of finite complements is a T_1 space which is not LH.

(2) Fix any two distinct points a, b in any infinite set X and define a topology on X as follows : each point of X-{a, b} is isolated, and the open nbds of both a and b are precisely the cofinite subsets of X, containing 'a' and 'b' respectively.

Then X is LH which is neither Hausdorff nor a US space.

(3) Let X be any uncountable set with the topology of countable complements. Then X is not LH; however every compact subset of X is countable, so that X is a KC space.

2. CHARACTERISATIONS OF LOCALLY HAUSDORFF SPACES

Definition. In a topological space X, a subset A of X is said to be locally closed, if every point x of A has a nbd U, such that $U \cap A$ is a closed subset of U.

It so turns out that locally closed subsets of X are precisely those that can be expressed as an intersectionof a closed set and an open set in X, so that their collection encompasses the families of open and closed subsets both.

Theorem 2.1. A topological space X is LH, if and only if, the diagonal $A(X) = \{(x, x) \in XxX, : x \in X\}$ is locally closed in the product space XxX.

Proof. First assume that X is LH, and let U_x be an open Hausdorff nbd of an arbitrarily selected point $x \in X$.

Then $(U_x x U_x) \cap A(X)$ becomes an open nbd of the point (x, x) in the subspace A(X). It is easy to see that the set $(U_x x U_x) \cap A(X)$ is closed in the space $U_x x U_x$ so that the set A(X) is locally closed in XxX.

Conversely suppose that A(X) is locally closed in XxX.

Choose any point x from X, arbitrarily; then there exist open nbds U_x and V_x of the point x from X, such that $(U_x x V_x) \cap A(X)$ is a closed subset of the subspace $U_x \times V_x$. It is clear that $U_x \cap V_x$ is an open Hausdorff nbd of the point x from X. Accordingly X is locally Hausdorff.

Since discrete spaces are LH, and there exists a space that is not LH (See Example 1.3 (1)], it is clear that LH-ness of a topological space is not continuous invariant; however it turns out to be a topological property. The following proposition tells us something more in view of Theorem 2.1. For any function $g : U \to V$, we set $Ker(g) = \{(u,v) \in UxU \cdot g(u) = g(v)\}$

Theorem 2.2. Suppose that f : X→Y is an open map from the space X onto the space Y with Ker(f) a locally closed subset of XxX; them Y is LH.

Proof. Choose any point x from X. Since (x, x) belongs to Ker(f), there exist open nbds U_x and V_x of x in X, such that $(U_x x V_x) \cap$ Ker(f) is closed in $U_x x V_x$. It is not difficult to check that the open nbd $f(U_x \cap V_x)$ of the point f(x) in Y is a Hausdorff subspace of Y. Hence Y is LH.

For a continuous map the converse of the above theorem is true.

Theorem 2.3. Suppose that f : X→Y is a continuous map from the space X into a LH space Y; then Ker(f) is locally closed in XxX.

Proof. Let (a,b) ∈ Ker(f) be arbitrarily chosen. Since Y is LH, there exists an open Hausdorff nbd U of the point f(a) = f(b) in Y. The continuity of f indicates that $f^{-1}(U) \times f^{-1}(U)$ is an open nbd of the point (a,b) in XxX. One can then easily check that $[f^{-1}(U) \times f^{-1}(U)] \cap$ Ker(f) is closed in $f^{-1}(U) \times f^{-1}(U)$. This ensures that Ker(f) is locally closed in XxX.

The combination of Theorems 2.2 and 2.3 therefore yields the following :

Theorem 2.4. Suppose f : X→Y is a continuous open map from the topological space X onto a space Y; then Y is LH, if and only if, Ker(f) is a locally closed subset of XxX.

The following theorem will be required to prove our main theorem in the concluding section of this paper.

Theorem 2.5. Suppose g,h : X→Y are two continuous maps from the space X into a LH space Y, then the set {x∈X: g(x) = h(x)} is locally closed in X.

Proof. Let A = {x∈X: g(x) = h(x)}; choose any point x from A and let U be an open Hausdorff nbd of the point g(x) = h(x) in Y. Then the set $g^{-1}(U) \cap h^{1}(U) \cap A$ becomes a closed nbd of the point x in the subspace $g^{-1}(U) \cap h^{-1}(U)$ which is evidently an open nbd of the point x from X. Consequently A is locally closed in X.

3. PRODUCTIVITY OF LOCALLY HAUSDORFF SPACES

Since the Tychonoff product of an arbitrary family of Hausdorff spaces is Hausdorff, the following is clearly visualised :

Theorem 3.1. The box product of an arbitrary family of LH spaces is an LH space.

In contrast these spaces are only finitely Tychonoff productive as proved in the following proposition :

Theorem 3.2. Suppose A is any non-empty set and for each 'a' in A, let X_a be a topological space and further let X be the Tychonoff product of the spaces X_a, for each a in A (i.e. X = $\Pi_{a\in A}X_a$); then X is an LH space, if and only if, for each a in A, X_a is an LH space, and for all but finitely many indices a from A, X_a is a Hausdorff space.

Proof. If the conditions are satisfied then X is clearly LH, which follows from the definition of the Tychonoff product topology and the fact that Hausdorffness is Tychonoff productive.

Conversely, assume that X is an LH space. Since LH-ness is a hereditary property and every hereditary property is projective, it follows that each of the component spaces is LH. If possible, let there exist an infinite sequence $X_{a_1}, X_{a_2}, \ldots, X_{a_n}, \ldots$ of spaces, none of which is Hausdorff. Then in each X_{a_i}, $i = 1,2,\ldots,n,\ldots$ one can choose a pair of distinct points x_{a_i} and y_{a_i}, such that each nbd of x_{a_i} meets each nbd of y_{a_i}. Choose a point x from X as follows :

$P_{a_i}(x) = x_{a_i}$, where for each a in A, P_a denotes the a^{th} projection map from the product space X to the component space X_a. Consider any basic open nbd, U of the point x from X. Let $b_1, b_2, \ldots, b_n$, be the finitely many points from A such that $P_{b_j}(U) \neq x_{b_j}$, $j = 1,2,\ldots,n$. Choose the point z from X in the following way :

$$\begin{aligned} P_a(z) &= y_a, \text{ for } a \in \{a_1, a_2, \ldots\} - \{b_j : j = 1,2,\ldots n\} \\ &= x_a, \text{ otherwise.} \end{aligned}$$

Then surely $z \neq x$, and z is a member of U. It is easy to see that every open nbd of the point z intersects every open nbd of the point x, in the space U. Hence U is not a Hausdorff subspace of X. Accordingly X cannot be LH, contrary to our assumption.

4. NON HAUSDORFF LOCALLY HAUSDORFF SPACES

For a Hausdorff space X, the one point compactification X^* need not be Hausdorff; X^* need not even be LH, viz., the spaces Q^* and S^* are not LH, where Q and S denotes the space of all rational numbers with the subspace topology of the reals and the Sorgenfrey line respectively. A close look into the inner topological structures of Q and S reveal that this is attributable to the fact that none of these spaces are locally compact at any of their points. This inspires us to answer the question when is X^*, LH for a Hausdorff space X as follows.

Theorem 4.1. If X be a Hausdorff space, then the one point compactification X^* is LH, if and only if, there exists a partition of $X = A \cup B$, where A and B are disjoint subsets of X with the property that B is compact and X is locally compact at each point of A.

Proof. Let $X^* = X \cap \{\infty\}$. First assume that X^* is LH. Then there exists an open Hausdorff nbd V of ∞ in X^* Choose $A = V - \{\infty\}$, and $B = X^* - V$; then A and B give us the partition with the required condition.

Conversely assume that there exists a partition of X with the above mentioned properties. Since X is Hausdorff, B is closed in X, so that $V = A \cap \{\infty\}$ is an open set in X^*. It suffices to verify that V is a Hausdorff space in X^*. Since X is already known to be Hausdorff, it is therefore enough to check that an arbitrary point 'a' from A and the point '∞' have disjoint open nbds in X^*. The local compactness of X at the point 'a' gives an open nbd U of a in X such that $cl_X U$ is compact, so that $(X^* - cl_X U)$ is an open nbd of ∞ in X^* disjoint with U.

The above theorem provides a method for constructing examples of non-Hausdorff LH spaces as follows.

Theorem 4.2. Suppose X is a Hausdorff space which is not completely regular, in which there exists at least one point of X having a compact nbd; let A and B be disjoint compact subsets of X with the property that X is not locally compact at any point of B and X is locally compact at each point of A; then the space $(A \cup B)^*$ is a non Hausdorff LH space.

Proof. A straight forward consequence of Theorem 4.1.

It may be mentioned that the hypothesis regarding X in the above theorem is not vacuous. Consider for example the regular Hausdorff space X, constructed by Mysior [See[5]]. Here X stands for the closed upper half plane of R^2, alongwith an additional point 'a'. All the points (x,y) with y>0, are assumed to be isolated; the basic open nbds of (x,0) contain (x,0) and all but finitely many points from the union of the following two segments :

$I_x = \{(x,y) : 0 \le y < 2\}$ and $J_x = \{(x + y, y) : 0 \le y < 2\}$.

The basic open nbds of 'a' have the form :

$U_n(a) = \{a\} \cup \{(x,y) : x > n\}, n = 1,2,.....$. 'a' is the only point in X at which X is not locally compact.

5. EPIMORPHISMS IN THE CATEGORY LHAUS OF ALL LOCALLY HAUSDORFF SPACES

Definition 5.1. In a category $\mathcal{C}$ a morphism $f : A \to B$ is said to be an epi morphism, if for any object C and for any pair of morphisms $g : B \to C$, and $h : B \to C$ from $\mathcal{C}$ the relation $g_0 f = h_0 f$ implies $g = h$.

A well known result in the category HAUS of all Hausdorff spaces states that the epimorphisms in HAUS are precisely those continuous functions $f : X \to Y$ in HAUS that satisfy $cl_Y f(x) = Y$. This leads to the characterisation of the epimorphisms in LHAUS in terms of "L-denseness" which is :

Definition 5.2. A subset of a topological space X is said to be L-dense in X, if whenever there exists a locally closed subset of X containing it then the locally closed subset is none else than X itself; i.e. $A \subseteq X$ is L-dense, if whenever $A \subseteq L$, where L is locally closed in X, then $L = X$.

It is clear that every L-dense subset of X is also dense, but the converse is not necessarily true [in Example 1.3(2), consider the set $X - \{a,b\}$.]

The following is a direct consequence of theorem 2.5 :

Theorem 5.3. If two continuous maps $f,g : X \to Y$, defined from a space X into a LH space Y agree on a L-dense subset of X, then $f = g$.

A characterisation of the epimorphisms in LHAUS is as follows :

Theorem 5.4. In LHAUS, a morphism $f : X \to Y$ is an epimorphism if and only if, f(X) is L-dense in Y.

Proof. If f(X) is an L-dense subset of Y, then in view of Theorem 5.3 f is clearly an epimorphism.

Conversely assume that f(X) is not L-dense in Y. Then there exists a locally closed subset S of Y with the property : f(X) is a subset of S and S is aproper subset of Y.

To show that $f : X \to Y$ is not an epimorphism in LHAUS, we shall construct an LH space Z and a pair of continuous functions $g,h: Y \to Z$ with the property that $g \ne h$, although $g_o f = h_o f$.

Let $Y_j = Y \times \{j\}$, $j = 1,2$ and $h_j : Y \to Y_j$ be defined as $h_j(y) = (y,j)$, $j = 1.2$. Consider the disjoint topological sum $Y_1 + Y_2$. Let $i_j : Y_j \to Y_1 + Y_2$, $j = 1,2$ be the inclusion maps, and finally let z be the quotient space of $Y_1 + Y_2$, obtained on identifying for each point y in S, the points (y,1) and (y,2). Let $q : Y_1 + Y_2 \to Z$ be the natural quotient map. Now take $g = q_o i_1 {}_o h_1$ and $h = q_o i_2 {}_o h_2$. Then one can easily see that $g,h : Y \to Z$ are both continuous maps such that $g_o f = h_o f$, but $g \neq h$; indeed for any point 'y' from $Y - S$, $g(y) \neq h(y)$.

To complete the proof we need to check that Z is an LH space and this is done by an evident extension of the argument in [9], page 256.

We close this article after posing the following problems :

(i) What are the extremal monomorphisms in LHAUS ?

(ii) What are the epireflective subcategories of LHAUS ?

REFERENCES

1. C. E. Aull, Separation of bicompact sets, Math. Ann. 158(1965), 197-202.
2. C.E. Aull and W.J. Thron, Separation axioms between T_0 and T_1 spaces, Indagationes Math. 24(1962), 26-37.
3. N. Bourbaki, Elements of mathematics : General Topology, 1966.
4. R. E. Hoffman, On Weak Hausdorff spaces, Archiv. Der. Math., (Basel) 32 (No. 5) (1979), 487-504.
5. A. Mysior, A regular space not completely regular Proc. Amer. Soc. 81 (No. 4) (1981), 652-653.
6. T. K. Mukherjee, On Weak Hausdorff Spaces, Bull. Cal. Math. Soc. 58 (1966) 153-157.
7. M. G. Murdeshwar and S. A. Naimpally, Semi Hausdorff spaces, Canad. Math. Bull. (1966), 353-356.
8. W. J. Thron, Topological Structures, Holt, Rinehart and Winston, New York-Toronto, 1966.
9. R. C. Walker, Stone-cech compactification, Springer Verlag, 1974.
10. A Wilansky, Between T_1 and T_2, Amer. Math. Monthly. 74 (1967), 261-266.
11. Stephen Willard, General Topology, Addison Wesley Reading, 1970

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ON Γ S - ACT

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Abstract. In this paper we introduce the notion of operator semigroup of a ΓS-act over a Γ-semigroup (S, Γ) and using this notion we study strictly cyclic ΓS-act and faithful transitive ΓS-act. We obtain some necessary and sufficient conditions for S to have faithful strictly cyclic ΓS-act and faithful transitive ΓS-act.

1. INTRODUCTION

Let S and Γ be two nonempty sets. S is called a Γ-semigroup [6] if there exist mappings $S \times \Gamma \times S \to S$, written as $(a, \alpha, b) \to a \alpha b$, and $\Gamma \times S \times \Gamma \to \Gamma$, written as $(\alpha, a, \beta) \to \alpha a \beta$, satisfying the following identities : $(a\alpha b)\beta c = a(\alpha b \beta)c = a\alpha(b\beta c)$ for all $a,b,c \varepsilon S$ and $\alpha, \beta \varepsilon \Gamma$. We call a Γ-semigroup S both sided if it further satisfies the identities. : $\alpha a (\beta b \gamma) = (\alpha a \beta) b \gamma = \alpha (a \beta b) \gamma$ for all $\alpha, \beta, \gamma \varepsilon \Gamma$ and $a, b \varepsilon S$. Throughout this paper we consider only both sided Γ-semigroup and call it simply Γ-semigroup. Some earlier work on both sided Γ-semigroup viz. embedding a Γ-semigroup in a semigroup with zero, on Γ-semigroup with the right and left unities, on prime radical and on radicals of Γ-semigroup may be found respectively in [2], [3], [4], and [5]. We shall denote a both sided Γ-semigroup by (S, Γ).

According to Sen and Seth (1988) [7] a right Γ S-act is a triple (M, Γ, S) (M is a nonempty set and S is a Γ-semigroup) together with a mapping from $M \times \Gamma \times S \to M$ written as $(m, \alpha, s) \to m\alpha s$ satisfying $(m\alpha s)\beta t = m\alpha(s\beta t)$ for all $m \varepsilon M$, $\alpha, \beta \varepsilon \Gamma$, $s,t \varepsilon S$.

In this paper we introduce the notion of ΓS-act over a both sided Γ-semigroup. We also introduce the notion of operator semigroup of a ΓS-act and study ΓS-act via operator semigroup. In 1961 Tully published a paper [8] dealing with those representations of a semigroup S which are strictly cyclic and more particularly those which are transitive. Here we introduce the notions of cyclic and transitive ΓS-act over a both sided Γ-semigroup and obtain some necessary and sufficient conditions for S to have faithful strictly cyclic ΓS-act and faithful transitive ΓS-act. We obtain that a necessary and sufficient condition for S, where (S,Γ) is a commutative conditionally unitary Γ-semigroup, to have a faithful transitive ΓS-act is that S is a Γ-group.

2. PRELIMINARIES

Definition 2.1. [7] Let S be a Γ-semigroup. A right Γ-congruence on S is an equivalence relation ρ on S such that $a\rho b$ implies that $a\alpha c \rho b \alpha c$ for all $\alpha \varepsilon \Gamma$ and $c \varepsilon S$.

Definition 2.2. [7] Let S be a Γ-semigroup. A right Γ-congruence ρ on S is modular if there exist $e \varepsilon S$ and $\alpha \varepsilon \Gamma$ such that $e\alpha a \rho a$ for all $a \varepsilon S$. e is called a left identity modulo ρ relative to α.

Definition 2.3. [3] Let (S, Γ) be a Γ-semigroup; M and N be the left operator semigroup and the right operator semigroup of (S, Γ). If there exists an element $[e, \delta] \varepsilon M$ ($[\delta, e] \varepsilon N$) such that $e\delta x = x$ ($x\delta e = x$) for all $x \varepsilon S$ then S is said to have the left unity $[e, \delta]$ (right unity $[\delta, e]$) Similarly we can define the left unity and the ritght unity of Γ.

Definition 2.4. [3] A Γ-semigroup (S, Γ) is said to be a Γ-semigroup with unities if S has left unity and right unity which are also right unity and left unity of Γ respectively.

Definition 2.5. [5] Let (S, Γ) be a Γ-semigroup. S is said to be commutative if for any $a, b \varepsilon S$ and $\Upsilon \varepsilon \Gamma$, $a\Upsilon b = b\Upsilon a$. Similarly the commutativity of Γ may be defined.

Definition 2.6. [5] A Γ-semigroup (S, Γ) is said to be commutative if both S and Γ are commutative.

Definition 2.7. [3] A Γ -semigroup (S, Γ) is called a Γ -group if for every $a, a_1 \varepsilon S$ and $\alpha, \alpha_1 \varepsilon \Gamma$ there exist $b, b_1 \varepsilon S$ and $\beta, \beta_1 \varepsilon \Gamma$ such that $s = b\beta a\alpha s$, $\Upsilon = \Upsilon b\beta a\alpha$, $\Upsilon = \beta_1 b_1 \alpha_1 a_1$ $s = s\beta_1 b_1 \alpha_1 a_1$ for all $s \varepsilon S$ and $\Upsilon \varepsilon \Gamma$.

Definition 2.8 [6] Let (S, Γ) be a Γ-semigroup. A nonempty subset I of S called a left (right) ideal of S if $S \Gamma I \subseteq I$ ($I \Gamma S \subseteq I$). A nonempty subset I of S is called a two sided ideal or simply an ideal of S if it is a left ideal as well as a right ideal of S. The ideals of Γ are defined analogously.

3. RIGHT Γ S-ACT

Definition 3.1. A right ΓS-act is a triple (M,Γ,S), where M is a nonempty set and (S, Γ) is a Γ-semigroup, together with a mapping $(m, \alpha, a) \rightarrow m\alpha a$ from $M \times \Gamma \times S$ to M such that $(m\alpha a)\beta b = m(\alpha a\beta)b = m\alpha(a\beta b)$ for all $m \varepsilon M$, $\alpha, \beta \varepsilon \Gamma$, $a, b \varepsilon S$. We shall denote a right ΓS-act (M, Γ, S) by $M_{\Gamma S}$ and call it simply a ΓS-act.

Example 3.2. Let S be the set of all mxn matrices, Γ be the set of all $n \times m$ matrices and M be the set of all $r \times n$ matrices over the ring of integers. Then (S, Γ) is a Γ-semigroup and (M, Γ, S) is a right ΓS-act with respect to usual matrix multiplication.

Example 3.3. Let A, B, C be three nonempty sets. $S = M(A, B)$ denotes the set of all mappings from A to B, $\Gamma = M(B, A)$ denotes the set of all mappings from B to A and $T = M(A, C)$ denotes the set of all mappings from A to C. Let $f, g \varepsilon S$ and $\alpha \varepsilon \Gamma$. If $f\alpha g$ and $\alpha f\beta$ denote the usual composite mappings then (S, Γ) is a Γ-semigroup. Let $t \varepsilon T$. If $t\alpha f$ denotes the usual composite map, then we can show that (T, Γ, S) is a right ΓS-act $[(t\alpha f)(a) = t\alpha(f(a)), a \varepsilon A]$.

Let (S, Γ) be a Γ-semigroup and $M_{\Gamma S}$ be a right ΓS-act. We define a relation ρ on $\Gamma \times S$ as follows: $(\alpha, a) \rho (\beta, b)$ if and only if $m\alpha a = m\beta b$ for all $m \varepsilon M$. Obviously ρ is an equivalence relation. Let $[\alpha, a]$ denote the equivalence class containing (α, a).

Let $R = \{[\alpha, a] : a \varepsilon S, \alpha \varepsilon \Gamma\}$. We define multiplication on R as follows : $[\alpha, a][\beta, b] = [\alpha, a\beta b]$. The multiplication on R is well defined and R is a semigroup with respect to this multiplication.

This semigroup R is called the right operator semigroup of the right ΓS-act M. It is evident that $[\alpha, a][\beta, b] = [\alpha a\beta, b]$.

Lemma 3.4. Let (S, Γ) be a Γ-semigroup. A nonempty subset M is a right Γ S-act if and only if M is a right R-act.

Proof. Let M be a right Γ S-act. We define a mapping from M x R to M by $m[\alpha, a] = m\alpha a$. Obviously this mapping is well defined and M is a right R-act. We can prove the converse by defining $m\alpha a = m[\alpha, a]$.

Definition 3.5. A Γ S-homomorphism from a Γ S-act $M_{\Gamma S}$ into another Γ S-act $M'_{\Gamma S}$ is a mapping $\theta : M \to M'$ such that $\theta (m\alpha a) = \theta (m)\alpha a$ for all $m \varepsilon M$, $\alpha \varepsilon \Gamma$, $a \varepsilon S$. Moreover if θ is one-one and onto, then we call it Γ S-isomorphism.

Definition 3.6 Let $M_{\Gamma S}$ be a ΓS-act. An equivalence ρ on M is a Γ S-congruence if $x \rho y$ implies that $x\alpha a \; \rho \; y\alpha a$ for all $\alpha \varepsilon \Gamma$ and $a \varepsilon S$.

Let ρ be a Γ S-congruence on a Γ S-act $M_{\Gamma S}$. The factor Γ S-act $M_{\Gamma S/\rho}$ consists of all the ρ classes $m \rho$ of M and the product of $m \rho$ by an element a of S and an element α of Γ is defined by $(m \rho) \alpha a = (m\alpha a) \rho$. If ρ and σ are two ΓS-congruences on $M_{\Gamma S}$ such that $\rho \supseteq \sigma$ then we can define $\rho^1 = {}^{\rho}/_{\sigma}$ on $M'_{\Gamma S} = M_{\Gamma S/\sigma}$ as follows : $\rho' = \{(m_1 a, m_2 a) \varepsilon M^1_{\Gamma S} \times M'_{\Gamma S} : (m_1, m_2) \varepsilon \rho\}$. Since ρ is a Γ S-congruence on $M_{\Gamma S}$, ρ' is a Γ S-congruence $M^1_{\Gamma S}$.

Conversely if ρ^1 is a Γ S-congruence on $M^1_{\Gamma S/\rho}$ and we define σ on $M^1_{\Gamma S}$ by $\rho = \{(m_1, m_2) \varepsilon M'_{\Gamma S} \times M_{\Gamma S} : (m_1\sigma, m_2\sigma) \varepsilon \rho'\}$. then ρ is a Γ S-congruence on $M_{\Gamma S}$, $\rho \supseteq \sigma$ and $\rho' = \rho/\sigma$.

Lemma 3.7. Let $M_{\Gamma S}$ be a Γ S-act. An equivalence relation σ on M is a Γ S-congruence if and only if σ is an R-congruence on the R-act M_R.

Lemma 3.8. Let $M_{\Gamma S}$ be a Γ S-act. $M_{\Gamma S/\rho}$ is a factor Γ S-act if and only if $M_{R/\sigma}$ is a factor R-act.

Proof. The proof follows from the above lemma.

Proposition 3.9. If σ is a Γ S-congruence on a Γ S-act $M_{\Gamma S}$ over a Γ-semigroup (S, Γ), then the mapping $\rho \to \rho/\sigma$ is a lattice isomorphism of the lattice of all Γ S-congruences ρ on $M_{\Gamma S}$ containing σ and the lattice of all Γ S-congruences on the factor Γ S-act $M_{\Gamma S/\sigma}$.

Lemma 3.10. Let $M_{\Gamma S}$ and $M^1_{\Gamma S}$ be two Γ S-acts and $\theta : M \to M'$ be a map. θ is a Γ S-homomorphism if and only if θ is an R-homomorphism.

Proof. The proof follows from the definition 3.5.

Theorem 3.11. Let (S, Γ) be a Γ-semigroup and θ be a Γ S-homomorphism from a Γ S-act $M_{\Gamma S}$ onto another Γ S-act $M^1_{\Gamma S}$ over (S, Γ). Then $\theta \; 0 \; \theta^{-1} = \rho$ (i.e., $m_1 \rho m_2$, $m_1, m_2 \varepsilon M$ if and only if $\theta (m_1) = \theta (m_2)$) is a Γ S-congruence on $M_{\Gamma S}$. If σ is any Γ S-congruence on $M_{\Gamma S}$ such that $\sigma \subseteq \theta \; 0 \; \theta^{-1}$ then $\theta' (m \sigma) = \theta (m)$ $(m \varepsilon M)$ defines a ΓS-homomorphism θ' from $M_{\Gamma S/\sigma}$ onto $M'_{\Gamma S}$ such that $\theta' 0 \sigma^4 = \theta$, where σ^4 is the natural Γ S-homomorphism from M Γ S to $M_{\Gamma S}/\rho$. If $\sigma = \theta \; 0 \; \theta^{-1}$, then θ' is a Γ S-isomorphism.

Proof. The proof is a routine matter of checking, so we omit it.

Definition 3.12. Let $M_{\Gamma S}$ be a Γ S-act. A nonempty subset B of M is called a Γ S-subact if $B \Gamma S \subseteq B$.

Definition 3.13. By a decomposition of a Γ S-act $M_{\Gamma S}$ we mean a partition of M into a set $\{P_i : i \varepsilon I\}$ of mutually disjoint Γ S-subacts P_i. If no such partition is possible with $|I| > 1$, then $M_{\Gamma S}$ is said to be indecomposable.

Lemma 3.14. Let $M_{\Gamma S}$ be a Γ S-act over a Γ-semigroup (S, Γ). B is a Γ S-subact if and only if B is an R-subact.

Proof. Let B be a Γ S-subact. Let b ε B and [α, a] ε R. Then b[α, a] = bαa ε B Γ S ⊆B. So B is an R-subact. Similar is the converse.

Lemma 3.15. Let $M_{\Gamma S}$ be a Γ S-act over a Γ-semigroup (S, Γ). M is a decomposable Γ S-act if and only if M is a decomposable R-act.

Proof. The proof follows from the definition 3.13 and the Lemma 3.14.

Theorem 3.16. There is a one to one correspondence between the decompositions of a Γ S-act $M_{\Gamma S}$ over a Γ -semigroup (S, Γ) and the Γ S-congruences σ on $M_{\Gamma S}$ with the property that mρmαa for all mεM and aεS, α ε Γ.

Proof. The proof follows from the above lemmas.

Theorem 3.17. Every Γ S-act over a Γ -semigroup (S, Γ) is uniquely decomposable into indecomposable Γ S-subacts.

Proof. The proof follows from above lemmas and the Theorem 11.3 of [1].

4. STRICTLY CYCLIC Γ S-ACT

Definition 4.1. A right Γ -congruence ρ on S where (S, Γ) is a Γ -semigroup is modular if there exist eεS and α ε Γ such that eαaρa for all aεS. e is called a left identity modulo ρ relative to α.

If S has a left unity then every right Γ-congruence on S is modular.

Definition 4.2. A Γ S-act $M_{\Gamma S}$ is called strictly cyclic if there exist an element m in M and an element α in Γ such that M = mαS. Such an element m is called a strict generator of M.

Remark. Let (S, Γ) be a Γ-semigroup. Then S is a Γ S-act. We denote it by $S_{\Gamma S}$. It is clear that the Γ S-congruences on the Γ S-act $S_{\Gamma S}$ are precisely the right Γ-congruences on S. Thus if ρ is a right Γ-congruence on S where (S, Γ) is a Γ-semigroup, we can form the Γ S-act $S_{\Gamma S/\rho}$.

Proposition 4.3. Let M = mαS be a strictly cyclic Γ S-act. Define a relation ρ on S by a ρ b if and only if mαa = mαb. Then ρ is a right Γ-congruence on S, there is a left identity modulo ρ relative to α, and $M_{\Gamma S} \cong S_{\Gamma S/\rho}$.

Proof. Obviously ρ is a right Γ-congruence on S. Since mεM = mαS, there exists e ε S such that m = mαe. Then mαeαa = mαa for all a in S, whence eαa ρ a for all a in S. Thus e is a left identity modulo ρ relative to α. Define θ : S → M by θ (a) = mαa. Now θ (aβs) = mα (aβs) = (mαa) βs = θ (a) βs. So θ is a Γ S-homomorphism of $S_{\Gamma S}$ onto $M_{\Gamma S}$ with kernel $\theta\ 0\ \theta^{-1} = \rho$.

By Theorem 3.11, $\theta' : S_{\Gamma S/\rho} \rightarrow M_{\Gamma S}$ defined by $\theta'(a\rho) = \theta(a)$ is a Γ S-isomorphism. Hence $M_{\Gamma S} \cong S_{\Gamma S/\rho}$.

Theorem 4.4. Every strictly cyclic Γ S-act is isomorphic to $S_{\Gamma S/\rho}$ for some modular right Γ-congruence ρ on S. Conversely, if ρ is a modular right Γ-congruence on S. then $S_{\Gamma S/\rho}$ is a strictly cyclic Γ S-act.

Proof. First part follows from the proposition 4.3. Conversely, let ρ be a modular right Γ-congruence on S and let e be a left identity modulo ρ relative to some $\alpha \varepsilon \Gamma$. Then for any a in S, $(e\rho)\alpha a = (e\alpha a)\rho = a\rho$. Hence $e\rho$ is a strict generator of $S_{\Gamma S/\rho}$ i.e. $S_{\Gamma S/\rho}$ is a strictly cyclic Γ S-act.

Definition 4.5. Kernel Γ S-congruence K of a Γ S-act $M_{\Gamma S}$ is defined as follows : aKb if and only if $m\alpha a = m\alpha b$ for all $m \varepsilon M$ and for all $\alpha \varepsilon \Gamma$, $a, b \varepsilon S$.

Definition 4.6. A Γ S-act $M_{\Gamma S}$ is said to be faithful if the Kernel Γ S-congruence K of $M_{\Gamma S}$ is the identity relation on S i.e., $m\alpha a = m\alpha b$ for all $m \varepsilon M$ and for all $\alpha \varepsilon \Gamma$ implies that $a = b$.

Proposition 4.7. Let ρ be a right Γ-congruence on S where (S, Γ) is a Γ-semigroup, e a left identity modulo ρ relative to α and K be the Kernel Γ S-congruence of the Γ S-act $S_{\Gamma S/\rho}$. Then aKb if and only if $cYa\ \rho\ cYb$ for all $c \varepsilon S$ and for all $Y \varepsilon \Gamma$. Moreover, K is the largest left Γ-congruence (and therefore the largest 2-sided Γ-congruence) contained in ρ. In particular, $S_{\Gamma S/\rho}$ is faithful if and only if there is no left (equivalently no 2-sided) Γ-congruence contained in ρ except the identity relation. This is equivalent to the condition that $a = b$ whenever for all $c \varepsilon S$, $\alpha \varepsilon \Gamma$, $c\alpha a\ \rho\ c\alpha b$.

Proof. Let K be the Kernel Γ S-congruence of the Γ S-act $S_{\Gamma S/\rho}$. Then aKb if and only if $(c\rho)Ya = (c\rho)Yb$ i.e., $(c\gamma a)\rho = (cYb)\rho$ if and only if $cYa\ \rho\ cYb$ for all $c \varepsilon S$ and for all $Y \varepsilon \Gamma$. Obviously K is a left Γ-congruence on S. Let aKb. Then $e\alpha a\ \rho\ e\alpha b$, $e \varepsilon S$, whence $a\rho b$. Hence $K \subseteq \rho$. Now suppose that τ is any left Γ-congruence on S contained in ρ. Then $a\tau b$ implies that $c\alpha a\ \tau c\alpha b$ for all $c \varepsilon S$, $\alpha \varepsilon \Gamma$, so $c\alpha a \rho c\alpha b$. Hence aKb. This shows that τ is contained in K. Remaining part now follows from the Definition 4.6.

Corollary 4.8. A necessary and sufficient condition for S where (S, Γ) is a Γ-semigroup to have a faithful strictly cyclic Γ S-act is that there exists a right Γ-congruence ρ on S such that there is a left identity modulo ρ relative to α and ρ contains no left Γ-congruence except the identity relation.

Proof. The corollary follows from Theorem 4.4 and Proposition 4.7.

Corollary 4.9. A sufficient condition for S where (S, Γ) is a Γ-semigroup to have a faithful strictly cyclic Γ S-act is that S has a left unity. If S is commutative then this condition is also necessary.

5. TRANSITIVE Γ S-ACT

Definition 5.1. A Γ S-act $M_{\Gamma S}$ is called transitive if for all $m \varepsilon M$ and for all $\alpha \varepsilon \Gamma$, $m\alpha S = M$ or equivalently, for all $m \varepsilon M$, $m \Gamma S = M$.

Proposition 5.2. Let $M_{\Gamma S}$ be a Γ S-act. M is a transitive Γ S-act if and only if M is a transitive R-act.

Proposition 5.3. Let (S, Γ) be a Γ-semigroup. A Γ S-act $M_{\Gamma S}$ is transitive if and only if M contains no proper Γ S-subact.

Remark. A transitive Γ S-act is evidently indecomposable and also strictly cyclic.

Proposition 5.4. Let (S, Γ) be a Γ-semigroup and ρ be a right Γ-congruence on S. Then the following are equivalent.

(1) $S_{\Gamma S/\rho}$ is transitive.

(2) For all a, b ε S there exists c ε S with aαc ρ b for α ε Γ.

(3) Each ρ-class meets each right ideal of S.

Proof. (1) $==>$ (2) Let $S_{\Gamma S/\rho}$ be transitive. Let a, b ε S. Then a ρ, b ρ ε $S_{\Gamma S/\rho}$. Hence there exists an element c ε S such that (a ρ) α c = b ρ for α ε Γ, i.e., (aαc) ρ = b ρ i.e. aαc ρ b. (1)$==>$ (3) Consider a right ideal A and a ρ-class b ρ. Let a ε A. Then aαc ρ b i.e., aαc ε b ρ for some c ∈ S and α ε Γ. Again a α c ε A. So A meets the ρ-class b ρ.

Either of (2) and (3) implies (1).

Before we prove the next theorem we consider the following example.

Example. 5.5. Let S and Γ be two nonempty sets. Then (S, Γ) is a Γ -semigroup where we define aαb = b and βaα = α. Here S as well as Γ has left unity but neither S nor Γ has right unity.

Definition 5.6. A Γ -semigroup (S, Γ) is said to be conditionally left unitary if, whenever S (or Γ) has left unity then Γ (resp. S) has right unity. Similarly we can define conditionally right unitary Γ -semigroup.

Example 5.7. Every Γ-semigroup (S, Γ) with unities is conditionally left unitary as well as conditionally right unitary.

Example 5.8. Every cancellative Γ-semigroup is conditionally left unitary and conditionally right unitary (Proposition 3.8, [3]).

Example. 5.9. Let S be a commutative semigroup. Then S is a both sided Γ -semigroup where Γ = S and aαb and αaβ denote the usual product of the elements in the semigroup S. Then (S,Γ) is conditionally left unitary and also conditionally right unitary.

If (S, Γ) is a commutative Γ -semigroup then the notions of the conditionally left unitary and the conditionally right unitary Γ-semigroup coincide and in this case we call such a Γ - semigroup conditionally unitary Γ-semigroup.

Theorem 5.10. A necessary and sufficient condition for S where (S, Γ) is a commutative conditionally unitary Γ-semigroup, to have a faithful transitive Γ S-act is that (S, Γ) is a Γ-group.

Proof. Let S have a faithful transitive Γ S-act $M_{\Gamma S}$. Since a transitive Γ S-act is strictly cyclic it follows from the Proposition 4.3, that $M_{\Gamma S}$ is isomorphic to $S_{\Gamma S/\rho}$ where ρ is a right Γ -congruence on S and there is a left identity modulo ρ relative to α i.e., eαa ρ a for all a ε S. Again, from the Proposition 5.4 (2), it follows that for all a, b ε S there exists c ε S with aαc ρ b.

Since $S_{\Gamma S/\rho}$ is faithful it follows from the Proposition 4.7 that ρ is the identity relation. Hence by the commutativity of (S, Γ) we say that S is simple and S has unity (left as well as right). Since (S, Γ) is conditionally unitary, Γ has also unity (left as well as right). Since S is simple, it follows from the Theorem 4.3 of [3], that M is simple and this implies that Γ is simple (Theorem 4.10, [3]). Thus S and Γ both are simple. Hence by the Proposition 5.7 of [3], it follows that (S, Γ) is a Γ -group.

Conversely, let (S, Γ) be a commutative Γ -group. Then S is simple. Obviously S is a Γ S-act. Now from the simplicity of S, it follows that the Γ S-act S is transitive. Also, since S has unities, it follows that Kernel Γ S-congruence K of $S_{\Gamma S}$ is the identity relation i.e., the Γ S-act $S_{\Gamma S}$ is faithful. Hence S has the faithful transitive Γ S-act $S_{\Gamma S}$.

REFERENCES

1. A. H. Clifford and G. B. Preston, The algebraic theory of semigroups, Amer. Math. Soc. 1 (1961), 2 (1967).
2. T. K. Dutta and N. C. Adhikari, Embedding a Γ -semigroup in a semigroup with zero, Bull. Cal. Math. Soc. 86, (No. 1) (1994), 1-6.
3. T. K. Dutta and N. C. Adhikari, On Γ -semigroup with the right and left unities, Soochow Jr. of Math. 19 (No. 4) (1993) 461-474.
4. T. K. Dutta and N. C. Adhikari, On Prime radical of Γ -semigroup, Bull. Cal. Math. Soc. 86 (No. 5) (1994), 437-444.
5. T. K. Dutta and N. C. Adhikari, On the radicals of Γ -semigroup, Math. Soc. 88 (No-3) (1996), 189-194.
6. M. K. Sen and N. K. Saha, On Γ -semigroup I, Bull. Cal. Math Soc. 78 (No. 3) (1986), 180-186.
7. M. K. Sen and A. Seth, Radical of Γ -semigroup, Bull. Cal. Math Soc. 80 (No. 3) (1988), 189-196.
8. E. J. Tully, Jr. Representation of a semigroup by transformations acting transitively on a set, Amer. J. Math. 83 (1961), 533-541.

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CATEGORY ANALOGUES OF TWO THEOREMS OF STEINHAUS

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1. INTRODUCTION

Let $A, B \subseteq R^n$ (the n-dimensional euclidean space). The 'distance set' (resp, 'difference set') of A and B is the set $\{||x - y||; x \varepsilon A, y \varepsilon B\}$ ($\{x - y, y - x ; x \varepsilon A, y \varepsilon B\}$), where $||.||$ stands for the usual norm in R^n. For $n = 1$, if we designate by R the set of real numbers and $A, B \subseteq R \backslash \{0\}$, then on taking mutual ratios instead of distances or differences between points we arrive at the notion called the ratio set of A and B and which is $\{\frac{x}{y}, \frac{y}{x}; x \varepsilon A, y \varepsilon B\}$. Steinhaus [3] proved many theorems related to the notions of distance (resp, difference) between points in sets of positive lebesgue measures in R and similar theorems were established by Bose-Majumdar [1] with ratios for subsets having non-zero abscissae. We state below two such results by Steinhaus [3].

Theorem A. If $\{A_n\}_{n=1}^{\infty}$ is any infinite sequence of lebesgue measurable sets of positive measures in R, then there exists an infinite sequence $\{a_n\}_{n=1}^{\infty}$ of distinct points such that $a_n \varepsilon A_n$ and their mutual distances are all rationals ($n=1, 2,3, \ldots$)

Theorem B. If E is any infinite lebesgue measurable subset in R, then there exists an enumerable set P composed of points whose mutual distances are rationals and a set Z of measure zero such that $P \subseteq E \subseteq P' \cup Z$, where P' represents the derived set of P.

In the case $A_n \subseteq R \backslash \{0\}$ ($n = 1, 2, 3, \ldots$), exact analogues of the above two therems with mutual distance replaced by mutual ratio holds. These were established by Bose-Majumdar [1]. However, it is easy to generalize theorem 1 and 2 in R^n and in situations where mutual difference between points arises.

A set $E \subseteq R^n$ is said to have the 'property of Baire' [2] if it can be expressed as the symmetric difference of an open set and a set of first category. The definition is applicable also in any topological space.

It is well-known that Baire-property resembles measurability of sets in many respects. So it is natural to ask whether the above results of Steinhaus (and also those by Bose-Majumdar) flows in the same vein when lebesgue measurability is replaced by Baire-property of sets. In the present context it is found that the answer to this runs in the affirmative and is shown in the general context of topological vector spaces. The articulation is carried out in such a manner so as to combine the representations of steinhaus and Bose-Majumdar into one whole.

Since finite dimensional euclidean spaces are particular cases of topological vector spaces, results of Steinhaus (also there corresponding forms with differences between points) and those of Bose-Majumdar are only some special cases of our constructions.

Let $(X, \oplus, \odot)$ be any topological vector space over a field K (which may be either R or the set $\mathbb{C}$ of complex numbers); moreover for our present purpose we choose X to be locally compact Hausdorff. We also denote (1) by O and 1 the additive identity (the origin) in X and the multiplicative identity in K. (ii) by $A \backslash B$ and $A \Delta B$ the difference and symmetric difference of two sets $A, B \subseteq X$.

(iii) by $\bar{A}$ the closure of any set A in X and (iv) for any set $L \subseteq K$ and $Y \subseteq X$ by $L \odot Y$ the set $\{\lambda \odot y ; \lambda \varepsilon L, y \varepsilon L\}$ and in case L consists entirely of invertible elements by L^{-1} the set $\{\lambda^{-1}, \lambda \varepsilon L\}$ and (v) for any set $A \subseteq X, x \varepsilon X$ and $\lambda \varepsilon K$ by $A_x^{(\lambda)}$ the set $\{(\lambda \odot y) \oplus x ; y \varepsilon A\}$. A set $E \subseteq X$ will be said to have the Baire-property if $E = G \Delta P$ where G is open and P is a set of first category in X. Equivalently, E has the Baire-property if $E=(G \backslash P) \cup Q$ where G is open and P, Q are first category sets in X. However, from now on all sets considered shall be either subsets of K or X. This would not be repeatedly mentioned for it will turn out clear from the context which situation we are refering to.

Theorem I. Let $\{B_n\}_{n=1}^{\infty}$ be any infinite sequence of second category sets with Baire-property in X. Let $F \subseteq K$, $D \subseteq X$ be countable dense subsets in the respective spaces. Then there exist sequences $\{\lambda_n\}_{n=1}^{\infty} \subseteq F$, $\{d_n\}_{n=1}^{\infty} \subseteq D$ and $\{x_n\}_{n=1}^{\infty} \subseteq X$ consisting of distinct elements such that $x_n \varepsilon A_n$ (n = 1, 2, 3, ...) and satisfying

$$(\lambda_1 \odot x_1) \oplus d_1 = (\lambda_2 \odot x_2) \oplus d_2 = \ldots = (\lambda_n \odot x_n) \oplus d_n = \ldots$$

Proof. We choose in addition to the sets B_n (n = 1, 2, 3, ...) an auxilliary set A_0 of second category with Baire property. We write $A_0 = (G_0 \backslash P_0) \cup Q_0$, G_0 is open P_0, Q_0 are sets of first category and such that $\bar{G}_0$ is compact in X (the choice is justified since X is locally compact hausdorff. Moreover as we shall see this additional choice in no way does affect the outcome of the result). We also set $A_n = B_n \backslash (F \backslash \{0\})^{-1} \odot D$ (n = 1, 2, 3, ...). Since $(F \backslash \{0\})^{-1} \odot D$ is countable, each A_n is again a second category set with the Baire-property. As usual we write $A_n = (G_n \backslash P_n) \cup Q_n$ where G_n is open and P_n, Q_n are first category sets (n = 1, 2, 3, ...). Since every set of first category is the union of a countable collection of nowhere dense sets, therefore

$P_n = \bigcup_{j=1}^{\infty} F_j^{[n]}$ where $F_j^{[n]}$ are nowhere dense sets (n = 0, 1, 2, 3, ... ; j = 1, 2, 3, ...).

To begin with, we consider the set $G0_{(p_0)}^{(\lambda_0)} \backslash \overline{F_1^{(0)}}_{(p_0)}^{(\lambda_0)}$ where is $\lambda_0 = 1$, $p_0 = 0$ Since the closure of every nowhere dense set is again nowhere dense, the set $G0_{(p_0)}^{(\lambda_0)} \backslash \overline{F_1^{(0)}}_{(p_0)}^{(\lambda_0)}$ non-empty open and so contains a closed neighbourhood H_0 of some point in X (since X is locally compact). Let J_0 be the interior of H_0. Since every neighbourhood of the origin is absorbing, F and D are dense respectively in K and x and the mapping $x \to \lambda \odot x$ is a homeomorphism, we can choose $\lambda_1 \varepsilon F$ and $p_1 \varepsilon D$ (distinct from λ_0 and p_0) such that $J0 \cap G1_{(P_1)}^{(\lambda_1)} \neq \phi$ and consequently it contains a basic open set O_1.

We next consider the set $O_1 \setminus \overline{F_2^{(0)}}{}^{(\lambda_0)}_{(P_0)} \bigcup \overline{F_1^{(1)}}{}^{(\lambda_1)}_{(P_1)}$. By the same argument as stated in the begining of the first step, the above set is non-empty open and so contains a closed neighbourhood H_1 of some point in X. Let J_1 be the interior of H_1. Again by the same reasoning as stated near the end of the first step, we can choose $\lambda_2 \varepsilon F$ and $p_2 \varepsilon D$ (distinct from λ_0, λ_1 and p_0, p_1 such that $J_1 \cap G^{(\lambda_2)}_{2(P_2)} \neq \phi$ and consequently it contains a basic open set O_2.

We next consider the set $O_2 \setminus \overline{F_3^{(0)}}{}^{(\lambda_0)}_{(P_0)} \bigcup \overline{F_2^{(1)}}{}^{(\lambda_1)}_{(P_1)} \bigcup \overline{F_1^{(2)}}{}^{(\lambda_2)}_{(P_2)}$ By the same argument as above, the set is non-empty open and so contains a closed neighbourhood H_2 of some point in X. Let J_2 be the interior of H_2. Again on the same lines as in above we can choose $\lambda_3 \varepsilon F$ and $p_3 \varepsilon D$ (distinct from λ_0, λ_1, λ_2 and p_0, p_1, p_2) such that $J_2 \bigcap G_{3(P_3)}^{(\lambda_3)}$ and consequently it contains a basic open set O_3.

Likewise we begin the n-th state starting with the set

$O_{n-1} \setminus \bigcup \overline{F_n^{(1)}}{}^{(\lambda_0)}_{(P_0)} \bigcup \overline{F_{n-1}^{(1)}}{}^{(\lambda_1)}_{(P_1)} \bigcup \dots \bigcup \overline{F_1^{(n-1)}}{}^{(\lambda n-1)}_{(P_{n-1})}$ where λ_1, λ_2, ……. λ_{n-1} (ε F and p_1, p_2, ………, p_{n-1} (ε D) are distinct elements and O_{n-1} is a basic open set contained in $J_{n-2} \bigcap G_{n-1(P_{n-1})}^{(\lambda_{n-1})}$. As in above the set $O_{n-1} \setminus \bigcup \overline{F_n^{(0)}}{}^{(\lambda_0)}_{(P_0)} \bigcup \overline{F_{n-1}^{(1)}}{}^{(\lambda_1)}_{(P_1)} \bigcup \cdot\cdot \bigcup \overline{F_1^{(n-1)}}{}^{(\lambda n-1)}_{(P_{n-1})}$ being non-empty open contains a closed neighbourhood H_{n-1} of some point in X. Let J_{n-1} be the interior of H_{n-1}. Again by the same arguments as followed in the previous procedures, we can choose $\lambda_n \varepsilon F$ and $p_n \varepsilon D$ (distinct from λ_1, λ_2, …,λ_{n-1} and p_1, p_2 …. p_{n-1}) such that $J_{n-1} \bigcap G_{n(p_n)}^{(\lambda_n)} \neq \phi$ and consequently it contains a basic open set O_n.

If the above procedure is continued indefiniltely, we get a decreasing sequence $\{J_n\}_{n=0}^{\infty}$ of closed neighbourhoods of points in X each of which is contained in the compact set $\overline{G}_0$.

Since X is Hausdorff if follow that $\bigcap\limits_{n=0}^{\infty} J_n \neq \phi$. But

$$\bigcap_{n=0}^{\infty} J_n \subseteq \bigcap_{j=1}^{\infty} (G_{0\,(p_0)}^{(\lambda_0)} \setminus \overline{F_j^{(0)}}{}^{(\lambda_0)}_{(P_0)}) \bigcap \{\bigcap_{n=1}^{\infty} \bigcap_{j=1}^{\infty} (G_{n(p_n)}^{(\lambda_n)} \setminus \overline{F_j^{(n)}}{}^{(\lambda_n)}_{(P_n)})\}$$

$$\subseteq (G_{0(p_0)}^{(\lambda_0)} \setminus \bigcup_{j=1}^{\infty} \overline{F_j^{(0)}}{}^{(\lambda_0)}_{(P_0)}) \bigcap \{\bigcap_{n=1}^{\infty} (G_{n(p_n)}^{(\lambda_n)} \setminus \bigcup_{j=1}^{\infty} \overline{F_j^{(n)}}{}^{(\lambda_n)}_{(P_n)})\}$$

$$\subseteq (G_{0(p_0)}^{(\lambda_0)} \setminus P_{0(p_0)}^{(\lambda_0)}) \bigcap \{\bigcap_{n=1}^{\infty} (G_{n(P_n)}^{(\lambda_n)} \setminus P_{n(P_n)}^{(\lambda_n)})\}$$

$$\subseteq A_{0(p_0)}^{(\lambda_0)} \bigcap A_{1(p_1)}^{(\lambda_1)} \bigcap \cdots \bigcap A_{n(p_n)}^{(\lambda_n)} \bigcap \cdots\cdots$$

Hence the set $\bigcap_{n=1}^{\infty} A_{n(p_n)}^{(\lambda_n)} \neq \phi$

Therefore there exist sequences $\{\lambda_n\}_{n=1}^{\infty} \subseteq F$, $\{d_n\}_{n=1}^{\infty} \subseteq D$ elements of which are distinct and also a sequence $\{x_n\} \subseteq X$ where $x_n \varepsilon A_n \subseteq B_n$ (n=1, 2, 3 ...) such that the relation $(\lambda \odot x_1) \odot d_1 = (\lambda_2 \odot x_2) \oplus d_2 = \ldots\ldots = (\lambda_n \odot x_n) \oplus d_n = \ldots\ldots$ is satisfied.

The elements $x_1, x_2 \ldots x_n \ldots$ are also distinct as by our choice $x_n \varepsilon A_n = B_n \setminus (F \setminus \{P_0\})^{-1} \odot D$ (n=1, 2, 3...). This proves the theorem.

In our next theorem, in addition to being locally compact and Hausdorff we also consider our topological vector space X to be second countable without the origin o being a point of isolation. As in theorem 1, we here also choose countable sets $F \subseteq K$ and $D \subseteq X$ dense in the respective spaces.

Theorem 2. Let B be a non-empty set with Baire-property. Then there exist a countable set $P = \{a_1, a_2, \ldots, a_n \ldots\}$ and a set H of first category such that $P \subseteq A \subseteq P' \cup H$ where P′ is the derived set of P and there exist $\lambda_1, \lambda_2 \ldots, \lambda_n, \ldots$ in K and $d_1, d_2 \ldots d_n$, in D which satisfy the relation.

$$(\lambda \odot a_1) \oplus d_1 = (\lambda_2 \odot a_2) \oplus d_2 = \ldots\ldots = (\lambda_n \odot a_n) \oplus d_n = \ldots\ldots$$

Proof. If B is of first category the choice of any countable set $P \subseteq B$ for which the above relation is satisfied fulfills our purpose. Let B be of second category. Let $\{U_n\}_{n=1}^{\infty}$ be the countable base in X. From the sets $U_n \cap B$ (n = 1, 2, 3 ....), we suppress those terms that are sets of first category. We are therefore left with a subsequence. $\{U_{n_K} \cap B\}_{K=1}^{\infty}$ whose terms are second category sets with Baire-property. Now using theorem 1, we can find infinite sequences $\{a_K\}_{K=1}^{\infty} \subseteq B$, $\{\lambda_K\}_{K=1}^{\infty} \subseteq F$ and $\{d_K\}_{K=1}^{\infty} \subseteq D$ of distinct points where $a_K \varepsilon U_{n_K} \cap B$ (k = 1, 2, 3 ....) and the relation $(\lambda_1 \odot a_1) \oplus d_1 = (\lambda_2 \odot a_2) \oplus d_2 = \ldots\ldots = (\lambda_n \odot a_n) \oplus d_n = \ldots.$ is satisfied. We set $P = \{a_1, a_2 \ldots, a_n, \ldots\}$ and let H be the union of those sets that are suppressed. Clearly H is a set of first category. Also P is countable since the origin being not an isolated point no other point of X is so. Let $\xi \varepsilon B \setminus H$. Since none of the sets from $U_n \cap A$ containing ξ has been suppressed, there exists a subsequence $\{U_{n_{k_j}}\}_{j=1}^{\infty}$ of basic open sets such that $\xi = \lim_{j \to \infty} a_{n_{K_j}}$. Therefore $\xi \varepsilon P'$ and hence $B \subseteq P' \cup H$. Therefore $P \subseteq B \subseteq P' \cup H$. This completes the proof of the theorem.

Remark 1. If B is a set of scecond category, the sequences $\{a_k\}_{k=1}^{\infty}$, $\{\lambda_k\}_{k=1}^{\infty}$, and $\{d_k\}_{k=1}^{\infty}$ can be chosen so as to consist of distinct points. Since, X is second countable the condition that o is not an isolated point can be replaced by the fact that X is an uncountable set.

Remark 2. If in particular, we set $A^{(\lambda)}$ and $A_{(x)}$ $(\lambda \varepsilon K, x \varepsilon D)$ as the sets $\{\lambda \oplus y, y \varepsilon A\}$ and $\{\lambda \odot y, y \varepsilon A\}$, then on using them instead of $A_x^{(\lambda)}$ in the proofs of theorem 1 and theorem 2 we arrive at two special forms of each of the above two theorems. In case X is the real line with its usual topology and vector space structure, K = D = Q (the set of rationals). these respectively are the Baire-category analogues of the above two theorems (theorem A and theorem B) of Steinhaus [3] and also their corresponding forms with ratios, whose measure-theoritic counterparts are the results of Bose-Majumdar [1]. Also a close introspection into the nature of the above proofs yields that a little technical generalization of theorem 1 and theorem 2 may be done on assuming $F^{-1} \odot D$ a set of first category in X.

REFERENCES

1. N. C. Bose-Majumdar, On some properties of sets with positive measures, Annali dell' Universita di Ferrara Scienze Matematiche, X, 1.
2. J. C. Oxtoby, Measure and Category, Springer-Verlag, 1980.
3. H. Steinhaus, Sur les distances des points des ensembles de measure positive, Fund Math 1 (1920), 93-104.

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REPRESENTATIONS OF Γ -SEMIGROUPS AND REGULAR Γ -SEMIGROUPS

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Abstract. In this paper we prove that any Γ -semigroup is monomorphic to the Γ -semigroup of mappings from a set into another set. We also give a faithful representation of regular Γ -semigroup which is closely analogous to Lallement's [2] faithful representation of regular semigroup.

1. INTRODUCTION

We know that the set of all bijective mappings from a nonempty set onto itself forms a group and the set of all mappings from a non-empty set into itself froms a semigroup under the composition of mappings as binary operation. Let us slightly change the situation by taking two different nonempty sets A and B. Let S be the set of all mappings from the set A into the set B. If $f, g \in S$ then the usual composition of f and g is not defined. But if α be a mapping from the set B into the set A then the usual composition of mappings $f\alpha g$ is defined and $f\alpha g$ belongs to S.

Moreover if Γ be the collection of all mappings from the set B into the set A then for any $f, g, h \in S$ and $\alpha, \beta \in \Gamma$, $(f\alpha g)\,\beta h = f\alpha\,(g\beta h)$. Keeping an eye on this very example Γ -semigroup is defined in [4] as follows : Let S and Γ be two non empty sets and μ be a mapping from $S \times \Gamma \times S$ into S. Let the element $(a, \alpha, b)\,\mu$ be denoted by $a\alpha b$. If $(a\alpha b)\,\beta c = a\alpha\,(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Then (S, Γ, μ) is called a **Γ -semigroup,** we denote this Γ -semigroup simply by (S, Γ). A Γ -semigroup (S, Γ) is said to be **regular** if for every element $a \in S$ we have $a \in a\Gamma S\, \Gamma a$ where $a\,\Gamma\, S\, \Gamma\, a = \{a\alpha b\beta a : b \in S \text{ and } \alpha, \beta \in \Gamma\}$. Let (S, Γ) be a Γ -semigroup and $\alpha \in \Gamma$. Now if we define $a o b = a\alpha b$ for all $a, b \in S$ then (S, o) becomes a semigroup We denote this semigroup by S_α. Let (S, Γ) and (S', Γ') be two Γ -semigroups. A pair of mappings (f, g) such that $f : S \to S'$ and $g: \Gamma \to \Gamma'$ is said to be a **homomorphism** from (S, Γ) into (S', Γ') if $(a\alpha b)f = (af)\,(\alpha g)\,(bf)$ for all $a, b \in S$ and $\alpha \in \Gamma$. If both f and g are one to one then (f, g) is said to be a **monomorphism.** In a Γ -semigroup (S, Γ) $\mathcal{L}$ and $\mathcal{R}$-relations are defined in [1] as follows : for $a, b \in S$ we say $a \mathcal{L} b$ iff $S\,\Gamma\, a \cup \{a\} = S\,\Gamma\, b \cup \{b\}$ and $a \mathcal{R} b$ iff $a\,\Gamma\, S \cup \{a\} = b\,\Gamma\, S \cup \{b\}$. Obviously $\mathcal{L}$ and $\mathcal{R}$ are equivalence relations on the set S.

2. A REPRESENTATION OF Γ-SEMIGROUPS

In this section we first give a list of examples to illustrate the diversity of mathematical systems that satisfy the axioms of Γ -semigroup. Then we give a representation theorem for Γ -semigroups.

Example 2.1. Let A and B be two non empty sets, $S = T(A, B)$ be the set of all mappings from the set A into the set B and $\Gamma = T(B, A)$ be the set of all mappings from the set B into the set A The usual mapping product of two elements of S cannot be defined. But if we take f, g from S and α from Γ then the usual mapping product $f\alpha g$ is defined. Also we find that $f\alpha g \varepsilon S$ and $(f\alpha g)\beta h = f\alpha(g\beta h)$ for all $f, g, h \varepsilon S$ and $\alpha, \beta \varepsilon \Gamma$. Thus (S, Γ) becomes a Γ-semigroup. We call this Γ-semigroup "The Γ-semigroup $T(A, B)$".

Example 2.2. Let S be the set of all m x n matrices and Γ be the set of all n x m matrices over the field of real numbers. If $A\alpha B$ denotes the usual matrix multiplication for $A, B \varepsilon S$ and $\alpha \varepsilon \Gamma$ then $A\alpha B \varepsilon S$ and $(A\alpha B)\beta C = A\alpha(B\beta C)$ for all $A, B, C \varepsilon S$ and $\alpha, \beta \varepsilon \Gamma$. Consequently (S, Γ) is a Γ-semigroup.

Example 2.3. Let S denote the set of all integers of the form $4n+3$ (where n is any integer) and Γ denote the set of all integers of the form $4n+1$ (where n is any integer). If $a\alpha c$ is defined as $a\alpha c = a+\alpha+c$ for all $a, c \varepsilon S$ and $\alpha \varepsilon \Gamma$ then (S, Γ) is a Γ-semigroup.

Example 2.4. Let S be a semigroup and I and Λ be two non-empty sets. Let Γ be the set of all Λ x I matrices over S and $M = I \times S \times \Lambda$. Let (i,x,λ), (j,y,μ) be any two elements of M and $P = (P_{\lambda j}) \in \Gamma$. If we define $(i,x,\lambda)P(j,y,\mu) = (i, xp_{\lambda j}y, \mu)$, then we can show that (M, Γ) is a Γ-semigroup. If Γ contains only one Λ x I matrix P then this Γ-semigroup is same as the Rees matrix semigroup over S [3, P-111].

Example 2.5. A semigroup can be considered as a Γ-semigroup in the following sense. Let S be an arbitrary semigroup. Let 1 be a symbol not representing any element of S. Let us extend the given binary operation in S to $S \cup \{1\}$ by defining $11=1$, $1a=a1=a$ for all a in S. It can be shown that $S \cup \{1\}$ is a semigroup with identity element 1. Let $\Gamma = \{1\}$. Putting $ab = a1b$ we can show that the semigroup S is a Γ-semigroup where $\Gamma = \{1\}$.

From example 2.5 it is clear that a Γ-semigroup can be considered as a generalisation of semigroup. Next we prove that any Γ-semigroup is monomorphic to the Γ-semigroup $T(A, B)$ of all mappings from a suitable set A into another set B.

Theorem 2.1. If (S, Γ) be a Γ-semigroup, $A = (S \times \Gamma) \cup \{1\}$ and $B=S$, where 1 is a symbol not belonging to S or Γ, then the Γ-semigroup (S, Γ) is monomorphic to the Γ-semigroup $T(A, B)$.

Proof. For each $a \varepsilon S$ let us define $\wp_a : (S \times \Gamma) \cup \{1\} \rightarrow S$ by $(x, \alpha)\wp_a = x\alpha a$ for all $x \varepsilon S$, $\alpha \varepsilon \Gamma$ and $1\wp_a = a$. For each $\alpha \varepsilon \Gamma$ let us define a mapping $\lambda_\alpha : S \rightarrow (S \times \Gamma) \cup \{1\}$ by $x\lambda_\alpha = (x, \alpha)$.

Then $(x, \beta)\wp_a\lambda_\alpha\wp_b = (x\beta a)\lambda_\alpha\wp_b = (x\beta a, \alpha)\wp_b = (x\beta a)\alpha b = x\beta(a\alpha b) = (x, \beta)\wp_{a\alpha b}$ and $1\wp_a\lambda_\alpha\wp_b = a\lambda_\alpha\wp_b = (a,\alpha)\wp_b = a\alpha b = 1\wp_{a\alpha b}$. Thus $\wp_a\lambda_\alpha\wp_b = \wp_{a\alpha b}$. Now we define a mapping $\phi : S \rightarrow T(A, B)$ by $ao = \wp_a$ and another mapping $\psi : \Gamma \rightarrow T(B, A)$ by $\alpha\psi = \lambda_\alpha$. Then $(a\alpha b)\phi = \wp_{a\alpha b} = \wp_a\lambda_\alpha\wp_b = (a\phi)(\alpha\psi)(b\phi)$. Therefore (ϕ, ψ) is a homomorphism from (S, Γ) into $(T(A, B), T(B, A))$. Moreover $a\phi = b\phi$ implies that $\wp_a = \wp_b$. Hence $1\wp_a = 1\wp_b$ which in turn implies that $a = b$. Again $\alpha\Psi = \beta\Psi$ implies that $\lambda_\alpha = \lambda_\beta$. Then for all $x \varepsilon S$, $x\lambda_\alpha = x\lambda_\beta$ and this gives $(x, \alpha) = (x, \beta)$. So we have $\alpha=\beta$. Thus (ϕ, ψ) is a monomorphism from (S, Γ) into $(T(A, B), T(B, A))$.

3. A REPRESENTATION OF REGULAR Γ -SEMIGROUPS

Here first we consider few examples to bring the idea of regular Γ -semigroup into focus.

Example 3.1. Let A = {1, 2, 3} and B = {4, 5} be two non empty sets. Let S = {x, y, z}, Γ = {α, β} where x, y, z are mappings from the set A into the set B and α, β are mappings from the set B into the set A. They are defined by

$$x = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \end{pmatrix}, y = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 5 & 5 \end{pmatrix}, z = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 4 & 5 \end{pmatrix}, \alpha = \begin{pmatrix} 4 & 5 \\ 1 & 1 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$$

Where $x = \begin{smallmatrix} 123 \\ 444 \end{smallmatrix}$ means that $1x = 4$, $2x = 4$

$3x = 4$ and others are similarly defined.

The composition table for the Γ -semigroup (S, Γ) is given by the following table where xαy means the usual composition of mappings.

S / Γ	α	β	Γ / S
x	x	x	x
x	y	y	y
x	x	x	z
y	x	x	x
y	y	y	y
y	x	y	z
z	x	x	x
z	y	y	y
z	x	z	z

Here $x\alpha y\,\beta x = x$, $y\alpha z\,\beta y = y$, $z\beta z\,\beta z = z$.

Therefore (S, Γ) is a regular Γ -semigroup. But we note that though (S, Γ) is a regular Γ -semigroup, S_α is not a regular semigroup, for there is no element p in S for which $z\alpha p\alpha z = z$ holds.

Also we note that S_β is a regular semigroup.

Example 3.2. Let M and N be two vector spaces over a field F. Then M x N is a vector space over the field F if we define for (m_1, n_1), (m_2, n_2) ε M x N and α ε F, $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$ and $\alpha (m_1, n_1) = (\alpha m_1, \alpha n_1)$.

A linear relation from M into N is defined as a subspace of $M \times N$ and the set of all such linear relations is denoted by $LR(M, N)$. Let $S = LR(M, N)$ and $\Gamma = LR(N, M)$. For $A, B \,\varepsilon\, S$ and $Y \varepsilon \Gamma$ we define $AYB = \{(m, n) \,\varepsilon\, M \times N$: there exists $(n_1, m_1) \,\varepsilon\, Y$ for which $(m, n_1) \,\varepsilon\, A$ and $(m_1, n) \,\varepsilon\, B\}$.

Then it can be shown that (S, Γ) is a Γ-semigroup. For $A \varepsilon LR\,(M, N)$ we define $A^{-1} = \{ (n, m) \,\varepsilon\, N \times M : (m, n) \,\varepsilon\, A \}$. Then we can show that $A^{-1} \,\varepsilon\, LR\,(N,M)$. Also $AA^{-1}A = A$. Hence A is regular for all $A \,\varepsilon\, LR\,(M, N)$. Thus (S, Γ) is a regular Γ-semigroup.

Let (S, Γ, μ) be a regular Γ-semigroup which we denote by (S, Γ). Let us define* . $S \times \Gamma \times S \rightarrow S$ by $(a, \alpha, b)^* = b\alpha a$ for $a, b \,\varepsilon\, S$, $\alpha \,\varepsilon\, \Gamma$ where $b\alpha a = (b, \alpha, a)\,\mu$. We can show that $(S, \Gamma, ^*)$ is also a regular Γ-semigroup. We denote this Γ-semigroup by $(S, \Gamma)^*$.

Let (S, Γ) and $(S'\ \Gamma')$ be two regular Γ-semigroups. Let us define $\mu : (S \times S') \times (\Gamma \times \Gamma') \times (S \times S') \rightarrow S \times S'$ by $((s, s'), (\alpha, \alpha'), (t, t'))\, \mu = (s\alpha t, s'\, \alpha'\, t')$ for all $(s, s'), (t, t') \,\varepsilon\, S \times S'$ and $(\alpha, \alpha') \,\varepsilon \Gamma \times \Gamma'$. Then we can show that $(S \times S', \Gamma \times \Gamma', \mu)$ is a regular Γ-semigroup. We denote this Γ-semigroup by $(S, \Gamma) \times (S', \Gamma')$.

Example 3.3. Let $S = \mathcal{P}T\,(A, B)$ denote the set of all partial mappings including the empty mapping from a non-empty set A into another non-empty set B and $\Gamma = \mathcal{P}T\,(B, A)$. Then it is easy to see that $(\mathcal{P}T(A, B), PT(B, A))$ is a regular Γ-semigroup if we define for $f, g \,\varepsilon\, S$ and $\alpha \,\varepsilon\, \Gamma$, $f\alpha g = \{ (a, b) \,\varepsilon\, A \times B$: there exist $b_1 \,\varepsilon\, B$, $a_1 \,\varepsilon\, A$ for which $(a, b_1) \,\varepsilon\, f$, $(b_1, a_1) \,\varepsilon\, \alpha$ and $(a_1, b) \,\varepsilon\, g\}$.

As $(\mathcal{P}T\,(A, B), \mathcal{P}T\,(B, A))$ is a regular Γ-semigroup by our above discussion $(\mathcal{P}T(A, B), \mathcal{P}T\,(B, A))^*$ is also a regular Γ-semigroup. Consequently $(\mathcal{P}T\,(A, B), \mathcal{P}T\,(B, A)) \times (\mathcal{P}T(A, B), \mathcal{P}T\,(B, A))^*$ is a regular Γ-semigroup.

A faithful representation of regular semigroups was given by Lallement in [2]. Here we give a faithful representation of regular Γ-semigroups.

Theorem 3.1. Let (S, Γ) be a regular Γ-semigroup. Let $A = S \times \Gamma$ and $B = S$. For each a in S and each θ in Γ, define

$\gamma_a = \{ ((a, \alpha), y) \,\varepsilon\, A \times B : y = a\alpha x$ and $(x, y) \,\varepsilon\, \mathcal{L} \}$,

$\delta_a = \{((x, \beta), y) \,\varepsilon\, A \times B : y = x\beta a$ and $(x, y) \,\varepsilon\, \mathcal{R} \}$,

$\lambda_\theta = \{ (s, (x, \theta)) \,\varepsilon\, B \times A\}$. Then (ϕ, ψ) where

$\phi\,.\ S \rightarrow \mathcal{P}T\,(A, B) \times \mathcal{P}T\,(A, B)$ and $\psi : \Gamma \rightarrow \mathcal{P}T\,(B, A) \times \mathcal{P}T\,(B, A)$ defined by $a\phi = (\delta_a, \Upsilon_a$ and $\theta\psi = (\lambda_\theta, \lambda_\theta)$ is a monomorphism from (S, Γ) into $(\mathcal{P}T\,(A, B), \mathcal{P}T\,(B, A)) \times (\mathcal{P}T\,(A, B), \mathcal{P}T\,(B, A))^*$.

Proof. Let $a, b \,\varepsilon\, S$ and $\theta \,\varepsilon\, \Gamma$. We prove that $(\delta_a \lambda_\theta \delta_b) = \delta_{a\theta b}$. For this let $(x, \alpha) \,\varepsilon$ dom $(\delta_a \lambda_\theta \delta_b)$ (where dom = domain). Then we have $(x, x\alpha a) \,\varepsilon\, \mathcal{R}$ and $(x\alpha a, x\alpha a\theta b) \,\varepsilon\, \mathcal{R}$ which implies that $(x, x\alpha a\theta b) \,\varepsilon\, \mathcal{R}$ since $\mathcal{R}$ is an equivalence relation. Consequently $(x, \alpha) \,\varepsilon$ dom $(\delta_{a\theta b})$.

Conversely if $(x, \alpha) \,\varepsilon$ dom $(\delta_{a\theta b})$ then $(x, x\alpha a\theta b) \,\varepsilon\, \mathcal{R}$. Consequently $x\, \Gamma\, S = x\alpha a\theta b\, \Gamma\, S \subseteq x\alpha a\, \Gamma\, S \subseteq x\, \Gamma\, S$. So we have $x\, \Gamma\, S = x\alpha a\, \Gamma\, S$. Therefore $(x, x\alpha a) \,\varepsilon\, \mathcal{R}$, since $\mathcal{R}$ is an equivalence relation it now follows that $(x\alpha a, x\alpha a\theta b) \,\varepsilon\, \mathcal{R}$. Hence $(x, \alpha) \,\varepsilon$ dom $(\delta_a \lambda_\theta \delta_b)$. Therefore dom $(\delta_a$

$\lambda_\theta \delta_b) = \text{dom}(\delta_{a\theta b})$. Also we have $(x, \alpha)\, \delta_a\lambda_\theta\, \delta_b = (x,\alpha),\delta_{a\theta b}$ for all $(x,\alpha) \in \text{dom}(\delta_a\lambda_\varepsilon\delta_b)$. Therefore $\delta_a\lambda_\theta\delta_b = \delta_{a\theta b}$. Next we prove that $\gamma_a\lambda_\theta\gamma_b = \gamma_{b\theta a}$. For this let $(x, \alpha)\ \varepsilon\ \text{dom}(\gamma_a\lambda_\theta\, \gamma_b)$ then $(x, a\alpha x)\ \varepsilon\ \mathcal{L}$ and $(a\alpha x, b\theta a\alpha x)\ \varepsilon\ \mathcal{L}$ So we have $(x, b\theta a\alpha x)\ \varepsilon\ \mathcal{L}$ since $\mathcal{L}$ is an equivalence relation. Consequently $(x, \alpha)\ \varepsilon\ \text{dom}(\gamma_{b\theta a})$. Conversely, let $(x, \alpha)\ \varepsilon\ \text{dom}(\gamma_{b\theta a})$ then $(x, b\theta a\alpha x)\ \varepsilon\ \mathcal{L}$. Consequently $S\ \Gamma\ x = S\ \Gamma\ b\theta a\alpha x \subseteq S\Gamma a\alpha x \subseteq S\Gamma\ x$. So $S\ \Gamma\ x = S\ \Gamma\ a\alpha x$ implying that $(x, a\alpha x)\ \varepsilon\ \mathcal{L}$. Since $\mathcal{L}$ is an equivalence relation, we have $(a\alpha x, b\theta a\alpha x)\ \varepsilon\ \mathcal{L}$. Hence $(x, \alpha)\ \varepsilon\ \text{dom}(\gamma_a\, \lambda_\theta\, \gamma_b)$. Therefore $\text{dom}(\gamma_{b\theta a}) = \text{dom}(\gamma_a \lambda_\theta \gamma_b)$. Also $(x, \alpha)\, \gamma_a\, \lambda_\theta\, \gamma_b = (x, \alpha)\, \gamma_{b\theta a}$, for all $(x, \alpha)\ \varepsilon\ (\text{dom}\ \gamma_{b\theta a})$. Now $(a\theta b)\phi = (\delta_{a\theta b}, \gamma_{a\theta b}) = (\delta_a\, \lambda_\theta\, \delta_b, \gamma_b\, \lambda_\theta\, \gamma_a) = (\delta_a\, \lambda_\theta\, \delta_b, (\gamma_a\, \lambda_\theta\, \gamma_b)^*) = (\delta_a, \gamma_a)\, (\lambda_\theta, \lambda_\theta)\, (\delta_b, \gamma_b) = a\ \phi\ \theta\ \psi\ b\phi$. Therefore (ϕ, Ψ) is a homomorphism from (S, Γ) into $(\mathcal{P}T(A, B), \mathcal{P}T(B, A)) \times (\mathcal{P}T(A, B), \mathcal{P}T(B, A))^*$. We now show that both ϕ and ψ are one to one. For this let $a\phi = b\phi$ then $(\delta_a, \gamma_a) = (\delta_b, \gamma_b)$. Consequently we have $\delta_a = \delta_b$ and $\gamma_a = \gamma_b$. Let $a'\ \varepsilon\ V^\beta_\alpha(a)$ for some $\alpha, \beta\ \varepsilon\ \Gamma$. Then $(a'\beta a, a)\ \varepsilon\ \mathcal{L}$ so $((a'\beta a, \alpha), a)\ \varepsilon\ \gamma_a = \gamma_b$.

Thus $a = b\alpha a'\beta a$. Again $(a\alpha a', a) \in \mathcal{R}$, so $((a\alpha a', \beta), a) \in \delta_a = \delta_b$. Therefore $a = a\alpha a'\beta b\ \varepsilon\ S\,\Gamma\, b$. Now for $b'\ \varepsilon\ V^\mu_\theta(b)$ for some $\theta, \mu\ \varepsilon\ \Gamma$. Now $(b\theta b', b)\ \varepsilon\ \mathfrak{R}$ so $((b\theta b', \mu), b)\ \varepsilon\ \delta_b = \delta_a$. Thus $b = b\theta b'\mu a\ \varepsilon\ S\ \Gamma\ a$. Consequently we have $S\ \Gamma\ b = S\ \Gamma\ a$, so $(a, b)\ \varepsilon\ \mathcal{L}$ Hence there exist $u\ \varepsilon\ S$, $\gamma\ \varepsilon\ \Gamma$ such that $b = u\gamma a = u\gamma a\alpha a'\ \beta a = b\alpha a'\ \beta a = a$. Hence ϕ is one to one. Next let $\theta_1\ \psi = \theta_2\ \psi$ for $\theta_1, \theta_2\ \varepsilon\ \Gamma$. Then $(\lambda_{\theta 1}, \lambda_{\theta 1}) = (\lambda_{\theta 2}, \lambda_{\theta 2})$. Hence $x\ \lambda_{\theta 1} = x\ \lambda_{\theta 2}$ for all $x\ \varepsilon\ B$. Therefore we have $(x, \theta_1) = (x, \theta_2)$ implying $\theta_1 = \theta_2$. Thus Ψ is one to one. Hence (ϕ, Ψ) is a monomorphism from the regular Γ-semigroup (S, Γ) into the regular Γ-semigroup $(\mathcal{P}T(A, B), \mathcal{P}T(B, A)) \times (\mathcal{P}T(A, B), \mathcal{P}T(B, A))^*$.

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REFERENCES

1. T. K. Dutta and T. K. Chatterjee, Green's Equivalences on Γ-semigroup, Bull. Cal. Math. Soc. 80 (1988), 30-35.
2. G. Lallement, Demi-groupes reguliers, Ann. Mat. Pura ed appl. 77 (1967), 47-129.
3. M. Petrich, Topics in semigroups, Pennsylvania State University, 1967.
4. M. K. Sen and N. K. Saha, On Γ-semigroup - I, Bull. Cal Math. Soc. 78 (1986), 180-186.

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A NOTE ON $\mathcal{H}$ RELATION ON AN INVERSE SEMIGROUP

M. K. Sen, H. X. Yang and Y. Q. Guo

ABSTRACT. *$\mathcal{H}$ relation is studied on an inverse semigroup and it is established that an inverse semigroup is cryptic if and only if it is group closed.*

$\mathcal{H}$ relation on an inverse semigroup S is not in general a congruence relation. If S is a group bound inverse semigroup, then also $\mathcal{H}$ relation may not be a congruence relation. In this short note we want to show that an inverse semigroup is cryptic [1] (ie, $\mathcal{H}$ is a congruence relation) if and only if it is group closed (i.e, $G_r(S)$ is a subsemigroup of S).

Definition 1. [2] A semigroup S is said to be group bound if some power of each element of S lies in a subgroup of S. An element a of a semigroup S is said to be a group element if a lies in a subgroup of S. We give an example of an inverse semigroup which is group bound but not cryptic.

Example 2. Let X = { 1, 2, 3 }, and $\mathcal{I}$(X) be the symmetric inverse semigroup on X. Since $\mathcal{I}$(X) is finite, $\mathcal{I}$(X) is group bound. Since

$\left\{\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix},\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}\right\}$ is subgroup of $\mathcal{I}$ (X). $\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$ is a group element. But

$$\begin{pmatrix}1&2\\1&2\end{pmatrix}\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}=\begin{pmatrix}1&2\\1&3\end{pmatrix},\begin{pmatrix}1&2\\1&2\end{pmatrix}\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}=\begin{pmatrix}1&2\\1&2\end{pmatrix}$$

Hence $\begin{pmatrix}1&2\\1&3\end{pmatrix}$ is not in the same $\mathcal{H}$ class as $\begin{pmatrix}1&2\\1&2\end{pmatrix}$. Consequently $\mathcal{H}$ is not a congruence relation on S.

Next we give an example of group bound inverse semigroup which is not a Clifford semigroup but $\mathcal{H}$ is a congruence relation.

Example 3. Let S = { a, b, c, d, o }. Define a binary operation on S by the following table :

	a	b	c	d	o
a	o	d	a	o	o
b	c	o	o	b	o
c	o	b	c	o	o
d	a	o	o	d	o
o	o	o	o	o	o

It can be checked easily that (S,.) is a group bound inverse semigroup on which $\mathcal{H}$ relation is a congruence relation but not a Clifford semigroup.

Theorem 4. Let S be an inverse semigroup. The following conditions are equivalent on S :

(i) $\mathcal{H}$ is a congruece relation on S.

(ii) $G_r(S)$ (the set of all group elements of S is a Clifford subsemigroup of S.

(iii) ($\forall e \in E(S)$, $\forall a \in G_r(S)$ ea, ae $\in G_r(S)$.

(iv) ($\forall e \in E(S)$, $\forall a \in G_r(S)$ ea = ae, i.e, $E(S) \subseteq C(G_r(S))$.

Proof. (i)⇒(ii). Let a, b $\in G_r(S)$. There exist idempotents e, f of S such that a $\in H_e$ and b $\in H_f$. Since $\mathcal{H}$ is a congruence relation, ab $\in H_{ef}$. So, $G_r(S)$ is a subsemigroup of S. Let e $\in$ E (S), a $\in G_r(S)$. We show that ea = ae. Assume that a $\in H_f$ for some $f \in E(S)$. Then ea $\mathcal{H}$ ef and ae $\mathcal{H}$ fe. Hence ea $\mathcal{H}$ ae which implies that $eaa^{-1} \mathcal{H} aea^{-1}$, where a^{-1} is the group inverse of a. Now eaa^{-1}, $aea^{-1} \in E(S)$. Consequently, $eaa^{-1} = aea^{-1}$ and this shows that $ea = (eaa^{-1})a = (aea^{-1})a = aa^{-1}ae = ae$.

(ii)⇒(iii). It is obvious.

(iii)⇒(iv). Suppose that ea $\in H_f$ and ae $\in H_g$., where f, g $\in$ E (S). There exists x $\in H_f$ such that eax = f. Then fe = ef = e(eax) = eax = f. Similarly, eg = ge = g. Now eae = (ea)e = f(ea)e = f(ae), eae = e(ae) = eg(ae) = g(ae). Let $(ae)^{-1}$ be the group inverse of ae. Then $f(ae)(ae)^{-1} = g(ae)(ae)^{-1}$, i.e, fg = g. Similarly we can show that fg = f. Hence f = g. Now, ea = (ea)f = (eae)f = e(ae)f = eae, and ae = f(ae) = f(eae) = f(ea)e = eae. Hence ea = ae.

(iv)⇒(i). Assume that a $\mathcal{H}$ b. There exist x, y $\in H_{a^{-1}a}$, u, v $\in H_{aa^{-1}}$ such that a = bx, b = ay, a = ub and b = va. Now, $ca = cub = cc^{-1}cub = cuc^{-1}cb$. This implies that ca $\in$ Scb. Again, $cb = cc^{-1}cb = c(c^{-1}c)va = cvc^{-1}ca$, which shows that cb $\in$ Sca. Hence ca $\mathcal{L}$ cb. Obviously, ca $\mathfrak{R}$ cb. Thus it follows that ca $\mathcal{H}$ cb. Similarly we can show that ac $\mathcal{H}$ bc. Therefore, $\mathcal{H}$ Is a congruence relation.

Definition 5. A semigroup S is said to be quasisemilattice if it is inverse, perodic and $\mathcal{H} = 1_s$.

Theorem 6. An inverse semigroup S is cryptic and group bound if and only if it is a quasi semilattice of groups (i.e, there exists a congruence ρ on S such that S/ρ is a quasisemilattice and each $e\rho$ is a group for every e $\in$ E (s)).

Proof. ($\Rightarrow$) This follows immediately.

($\Leftarrow$). Suppose that there exists a congruence ρ such that S/ρ is a quasi semilattice and each $e\rho$ is a group for every $e \in E(S)$. Let $a \in S$. By the hypothesis, there exists a natural number n such that $(a\rho)^n$ is an idempotent. By Lallement's Lemma, we have that $a^n\rho = e\rho$ for some $e \in E(S)$. Since $e\rho$ is a group, a^n lies in a subgroup of S. So S is group bound. Obviously ρ is idempotent seperating, and hence $\rho \subseteq \mathcal{H}$. Let $(a,b) \in \mathcal{H}$. Then $a\rho \, \mathcal{H} \, b\rho$. Since S/ρ is a quasisemilattice, the relation $\mathcal{H}$ is the identity relation on S/ρ. Hence $a\rho = b\rho$, showing that $(a,b) \in \rho$. Thus it follows that $\mathcal{H} = \rho$. Consequently, $\mathcal{H}$ is a congruence relation on S.

REFERENCES

1. N.R. Reilly, Minimal non-cryptic varieties of inverse semigroups, Quart. J. Math. Oxford (2) 36 (1985) 467-487.
2. T.E. Hall and W. D. Munn, Semigroups satisfying minimal conditions II, Glasgow Math. J. 20 (1979), 133-140.

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π-GROUP CONGRUENCE ON AN EVENTUALLY REGULAR SEMIGROUP

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1. INTRODUCTION

A semigroup is eventually regular if for every $a \in S$ there exists a positive integer m such $a^m \in a^m Sa^m$. We shall denote by RegS the set of all regular elements of S and by E(S) the set of all idempotents of S. If $x \in$ RegS, V(x) will denote the set of inverses of x.

An eventually regular semigroup S is called a π-group if $|E(S)| = 1$. It is easy to see that a π-group is nil-extension of a group. *A* nonempty subset *A* of a semigroup S is said to be *N*-subset if for any $x,y \in S^1$, $xAy \subset A$ whenever $xAy \cap A \neq \emptyset$. If a N-subset *A* of S is a subsemigroup of S, we will say that *A* is a *N*-subsemigroup of S. If E(S) $\subset A$, we say *A* is full.

Congruences on an eventually regular semigroup are discussed in [4], [5], [6] using the similar method which is used in regular semigroup. In this paper we first give some properties of the *N*-subset of S and then we describe the π–group congruence on an eventually regular semigroup. At the end we will prove that the π–group congruence is just the group congruence when S is regular and this will give a new description of the group congruence on a regular semigroup.

2. PROPERTIES OF *N-SUBSETS*

Theorem 2.1. Let *S* be a semigroup.

(1) If ρ is a congruence on *S*, then every ρ-class of *S* is a *N*-subset of *S*.

(2) Let *A* be a *N*-subset of *S*, define a binary relation on *S* by

$$Q_A = \{(x,y) \in S \times S : x = y \text{ or } x,y \in A\}$$

Let ρ_A denote the congruence generated by Q_A, then *A* is a congruence class of ρ_A and ρ_A is the smallest congruence on *S* which makes *A* as its congruence class.

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(3) If $\{A_i : i \in I\}$ is a family of N-subsets of S and $A = \bigcap_{i \in I} A_i$ is nonempty, then A is a *N-subset of S.*

(4) Let S_1 be a semigroup and $\phi : S \to S_1$ be an epimorphism from S to S_1. Then the inverse image $A_1\phi^{-1}$ of a N-subset A_1 of S_1 is a N-subset of S.

(5) Let A be a N-subset of S. Then A is a N-subsemigroup of S iff there exist x and y in A *such that* $xy \in A$.

(6) If the intersection of a subsemigroup T and a N-subset A of S is nonempty, then $A \cap T$ is a N-subset of T.

Proof. (1) For every $a \in S$, $a\rho$ denotes the ρ-class containing a. Let $x,y \in S^1$ such that $x(a\rho)y \cap (a\rho) \neq \varnothing$, then $a\rho b$ for some $b \in a\rho$ and $(xby)\rho a$. Thus for every $c \in a\rho = b\rho$, $(xcy)\rho\ (xby)\rho a$ and $x(a\rho)y \subset a\rho$

(2) First it is easy to see $a\rho_A b$ for $a,b \in A$. Suppose now $x \in S$ and $x\rho a$ for some $a \in A$. Then $x = a$ or there exist $z_1, z_2 \ldots, z_{n-1} \in S$ such that $(z_i, z_{i+1}) \in Q_A^C$ for $i = 0,1,2,\ldots, n-1$, *where* $z_0 = a$, $z_n = x$ and $z_i \neq z_{i+1}$ for $i = 0,1,\ldots n-1$. If $x = a$ then $x \in A$.

Otherwise, if $x \neq a$, since $(a = z_1) \in Q_A^C$, there exist $x_1, y_1 \in S_1$ such that $a = x_1 b_1 y_1$, $z_1 = x_1 c_1 y_1$, where $(b_1 c_1) \in Q_A$ for some $b_1, c_1 \in S$. Thus $b_1, c_1 \in A$ since $a = z_1$ implies $b_1 \neq c_1$. Then

$$a = x_1 b_1 y_1 \in x_1 A y_1 \cap A \neq \varnothing \text{ and thus } x_1 A y_1 \subset A$$

So that $z_1 = z_1 c_1 y_1 \in A$.

Using the same deduction we obtain that $z_2, z_3, \ldots z_{n-1} \in A$, so that $x \in A$ and then A is a congruence class of ρA.

Now suppose that ρ is a congruence on S such that A is a congruence class of ρ. Suppose $x,y \in S$ and $(x,y) \in Q_A$, then $x = y$ or $x,y \in A$. So that $(x,y) \in \rho$, thus $Q_A \subset \rho$ and then $\rho A \subset \rho$.

(3) Follows from the definition of N-subset easily.

(4) Since ϕ is an epimorphism, the binary relation defined by $\rho = \{(x,y) : (x\phi, y\phi) \in \rho A_1\}$ is a congruence on S. Let $a,b \in A_1\phi^{-1}$, then $(a\phi, b\phi), \in \rho A_1$, and thus $(a,b) \in \rho$. On the other hand, if $a \in A_1\phi^{-1}$, $x \in S$ and $x\rho a$, then $(x\phi, y\phi) \in \rho A_1$, that is $x\phi \in (a\phi)\,\rho A_1 = A_1$, thus $x \in A_1\phi^{-1}$. So that $A_1\phi^{-1}$ is a congruence class of ρ and then $A_1\varphi^{-1}$ is a N-subset of S by (1).

(5) Suppose $x,y \in A$ such that $x,y \in A \cap A \quad \varnothing$. So that $x A \subset A$ since A is a N- subset of S. Now for every $b \in A$, $xb \in A$ which implies $Ab \cap A \neq \varnothing$ and then $Ab \subset A$. This A is a subsemigroup of S.

(6) Suppose $x,y \in T^1$ such that $[x(A \cap T)y] \cap (A \cap T) \quad \varnothing$ and thus $x(A \cap T)y \subset xAy \subset A$, so that $x(A \cap T)y \subset A \cap T$ since obviously $x(A \cap T)y \subset T$.

Corollary 2.2. If a N-subset A of a semigroup S contains idempotents, then it is a N-subsemigroup. Conversely, every N-subsemigroup of an eventually regular semigroup contains idempotents.

Proof. Let $e = e^2 \in A$, then by (5) of Theorem 2.1, A is a N-subsemigroup of S. Conversely, If A is a N-subsemigroup of S, and $a \in A$, then $a^2 \in A$. By (2) of Theorem 2.1, $A = a\rho_A \in E(S/\rho_A)$. So that there exists an idempotent $e \in E(S)$ such that $e\rho_A = a\rho_A = A$ since S is eventually regular.

3. π-GROUP CONGRUENCE ON AN EVENTUALLY REGULAR SEMIGROUP

Definition 3.1. Let S be a semigroup and $a,b \in S$, we say b devides $a(b/a)$, if $a = xby$ for some $x,y \in S^1$.

In this section S will be always an eventually regular semigroup unless otherwise stated.

Lemma 3.1. Let m be the positive integer such that $a^m \in$ Reg S. and If $(a^m)' \in V(a^m)$, then $a^i (a^m)' a^j \in E(S)$, $i + j = m$.

Lemma 3.2. Let A be a full N-subsemigroup of S and $a \in S$. If there exists an element $u \in A$ such that u/a then $(a\vartheta)\rho_A = a\rho_A = (\vartheta a)\rho_A$ for every $\vartheta \in A$.

Proof. Since A is a full N-subsemigroup of S, then it is the idempotent of S/ρ_A by Theorem 2.1 (2).

From u/a, we have $a = xuy$ for some, $x,y \in S^1$. Let $n \in N$ such that $y^n \in \mathrm{Reg}S$ and $(y^n)' \in V(y^n)$.

$$\begin{aligned}(a\vartheta)\rho_A &= (xuy\vartheta)\rho_A \\ &= [xuy\,(y^n)'y^m\vartheta]\rho_A && (\text{since } (y^n)'y^m \in E(S) \subset A \\ &= [xuy\,(y^n)'y^m]\rho_A && (\text{since } \vartheta \in A) \\ &= [xu\,(y\,(y^n)'y^{n-1})y]\rho_A \\ &= (xuy)\rho_A && (\text{since } y\,(y^n)'y^{n-1} \in E(S) \subset A \\ &= a\rho_A\end{aligned}$$

Similarly, $a\rho_A = (\vartheta a)\,\rho_A$.

Lemma 3.3. Let S be a semigroup and A be a N-subsemigroup of S. Then for any $a,b \in S$, if $a\rho_A b$ and $a \neq b$, then there exist $u,\vartheta \in A$ such that u/a and u/b.

Proof. By the definition of ρ_A, there exist $z_1, z_2 \ldots, z_{n-1} \in S$ such that $(z_i, z_{i+1}) \in Q_A^C$, and $z_i \neq z_{i+1}$ for $i = 0,1,\ldots, n-1$. where $z_0 = a, z_n = b$.

Since $(a,z_1) \in Q_A^C$, there exist $x,y \in S_1$, $(u,w) \in Q_A$ such that $a, = xuy$, $z_1 = xwy$. Thus $u \neq w$ since $a \neq z_1$ and then $u,w \in A$ and u/a.

Using the same method we can prove that there exists $\vartheta \in A$ such that ϑ/b.

Theorem 3.4. Let A be a full N-subsemigroup of S. Then

(1) ρ_A ia a π-group congruence on S.

(2) For any $a,b \in S$ $a\rho_A b$ iff $a = b$ or there exist $u,\vartheta,\omega \in A$ and $e \in E(S)$ such that $u \mid a$, $\vartheta \mid b$ and $be = wa$.

Proof. (1) Since A is a ρ_A-class by Theorem 2.1 and S/ρ_A is obvious eventually regular. Then ρ_A is a π —group congruence of S since $E\ (S) \subset A$.

(2) If $a\rho_A b$ and $a \neq b$, by Lemma 3.3, there exist $u, \vartheta \in A$ such that $u \mid a$, $\vartheta \mid b$ Now Let m be the positive integer such that $a^m \in$ Reg$S$ and $(a^m)' \in V\ (a^m)$, then

$[a^m\ (a^m)]\ \rho_A\ [ba^{m-1}\ (a^m)']$ and thus b$a^{m-1}\ (a^m)' \in$ A

Since $a^m\ (c^m)' \subset E\ (S) \subset A$ and A is a ρ_A-class. Let $e = a^{m-1}\ (a^m)'\ a \in E\ (S)$, $w = ba^{m-1}\ (a^m)'$, then $wa = be$.

Conversly, suppose a = b the $a\rho_A b$. Otherwise, there exist u, v $w \in A$ and e $\in$ E (S), such that $u \mid a$, $\vartheta \mid b$, $wa = be$. By lemma 3.2, $a\rho_A = (wa)\ \rho_A = (be)\ \rho_A = b\rho_A$.

Theorem 3.5. (1) Let ρ be a π – group congruence on S, then the kernel of ρ is a full N-subsemigroup of S and $\rho_{\ker\rho} \subset \rho$.

(2) If A is the intersection of all the full N-subsemigroups of S, the ρ_A is the smallest π—group congruence on S.

(3) For every $a \in$ S, $a\rho_A \in$ Reg (S/ρ_A) if and only if there exists $u \in A$ such that $u \mid a$.

(4) For every $a \in$ S, if $a\rho_A$ contains more than one element, then $a\rho_A \in$ Reg (S/ρ_A)

(5) Let A be a full, and closed N-subsemigroup of S. If there exists a $\in$ S such that for every u $\in A$, $u \mid a$. Then there exists a π—group congruence on S such that $\rho_A \subset \rho$ and $\rho_A \doteqdot \rho$.

Proof. (1) Since ker$\rho = \{a \in S : a\rho e$ for some $e \in E\ (S)\} = f\rho$ for every $f \in E\ (S)$. By theorem 2.1 and corollary 2.2, kerρ is a full N-subsemigroup and $\rho_{\ker\rho} \subset \rho$.

(2) It is the smallest π-group congruence, since A is the smallest full N-subsemigroup.

(3) Let $a\rho_A \in$ Reg $(S \mid \rho_A)$. If $a \in$ Reg S, then e $= a'a \in$ E (S) $\subset$ A, where $a' \in V\ (a)$. It follows that $a = a$e and then e$\mid a$.

It $a \notin$ Reg S, let $n \in N$ be such that $a^n \in$ Reg S and $(a^n)' \in V\ (a^n)$, then

$a^n\ (a^n)'\ a \in$ Reg S and $a^{n-1}\ (a^n)' \in V\ (a^n(a^n)'a)$

Thus $a \neq a^n(a^n)'a$. Now since Reg $(S/\rho A)$ is a subgroup of S/ρ_A with identity $(a^n(a^n)'a)\rho_A$ *then* $a\rho_A = (a^n(a^m)'a)\rho_A$. By lemma 3.3, There exists $u \in A$, such that $u \mid a$.

Conversely Suppose there exists $u \in A$ such that $u \mid a$, and $a = xuy$ for some x, y $\in S^1$. Let m, n be the positive integers such that y^m and a^n are regular, respectively. Suppose $(a^n)' \in V\ (a^n)$, and $(y^m)' \in V\ (y^m)$. It is easy to see $a^n\ (a^n)'a \in$ Reg S. Thus

$$\begin{aligned} a\rho_A &= (xuy)\rho_A = (xuy^{m'}\ (y^m)'y)\ \rho_A \qquad \text{(since } y^m\ (y^m)' \in E\ (S) \subset A) \\ &= (xuyy^{m-1}\ (y^m)'y)\rho_A \\ &= [xuy\ (y^{m-1}(y^m)'y)(a^{n-1}(a^n)^1a)]\rho_A \\ &= (xuya^{n-1}(a^n)'a)\rho_A \\ &= (a^n\ (a^n)'a)\ \rho_A \in \text{Reg } (S/\rho_A) \end{aligned}$$

(4) Suppose $a \neq b$ and $a\rho_A b$, for $a, b \in S$. By lemma 3.3, there exist $u, v \in A$ such that $u \mid a$, $v \mid b$ and then $a\rho_A \in \mathrm{Reg}\,(S/\rho_A)$ by (3). Equivalently, we know that if $a\rho_A$ is not a regular element in S/ρ_A, then $a\rho_A$ contains only one element.

(5) Recall that a subset A of an eventually regular semigroup is self-conjugate, if

$$aAa^{n-1}(a^n)' \subset A, \text{ and } a^{n-1}(a^n)'Aa \subset A$$

where n is the positive integer such that $a^n \in \mathrm{Reg}S$, and $(a^n)' \in V(a^n)$

Now, since $a^n(a^n)'$, $a(a^n)'a^{n-1} \in E(S) \subset A$. It follows that

$$a[(a^n)'a^n]a^{n-1}(a^n)' = a[(a^n)'a^{n-1}](a^n)(a^n)' \in A$$

and then $aAa^{n-1}(a^n)' \cap A \neq \varnothing$. so that $aAa^{n-1}(a^n)' \subset A$.

Similiarly, we have $a^{n-1}(a^n)'Aa \subset A$. Again since A is closed we obtain that

$$\rho = \{(x,y) \in S \times S : xu = vy \text{ for some } u,v \in A\}$$

is a group congruence on S by [5], and $\ker\rho = A$. Thus $\rho A \subset \rho$. Since there exists $a \in S$, $u \mid a$ for every $u \in A$, and this implies $a\rho A \notin \mathrm{Reg}\,(S \mid \rho_A)$. Therefore $\rho_A \neq \rho$.

4. GROUP CONGRUENCE ON REGULAR SEMIGROUP

In this section, S will denote a regular semigroup. Let A be a full, self-conjugate and closed subsemigroup of S. In [3], D. R, Latorre showed that the relation defined by

$$\sigma_A = \{(a,b) \in S \times S : au = vb \text{ for some } u, v \in A\}$$

is a group congruence on S. Now, we will describe the group congruence on S by the full N-subsemigroup of S. First it is easy to show the following theorem.

Theorem 4.1. Let A be a full N-subsemigroup of S. Then ρ_A is a group congruence on S.

Theorem 4.2. Let A be a full N-subsemigroup of S. Then for any $a,b \in S$, the following conditions are equivalent.

(1) $a\rho_A b$;

(2) there exists $b_l \in V(b)$ such that $b_l a \in A$;

(3) there exists $e \in E(S)$ and $u \in A$ such that $ea = bu$

Proof.(1) implies (2). If $a\rho_A b$; then $(b_l a)\ \rho_A\ (b_l b)$ for all $b_l \in V(b)$ Then $b_l a \in A$ since $b_l b \in E(S) \subset A$ and A is a ρ_A-class.

(2) implies (3). Let $u = b_l a$, $e = bb_l$. Then $bu = bb_l a = ea$.

(3) implies (1). It is easy to see $a\rho_A = (ea)\rho_A = (bu)\rho_A = b\rho_A$ since ρ_A is a group congruence and $u \in A$.

Theorem 4.3. Let A be a full subsemigroup of S. Then the following conditions are equivalent.

(1) A is a N-subsemigroup.

(2) A is unitary and self-conjugate

(3) A is closed and self-congugate

(4) A is unitary and symmetric.

Proof. (1) implies (2). Let $a \in A$, $x \in S$ and $ax \in A$. Then $Ax \cap A \neq \varnothing$ and then $Ax \subset A$. Let $x_l \in V(x)$, then $xx_l \in E(S) \subset A$. So that $x = xx_lx \in Ax \subset A$. We also have $xa \in A$ implies $x \in A$. Thus A is unitary.

Let $x \in S$ and $x_l \in V(x)$. Then $x_l(xx_l)x = x_lx \in E(S) \subset A$ and thus $Ax \cap A \neq \varnothing$ So $x_lAx \subset A$ which implies that A is self-conjugate.

(2) implies (3). It is obvious that $A \subset Aw$. Let $x \in Aw$,Then there exists $u \in A$ such that $ux \in A$ and then $x \in A$ since A is unitary. So $A = Aw$.

(3) implies (4). Let $x,y \in S$ be such that $xy \in A$. Suppose $x_l \in V(x)$, then $x_lxyx \in x_lAx \subset A$, and then $yx \in A$ since $xx_l \in E(S) \subset A$ and A is closed. Thus A is symmetric.

Let $a \in A$, $x \in S$ and $ax \in A$, then $x \in A$ since A is closed. If $xa \in A$ then $ax \in A$ since A is symmetric. Thus $x \in A$, which implies that A is unitary.

(4) implies (1). $x, y \in S^1$ and $xAy \cap A \neq \varnothing$. Then there exists $a \in A$ such that $xay \in A$, and then $ayx \in A$ by the symmetry of A. So $yx \in A$, since A is unitary. For every $b \in A$, $byx \in A$ and then $xby \in A$. So $xAy \subset A$.

REFERENCES

1. J. Howie, An introduction to semigroup theory, Academic Press, 1976.
2. S. Bogdanovic, Semigroups with a system of subsemigroups, *Novisad* ; 1985.
3. D. R. Latorre, Group congruence on regular semigroup, *Semigroup Forum,* 24 (1982) 327-340.
4. P. Protic and S. Bogdanvoic, Some congruences on a strongly π-inverse r-semigroup, *Zbor Rad, PMF. Novisad,* 2 (1985), 79-89.
5. S. Hanumantharao and P. Lokshmi, Group congruences on eventually regular semigroup, *J. Austral. Math. Soc (series A),* 45 (1988) 320-325.
6. S. Hanumantha Rao and P. Loksshmi, The least semilattic of group congruence on an eventually regular semigroup, *semigroup Forum Vol.* 42 (1991) 107-111.

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EXISTENCE AND UNIQUENESS OF AN INVARIANT INTEGRAL ON A MORE GENERAL CLASS OF FUNCTIONS IN A GENERALIZED TOPOLOGICAL GROUP

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ABSTRACT : In this note, we initiate θ-Harr integral on a more general class of functions than the class of real valued continuous functions with compact support, in a locally θ-H-closed [6], θ-topological group [4]. In this space, which is more general than locally compact topological group, we establish the existence and essential uniqueness of θ-Haar integral.

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Key Words and Phrases : H-set, θ-open, Urysohn, locally-θ-H-closed, θ-CR, θ-T_2, θ-Haar integral, continuous function with H-set support.

1. INTRODUCTION

Veličko, [7] initiated the concepts of H-sets and θ-open sets and Fomin [2] introduced weaker forms of continuity under the terminology θ-continuity. Using such concepts the notion of locally θ-H-closedness [6] and θ-topological group [4] were introduced. It can be noted that every compact set is an H-set but not conversely; so it is natural that the class $\mathcal{H}$ of all real valued continuous functions with H-set support is larger than the class of all real valued continuous functions with compact support. So our objective would be to take these general class of functions under the purview of integration theory. In this paper we have shown the existence of an invariant integral on $\mathcal{H}$ and that too in a generalized topological group.

2. DEFINITIONS AND PREREQUISITES

For the definition of H-sets, θ-poen sets, θ-continuous function, θ-homeomorphism and Urysohn space we refer to the book of Porter and Woods [5]. A topological space X is said to be locally θ-H--closed [6] if every point x of X has a θ-open set containing x whose closure is an H-set. A topological space X is said to be θ-completely regular

[6] (θ-CR, for short) if for any point p and a θ-closed set H in X with $p \notin H$, there exists a continuous function $f : X \to [0, 1]$ such that $f(p) = 0$ and $f(H) = 1$; X is called θ-T_2 [6] if any two distinct points can be separated by θ-open sets Clearly every θ-T_2 space is Urysohn and H-set in a Urysohn space is θ-closed [3]. In [3] it was established that there exists a space X which is locally θ-H-closed, θ-CR, θ-T_2 but non-regular, non-locally compact, non-H-closed. A group X with multiplication map m and inversion map i, endowed with topology τ, is called a θ-topological group [4] if m and i are θ-continuous and each left as well as each right translation is an open map. Clearly every topological group is a θ-topological group but not conversely [4]. A real valued function f on X is said to have H-set support if there exists an H-set K of X outside of which f vanishes.

3. EXISTENCE OF θ-HAAR INTEGRAL

Let X be a locally θ-H-closed θ-T_2, θ-CR, θ-topological group, $\mathcal{H}$ be collection of all real valued continuous functions on X with H-set support and $\mathcal{H}^+$ be the subclass of $\mathcal{H}$ consisting of all $f \in \mathcal{H}$ for which $f \geq 0$ and f is not identically zero i.e.

$$\mathcal{H}^+ = \{f \in \mathcal{H} : f \geq 0 \text{ and } f \neq 0\}$$

It can be easily shown that for any $a \in x$, the function $x \to a^{-1}x$ is θ-continuous and infact θ-homeomorphism. Further for any $a \in x$ and for any real valued function f on X, the function $f_a(x) = f(a^{-1}x)$ is θ-continuous iff f is θ-continuous. Also f_a is a function having H-set support iff f is so.

Definition 3.1. A θ-Haar integral I for X is a +ve linear form on $\mathcal{H}$ not identically zero such that $I(f_a) = I(f)$ for all $f \in \mathcal{H}$ and for all $a \in X$.

Our goal is to show the existence of an non-zero linear functional I on $\mathcal{H}^+$ s.t. $I(f_a) = I(f)$ for all $f \in \mathcal{H}^+$ and then extend the funtional to the whole of $\mathcal{H}$ as follows:

Let $f \in \mathcal{H}$ then f^+ and $f^- \in \mathcal{H}^+$ and hence $I(f^+)$ and $I(f^-)$ exist. Define the extension $I(f) = I(f^+) - I(f^-)$.

The following lemma is the basis for showing the existence of θ-Haar Integral.

Lemma 3.2. For $f, h \in \mathcal{H}^+$, there exist $a_1, \ldots, a_n$ of X and $c_i > 0$, $i = 1, 2, \ldots, n$ such that $f \leq \sum_{i=1}^{n} c_i h_{a_i}$

Proof. Since $h \in \mathcal{H}^+$, h is not identically zero on X and hence for some $x \in X$, $h(x) > 0$, then the set $V = \{x \in X : h(x) > 0\}$ is non-empty θ-open set as $x \in X$ and h is θ-continuous. Again X being locally-θ-H-closed, there exists a θ-open set U such that $x \in U \subset \overline{U} \subset V$ and is an H-Set ([6]). Now h being θ-continuous, $h(\overline{U})$ is an H-set in R, and hence a compact subset of R So h is bounded on $\overline{U}$

and attains its bounds on $\overline{U}$ and hence there exists $\eta > 0$ (where η = g. 1.b. of h on $\overline{U}$) such that $h(x) \geq \eta$ on U.

Now f being in $\mathcal{H}^+$, there exist an H-set K such that f = 0 on X – K. The set $\{aU : a \in X\}$ is an θ-open covering of the H-set K (each translation being θ-homorphism). So there exist $a_1, \ldots, a_n$ such that $K \subset \bigcup_{i=1}^{n} a_i U$. If we take M = 1.u.b. of f, $c_i = \frac{M}{\eta}$ for i = i = i, 2, ... , n, then $f(x) \leq \sum_{i=1}^{n} c_i h_{a_i}(x)$.

Definition 3.3. For f, g $\in \mathcal{H}^+$ we define

$$(f : g) = \text{g.1.b} \{ \sum_{i=1}^{n} c_i : f(x) \leq \sum_{i=1}^{n} c_i g_{a_i}(x).\ c_i > 0\}$$

in view of the above lemma, for any two functions f, g $\in \mathcal{H}^+$, (f : g) exists and is finite. Also (f : g) > 0 and $(f : g) \geq \frac{M}{m}$ where M = 1.u.b of f and m = 1.u.b. of g. Further it satisfies the following properties.

(i) $(f_a : g) = (f : g)$ for any $a \in x$

(ii) $(cf : g) = c\,(f : g)$ for $c \geq 0$

(iii) $(f + g : h) \leq (f : h) + (g : h)$, $\forall f, g, h \in \mathcal{H}^+$

(iv) $f \leq g \Rightarrow (f : h) \leq (g : h)$

(v) $(f : h) \leq (f : g)\,(g : h)$

(vi) $\frac{1}{(h:f)} \leq \frac{(f:g)}{(h:g)} \leq (f : h)$

Let $f_0 \in \mathcal{H}^+$ be fixed and for any f, g $\in \mathcal{H}^+$, we define $I_g(f) = \frac{(f:g)}{(f_0:g)}$. Since $(f : g) \leq (f : f_0)\,(f_0 : g)$ it follows that $0 \leq I_g(f) \leq (f : f_0)$ for every $f \in \mathcal{H}^+$. Let $X_f = [o, : (f, f_0)]$. Let $Y = \underset{f \in \mathcal{H}^+}{\pi} X_f$. Clearly Y is compact. For each θ-open set V containing e, let $F_V = \text{cl}\,\{I_g \in Y$: Support of g is in V}. Such a g always exists as X is locally θ-H-closed and θ-CR. So F_V is non-empty. T}. he family $\{F_V$: V is θ-open set containing e} has finite intersection property and Y being compect,

$\cap$ {F_V : V is θ-open set containing e} $\neq \phi$. Let I = (I (f)), where I (f) is the f-th co-ordinate of I belongs to the intersection. We shall show that I is the required functional on $\mathcal{H}^+$.

Since I = (I (f)) $\in \cap$ {F_V : V is θ-open set cintaining e} so for any $\varepsilon > 0$ and $f_1, f_2, \dots, f_n \in \mathcal{H}^-$, the basic open set $(I(f_1) - \varepsilon, I(f_1) + \varepsilon) \times \dots \times (I(f_n) - \varepsilon, I(f_n) + \varepsilon)$

$\times \underset{f \neq f_1, \dots\dots, f_n}{\pi X_f}$

Contains a point $(I_g (f)) \in \{I_g \in Y$: Support of g is in V}. Hence for i = 1, 2, ... n, $| I_g (f_i) - I(f_i) | < \varepsilon$. .. .. (B)

Since the following properties (i)— (iv) below hold for $I_0 = I_g$, it can easily be obtained (i)— (iv) for $I_0 = I$.

(i) $I_0 \neq 0$, $I_0 (f) > 0$ if $f \neq 0$

(ii) $I_0 (f_a) = I_0 (f)$

(iii) $I_0 (cf) = cI_0 (f)$, $c \geq 0$

(iv) $I_0 (f_1 + f_2) \leq I_0 (f_1) + I_0 (f_2)$.

In order to show I is left invariant non-zero +ve linear functional on $\mathcal{H}^+$ it is sufficient to show that the reverse inequality in (iv) holds for $I_0 = I$.

For this we need two Propositions :-

Proposition 3.4. If $f \in \mathcal{H}$, then given any $\varepsilon > 0$, there exists a θ-open set U containing e such that $| f(z) - f(g) | < \varepsilon$ wherever $gz^{-1} \in U$.

Proof. We define $U^* = \{y \in x : | f(yz) - f(z) | < \varepsilon$ for all $z \in X\}$. Since $f \in \mathcal{H}$, there exists an H-set K such that f = 0 on X − K. As X is locally θ-H-closed, there exists an θ-open set L containing e such that $\bar{L}$ is an H-set Now the inversion map being θ-homeomorphism, L^{-1} is θ-open sat containing e as well as $(\bar{L})^{-1}$ is an H-set (as θ-continuous mapping carries H-set into an H-set). So $V = L \cap L^{-1} \subset \bar{L} \cap (\bar{L})^{-1}$. X being an θ-$T_2$ and hence Urysohn, the H-set $\bar{L} \cap (\bar{L})^{-1}$ is θ-closed. $\bar{V}$ being closure of an θ-open set implies $\bar{V}$ = θ-closure of V and as X is Urysohn, θ-closure of V is θ-closed set. So $\bar{V}$ is a θ-closed set contained in a H-set $\bar{L} \cap (\bar{L})^{-1}$. So $\bar{V}$ is H-set. Furthermore as the multiplication map is θ-continuous and $K \times \bar{V}$ is and H-set, then $K.\bar{V}$ is H-set. Clearly $K \subset K\bar{V}$. Given any $g \in K\bar{V}$, f being continuous and hence θ-contnuous, there exist θ-open sets V_g containing g and U_g containing e such that $| f(yz) - f(z) | < \varepsilon$ wherever $y \in U_g$ and $z \in V_g$. Since $K\bar{V}$, is an H-set, there exist $g_1, g_2, \dots, g_n \in K\bar{V}$ such that $K\bar{V} \subset \bigcup_{i=1}^{n} V_{g_i}$. Define $U = U_{g_i} \cap \dots \cap U_{g_i} \cap V$.

Then U is an θ-open set containing e. We shall show that $U \subset U^*$. Let $Y \in U$ and $z \in X$. If $z \in K\bar{V}$ then $z \in V_{g_i}$ for some i. Since $Y \in U_{g_i}$ it follows that $|f(yz) - f(z)| < \varepsilon$. So $Y \in U^*$. On the otherhand if $z \notin K\bar{V}$, then $z \notin K$, hence $f(z) = 0$; since $yz \notin K$ ($yz \in K$ would imply $z \in Ky^{-1} \subset KV \subset K\bar{V}$,— a contradiction), $f(yz) = 0$. So $|f(yz) - f(z)| = 0 < \varepsilon$. So $y \in U^*$. Therefore $U \subset U^*$ and hence the proposition.

Proposition 3·5. Let $f_1, f_2 \in \mathcal{H}^+$ and $\varepsilon > 0$. Then there exists a θ-open set U containing e such that for all $g \in \mathcal{H}^+$ having support in U, $I_g(f_1) + I_g(f_2) \le I_g(f_1 + f_2) + \varepsilon$.

Proof. Clearly $f_1 + f_2 \in \mathcal{H}^+$. Let K be the H-set s. t. $f_1 + f_2$ vanishes outside K. Again X being locally θ-H-closed and θ-CR, there exists $f \in \overset{+}{\mathcal{H}}$ such that $f'(K) = 1$. Let δ_1 be arbitrary positive number and let $f = f_1 + f_2 + \delta_1 f'$. Clearly $f \in \overset{+}{\mathcal{H}}$. We define two functions

$$h_i(x) = \frac{f_i(x)}{f(x)} \text{ if } f(x) \neq 0$$

$$= 0 \text{ otherwise}$$

for i = 1, 2.

Since $h_i(x) \in \mathcal{H}^+$ given any $\delta_2 > 0$ by Proposition 3.4, there exist a θ-open set U containing e s. t.

$$|h_i(z) - h_i(a)| < \delta_2$$

whenever $az^{-1} \in U$ and $a, z \in X$.

Let $g \in \overset{+}{\mathcal{H}}$ be such that its support lies in U and we consider $f(x) \le \sum_i c_i g(z_i^{-1} z)$. If $z_i^{-1} z$ is such that $g(z_i^{-1} z) \neq 0$ then $z_i^{-1} z \in U$ and therefore,

$$|h_i(z) - h_i(z_i)| < \delta_2$$

But $f_i(z) = f(z)\, h_i(z) \le \sum_i c_i\, g(z_i^{-1} z)\, h_i(z)$

$\le \sum_i c_i\, g(z_i^{-1} z)(h_i(z_i) + \delta_2)$. Therefore

$(f_i : g) \le \sum_i c_i (h_i(z_i) + \delta_2)$

Hence $(f_1 : g) + (f_2 : g) \le \sum_i c_i (h_1(z_i) + h_2(z_i) + 2\delta_2) \le \sum C_i (1 + 2\delta_2)$ (as $h_i(z_i) + h_2(z_i) \le 1$)

This implies, $(f_1 : g) + (f_2 : g) \le (f : g)(1 + 2\delta_2)$

For a fixed non-zero $f_o \in \mathcal{H}^+$ if we divide the above inequality by $(f_o : g)$ then $I_g(f_1) + I_g(f_2) \le I_g(f)(1 + 2 d_2)$ ... ... (A) Also it is easy to show that $I_g(f) \le I_g(f_1) + I_g(f_2) + \delta_1 I_g(f')$. Therefore $I_g(f_1) + I_g(f_2) \le (1 + 2\delta_2)(I_g(f_1) + I_g(f_2) + \delta_1 I_g(f'))$. Since δ_1 are arbitraty we can choose δ_1 and δ_2 in such a manner such that $\{2\delta_2 (I_g(f_1) + I_g(f_2) + \delta_1 I_g(f')) + \delta_1 I_g(f')\} < \varepsilon$. So $I_g(f_1) + I_g(f_2) \le I_g(f_1 + f_2) + \varepsilon$ (from relation B) Hence the proposition.

For a given $\varepsilon > 0$ from relation (B) (before Proposition 3.4), we have,

$|I(f_1 + f_2) - I_g(f_1 + f_2)| < \varepsilon$.

By Proposition 3.5, $I_g(f_1) + I_g(f_2) \leq I_g(f_1 + f_2) + \varepsilon$.

Therefore $I(f_1) + I(f_2) \leq I_g(f_1) + I_g(f_2) + 2\varepsilon$

$\leq I_g(f_1 + f_2) + 3\varepsilon \quad \leq I(f_1 + f_2) + 4\varepsilon$.

Since ε is arbitrary, we have

$I(f_1) + I(f_2) \leq I(f_1 + f_2)$.

So I is additive.

Therefore I can be extended to $\mathcal{H}$ and that extended I will be our required θ-Haar Integral.

Hence we get a theorem.

Theorem 3·6. Let X be a locally θ-H-closed θ-T_2, θ-CR, θ-topological group. Then there exists a non-trivial θ-Haar integral on $\mathcal{H}$.

4. UNIQUENESS OF θ-HAAR INTEGRAL

Suppose J is any other θ-Haar integral for X. We shall show that J = CI for a suitable constant C > 0. The following lemma is important and as the proof is quite similar to the proof of Lemma 15 Art. 76 [1]. we shall omit the Proof.

Lemma 4.1. If K is an H-set of X, and $f, f' \in \mathcal{H}^+$ such that support of f lies in K and $f' = 1$ on K then given any $\varepsilon > 0$, there exists a θ-open set U containing e such that

$$(f : g) \leq \varepsilon (f' : g) + \frac{J(f)}{J(g)}$$

for every $g \in \mathcal{H}^+$ for which $g(x^{-1}) = g(x)$, $\forall\, x \in X$ and which vanishes outside U.

Theorem 4.2. If J is any θ-Haar integral for G, then J = C I for a suitable C > 0.

Proof. Let $f_1, f_0 \in \mathcal{H}^+$, where f_0 is fixed and K_i be the H-sets of X, such that f_i vanishes outside K_i for i = 1, 0. Again as X is locally-θ-H-closed, θ-CR and θ-T_2, there exists $f_i' \in \mathcal{H}^+$ such that $f_i' = 1$ on K_i for i = 1, 0. Let $\varepsilon > 0$. Then by Lemma 4.1 there exists a θ-open set V_i containing e such that

$$(f_i : g) \leq \varepsilon (f'_i : g) + \frac{J(f_i)}{J(g)} \quad \dots\dots\dots (1)$$

for every $g \in \mathcal{H}^+$ which is such that $g(x^{-1}) = g(x)$, for all $x \in X$ and vanishes outside V_i for i = 1, 0. It is easy to see that

$$J(f_i) \leq (f_i : g)\, J(g) \quad \dots \quad \dots \quad \dots (2)$$

Let C_i be the H-set support of $f_i' \in \mathcal{H}^+$. Since X is locally θ-H-closed θ-CR, θ-T_2, there exists $f_i'' \in \mathcal{H}^+$ such that $f_i'' = 1$ on C_i. Now we choose $\delta > 0$ so that $\delta\ (f_i'' : f'_i) < 1$.

Then by Lemma 4.1, there exists θ-open sets U_i containing e such that $(f'_i : g) \leq \delta\ (f_i'' : g) + \dfrac{J(f_i')}{J(g)}$

for every $g \in \mathcal{H}^+$ which is such that $g(x^{-1}) = g(x)$ for all $x \in X$ and vanishes outside U_i.

Hence using $(f_i'' : g) \leq (f_i'' : f_i')\ (f_i' : g)$, we get

$$(f'_i : g)\ J(g)\ [1 - \delta\ (f_i'' : f'_1)] \leq J\ (f_i')$$

Since $1 - \delta\ (f_i'' : f_i')$ is independent of g and is greater than zero we have

$(f_i' : g)\ J(g) \leq \dfrac{J(f_i')}{1-\delta(f_i'', f_i')} = M_i$ (say), for i = 1, 0. If $U = U_1 \cap U_0$ and $M = \text{Max}\ (M_1, M_0)$ then

$$(f_i' : g)\ J(g) \leq M \quad \text{..............................(3)}$$

So if $W = V_1 \cap V_0 \cap U$ and if $g \in \mathcal{H}^+$ is such that $g(x^{-1}) = g(x)$ for all $x \in X$ and vanishes outside W, then (1) – (3) holds simultaneously. Therefore

$$\frac{J(f_1)}{\varepsilon(f_0' : g)\, J(g) + J(f_0)} \leq \frac{(f_1 : g)}{(f_0 : g)} \leq \frac{\varepsilon(f_1' : g)\, J(g) + J(f_1)}{J(f_0)}$$

Now from (3),

$$\frac{J(f_1)}{\varepsilon M + J(f_0)} \leq \frac{(f_1 : g)}{(f_0 : g)} \leq \frac{\varepsilon M + J(f_1)}{J(f_0)}.$$

From relation (B) given before Proposition 3.4, we get a sequence $g_n \in \mathcal{H}^+$ such that $g_n(x^{-1}) = g_n(x)$ for all $x \in X$ and g_n vanishes outside W, and $I_{g_n}(f_1)$ converges to $I\ (f_1)$.

$$\text{So, } \frac{J(f_1)}{\varepsilon M + J(f_0)} \leq I\ (f_1) \leq \frac{\varepsilon M + J(f_1)}{J(f_0)}$$

If we let $\varepsilon \to 0$ then

$$I\ (f_1) = \frac{J(f_1)}{J(f_0)}$$

i.e. $J(f_1) = J(f_0)\ I(f_1)$.

and this holds for every $f_1 \in \mathcal{H}^+$.

So $J(f) = J\ (f_0)\ I\ (f)$ for every $f \in \mathcal{H}^+$.

i.e. $J(f) = CI\ (f)$.

REFERENCES

1. S. K. Berbarian, Measure and Integration, the Macmillan Company, 1967.
2. S. V. Fomin, Extensions of topological spaces, Ann, Math, 44(1943), 471-480.
3. S. Ganguly, C. K. Basu and S. Sinha Roy, A new regular measure on a topological space I, Far East J. Math. Sci., Special vol. (1997), Part-1, 35-47.
4. S. Ganguly, C. K. Basu and S. Sinha Roy, Existence of an invariant measure in a generalized topological group, Bull. Cal. Math. Soc. 89 (1997) 131-134.
5. J. R. Porter and R. W. Woods, Extensions and absolutes of Hausdorff spaces, Springer Verlag, 1988.
6. S.Sinha Roy and S. Bandyopadhyay, On θ-Completely regular and locally θ-H-closed spaces, Bull. Cal. Math. Soc., Bull. Cal. Math. Soc. 87 (1995) 19-28.
7. N. V. Veličko, H-closed topological spaces, Mat. Sb. 70(112) (1966) 98-112 Amer. Math. Soc. transl. (2) 78 (1969) 103-118.

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ON SEVERAL CLASSES OF ORTHODOX Γ — SEMIGROUPS

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ABSTRACT : In this paper the concepts of orthodox Γ—regular semigroups, orthogroup Γ—semigroups, and rectangular Γ—groups are introduced and studied. Some characterizations of these Γ—semigroups are obtained. Also the structure theorm of orthogroup Γ—semigroups is given. These notions are generalization of the corresponding ones in semigroups.

Key word : Union of groups, union of Γ—groups, orthodox semigroups, orthodox Γ—somigroups

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1. PRELIMINARIES

A set M is said to be a Γ—semigroup (due to Sen and Saha [1]) if the following conditions are satisfied :

(i) $M = \{a,b,c\ldots\}$ and $\Gamma = \{\alpha,\beta,\gamma\ldots\}$ are two nonempty sets.

(ii) $a\alpha b \in M$ for all $a, b \in M$ and all $\alpha \in \Gamma$.

(iii) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and all $\alpha, \beta \in \Gamma$.

Let M be a Γ— semigroup and α an element of Γ. We define the product $a.b = a\alpha b$, then (M, .) is a semigroup, denoted by M_α, called a related semigroup of M. If some one (so every, due to [1]) related semigroup of M is a group, then we say M is a Γ-group. An element a of M is said to be regular if $a \in a\Gamma M\Gamma a$, where $a\Gamma aM\Gamma a = \{(a\alpha b)\beta a := b \in M, \alpha,\beta \in \Gamma\}$. A Γ— semigroup is said to be regular if every element of M is regular (due to [1]. Binary relations $\mathcal{J}$, $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ on M (due to [7]). which are analogous of Green's relations in a semigroup, are defined as follows :

(i) a $\mathscr{J}$ b iff (a) = (b),
(ii) a$\mathscr{L}$ b iff $(a)_1 = (b)_1$,
(iii) a$\mathscr{R}$ b iff $(a)_r = (b)_r$,
(iv) a $\mathscr{H}$ b iff a$\mathscr{L}$ b and a$\mathscr{R}$ b
(v) a$\mathscr{D}$ b iff there exists c ∈ M such that a$\mathscr{L}$ c and c$\mathscr{R}$ b.

Where $(a)_r = a\,\Gamma\, M \cup \{a\}$, $(a)_l = M\,\Gamma\, a \cup \{a\}$, $(a) = a\,\Gamma\, M \cup M\,\Gamma\, a \cup M\Gamma a\Gamma M \cup \{a\}$. Throughout this paper E_α denotes the set {e : eαe = e, e ∈ M} and $V_\alpha^\beta(a)$ denotes the set {b : b ∈ M, aαbβa = a and bβaαb = b} for all a ∈ M and all α, β ∈ Γ. An element of E_α is said to be an α—idempotent of M.

The following known results will be used in this paper. A Γ— semigroup is said to be a Γ— regular semigroup if its every related semigroup is regular.

Lemma. 11. (due to [3]) A regular Γ— semigroup is a Γ— regular semigroup if and only if $V_\alpha^\beta(e) \neq \phi$ and $V_\beta^\alpha(e) \neq \phi$ for all α, β ∈ Γ and for all e ∈ E_α.

Lemma. 12. (due to [3]) In a Γ — semigroup M for all a ∈ M and all α, β ∈ Γ, $V_\alpha^\beta(a) \neq \phi$ if and if M is regular, and $V_\alpha^\beta(e) \neq \phi$ and $V_\beta^\alpha(e) \neq \phi$ for all e ∈ E_α.

Let M be a Γ—semigroup, define the binary operation on M × Γ(Γ×M, respectively) by

$(\forall\ (a, \alpha), (b, \beta) \in M \times \Gamma)\ (a, \alpha)(b, \beta) = (a\ \alpha\ b, \beta)$

$((\forall\ (\alpha, a), (\beta, b) \in \Gamma \times M)\ (\alpha, a)\ (\beta, b) = (\alpha, a\beta b)$ respectively),

then M × Γ (Γ × M, respectively) is a semigroup (due to [4]). The semigroups M × Γ and Γ × M are called left and right operator semigroups of M, respectively.

Lemma. 13. (due to [4]) Let M be a regular Γ—semigroup, then M is an orthodox Γ— semigroup if and only if the set Reg (M × Γ) of regular elements of M × Γ are orthodox subsemigroups of M × Γ and Γ × M, respectively.

Lemmal. 14. (due to [3]) Let M be a regular Γ—semigroup then for all (a, α) ∈ M × Γ, $V(a, \alpha) = \cup\ \{(x,\beta) : x \in V_\alpha^\beta(a)\}$.

$\beta \in \Gamma,\ V_\alpha^\beta(a) \neq \varnothing$

2, ORTHODOX Γ— REGULAR SEMIGROUP

According to Sen and Saha [2] a Γ—semigroup M is said to be an orthodox Γ—semigroup if M is a regular Γ—semigroup and $(E_\alpha\ \alpha\ E_\beta) \cup ((E_\beta\ \alpha\ E_\alpha) \subseteq E_\beta$ holds for all α, β ∈ Γ. Γ—orthodox semigroup, i. e. its every related semigroup M_α. (α ∈ Γ) is an orthodox semigroup, had been introduced and the following three examples are given by Zhao Xian Zhong in [3].

Example 2.1. Let Q* denote all non—zero rational numbers and Γ denote all positive integers. Define that aαb is the product of three numbers $|a|$, α, and b, then Q* is both an orthodox Γ—semigroup and a Γ—orthodox semigroup.

Example 2.2. Let M be the set of all positive integers. Suppose that $\Gamma = M$, define $a\alpha b = [a, \alpha, b]$ = the l. c. m of a, α and b, for all a, b $\in$ M and all $\alpha \in \Gamma$. Then M is an orthodox Γ—semigroup but M is not a Γ—orthodox semigroup.

Example 2.3. Suppose that $I = \Lambda = \{1,2\}$, $G = \{e,g,g^2,g^3\}$ is the cyclic group of order four generated by g, and

$$P = (P_{ij}) = \begin{pmatrix} g & g^3 \\ g^2, & e \end{pmatrix}, \quad Q = (q_{ij}) = \begin{pmatrix} g & g^2 \\ g^3, & e \end{pmatrix}$$

are two 2×2 matrices on G. Take $M = G \times I \times \Lambda$, $\Gamma = \{\alpha, \beta\}$ and define $(a, i, \lambda)\alpha(b, j, \mu) = (ap_{\lambda j}b, i, \mu)$ and $(a, i, \lambda)\,\beta\,(b, j, \mu) = (aq_{\lambda j}b, i, \mu)$, Then M is a Γ—orthodox semigroup, but M is not an orthodox Γ—semigroup.

It is illustrated by the above examples that the class of Γ—orthodox semigroups and the class of orthodox Γ—semigroups are not contained within each other, and the intersection of these two classes is not empty. Now we ask : what is this intersection? The answer will be given in the following.

Definition 2.4. A Γ—semigroup M is said to be an orthodox Γ—regular semigroup if M is an orthodox Γ—semigroup and its every related semigroup M_α. $(\alpha \in \Gamma)$ is regular.

Proposition 2.5. A Γ—semigroup M is an orthodox Γ—regular semigroup if and only if M is both an orthodox Γ—semigroup and Γ—orthodox semigroup.

Proof : Omitted.

The following proposition 2.6 will be directly obtained from Lemma 1.3.

Proposition 2.6. A Γ—semigroup M is an othodox Γ—regular semigroup if and only if its right operator semigroup ΓxM and its left operator semigroup MxΓ are orthodox.

Proof : Omitted.

Lemma 2.7. If M is an orothodox Γ—regular semigroup, then $E_\alpha\,\beta\,E_\beta = E_\beta\,\beta\,E_\alpha = E_\alpha$ holds for all α, $\beta \in \Gamma$.

Proof : For any α, $\beta \in \Gamma$. and any $e \in E_\alpha$ there exists $x \in M$ such that $x \in V_\alpha^\beta$ (e) by lemma 1.1. This implies $e = e\alpha x\beta e$ and $e\alpha x \in E_\beta$, hence, $E_\alpha \subseteq E_\beta\beta E_\alpha$. Since M is an orthodox Γ—semigroup, we have $E_\beta\beta E_\alpha \subseteq E_\alpha$, and so $E_\alpha = E_\beta\beta E_\alpha$. The proof of $E_\alpha = E_\beta\beta E_\alpha$ is similar.

Lemma 2.8. Let M be a regular Γ—semigroup. If for all α, $\beta \in \Gamma$; $E_\alpha\beta E_\beta = E_\beta\beta E_\alpha = E_\alpha$ holds, then M is an orthodox Γ—regular semigroup.

Proof : M is obviously an orthodox Γ—semigroup. To prove the remaining half, suppose that for any α, $\beta \in \Gamma$, $E_\alpha\beta E_\beta = E_\beta\beta E_\alpha = E_\alpha$. Then for any (but fixed) $e \in E_\alpha$. there exist $x \in E_\alpha$ and $y \in E_\beta$ such that $x\beta y = e$. Hence in particular $e\beta y = e$ and $e = e\beta(y\alpha e) = e\beta(y\alpha e)\alpha e$. By $y\alpha e \in E_\beta\alpha E_\alpha = E_\beta$ we have that $(y\alpha e)\alpha e\beta(y\alpha e) = (y\alpha e)\beta(y\alpha e) = y\alpha e$. This implies $y\alpha e \in V_\beta^\alpha$ (e) and V_β^α (e) $\neq \varnothing$.

Similarly, we have V_α^β (e) ≠ ∅. Now from lemma 1.1, it follows that M is a Γ—regular semigroup.

The following thorem follows from 2.7 and lemma 2.8.

Theorem 2.9. A regular Γ—semigroup is an orthodox Γ—regular semigroup if and only if for all α, β ∈ Γ, $E_\alpha \beta E_\beta = E_\beta \beta E_\alpha = E_\alpha$.

Let us now prove the following proposition which will provide the recipe for manufacturing the class of orthodox Γ—semigroups.

Proposition 2.10. Let (S, o) be an orthodox semigroup. Suppose that M = S, and Γ is a nonempty subset of S, and aαb is the product of three elements a, α, b in the semigroup (S, o). Then we have

(i) M is an orthodox Γ—semigroup.

(ii) If S contains the identity e and Γ is an $\mathscr{H}$—class containing e in S, then M is an orthodox Γ— regular semigroup.

Proof : (i) M is obviously a Γ—semigroup. To prove the remaining half, notice that for amy α, β ∈ Γ, a ∈ E_α, and b ∈ E_β, we have aα,αa, bβ, βb ∈ E(S), where E(S) denotes the set of idempotents of S. Since S is orthodox, we now find that (aαb) β(aαb) = ((aα) (bβ)) ((aα) (bβ)) b = (aαbβ)b = aα(bβb) = aαb. Similarly, (bαa)β(aαa) = bαa. Thus $E_\alpha \alpha E_\beta \subseteq E_\alpha$ and $E_\beta \alpha E_\alpha \subseteq E_\beta$. Hence M is an orthodox Γ—semigroup as required.

(iii) It is only needed that M is a Γ—regular semigroup, by (i). For any a ∈ M, there exist x ∈ M and α, β∈Γ such that a = aαxβa, since M is a regular Γ_ semigroup. Now, note that the $\mathscr{H}$—class H_e is a subgroup in S and Γ = H_e., we have a = $a\delta\delta^{-1} \alpha x \beta \delta^{-1} \delta a$ = aδzδa for all δ∈Γ, where z = $\delta^{-1} \alpha x \beta \delta^{-1}$ and δ^{-1} is the inverse of δ in group H_e. Hence M_δ is regular for all δ∈Γ, that is, M is a Γ—regular semigroup as required.

3. ORTHOGROUP Γ—SEMIGROUPS

Let M be a Γ—semigroup. If for some α∈Γ, the related semigroup M_α (α∈Γ) is a group, then every related semigroup is a group (due to Sen [1]), and M is called a Γ—group. So the left operator semigroup Γ × M is a union of groups. Moreover, we can show that Γ × M is simple. Hence Γ × M is a completely simple semigroup. In this section we want to generalize these results.

Definition 3.1. A Γ—semigroup M is said to be an orthogroup Γ—semigroup if M is an orthodox Γ—regular semigroup and for some α∈Γ, the related semigroup M_α, is a union of groups.

It is clear that every Γ—group is an orthogroup Γ—semigroup. But it will be shown in example 3.7 that not every orthodox Γ—regular semigroup is an orthogroup Γ—semigroup.

Let M be a Γ—semigroup, if for some α∈Γ, the related semigroup M_α is a union of groups, M_β is not necessarily a union of groups for each β∈Γ. For example, suppose that Γ is the set $\{\alpha, \beta\}$, and M is a Γ—semigroup with the related semigroup M_α. being a o—group $G \cup \{o\}$, and the related semigroup M_β being a null semigroup. Then M_α is a union of groups, but the M_β is not a union of groups. We prove that every related semigroup of an orthogroup Γ—semigroup is a union of groups. To do this the following results are needed.

Lemma 3.2 Let M be a Γ—regular semigroup and $M \times \Gamma$ denote its left operator semigroup. Then for all $(a, \alpha) \in M \times \Gamma$ we have

(i) $L(a, \alpha) = \{(x, \alpha) : x \in L_a\} = (L_a, \alpha)$,

(ii) $R(a, \alpha) = \{(y, \beta) : y \in R_a, \beta \in \Gamma\} = (R_a, \alpha)$,

(iii) $H(a, \alpha) = \{(z, \alpha) : z \in H_a\} = (H_a, \alpha)$

where $L(a, \alpha)$ ($R(a, \alpha)$, $H(a, \alpha)$) denotes L—class (R—class, H—class) containing the element (a, α) of $M \times \Gamma$, L_a (R_a, H_a) denotes L—class (R—class, H—class) containing the element a in M.

Proof. Omitted :

Lemma 3.3 Let M be an othodox Γ—regular semigroup, and $M \times \Gamma$denotes its left operator semigroup, and B denotes the set of idempotents of $M \times \Gamma$. Then we have

(i) $J^{\beta}_{(e,\alpha)} = \bigcup_{\beta\in\Gamma} (V^{\beta}_{\alpha}(e), \beta)$ for all $(e, \alpha) \in B$

(ii) For all $(e, \alpha), (f, \beta) \in B$, if $(f, \beta) \in R(e, \alpha)$, then $(f, \beta) \in J^{\beta}_{(e,\alpha)}$, where $J^{\beta}_{(e,\alpha)}$denotes J—class containing (e, α) in B and $R(e, \alpha)$ denotes R—class containing (e, α) in $M \times \Gamma$.

Proof, (i) By Lemma 1.3 and Lemma 1.4 B is a band, and so $J^{\beta}_{(e,\alpha)} = V(e, \alpha)$ $= \bigcup_{\beta\in\Gamma} (V^{\beta}_{\alpha}(e), \beta)$, since M is Γ—regular semigroup.

(ii) If $(f, \beta) \in R(e, \alpha)$, then there exist (x, β) and $(y, \alpha) \in M \times \Gamma$ such that $(e, \alpha)(x, \beta) = (f, \beta)$, and $(f, \beta)(y, \alpha) = (e, \alpha)$. Hence in particular $e\alpha x = f$ and $f\beta y = e$. Moreover, $e\alpha f = f$ and $f\beta e = e$. Notice that $e \in E_\alpha$ and E_β, thus we have $f\beta e\alpha f = f$ and $e\alpha f\beta e = e$, that is, $f \in V^{\beta}_{\alpha}(e)$. By (i), $(f, \beta) \in J^{\beta}_{(e,\alpha)}$, as required.

Proposition 3.4. The left operator semigroup $M \times \Gamma$ of an orthogroup Γ—semigroup M is a union of groups.

Proof. Let M be an ortho group Γ—semigroup, and $M \times \Gamma$ denote its left operator semigroup, and B denote the set of idempotents of $M \times \Gamma$. Since M is a Γ—regular semigroup, every $\mathfrak{R}$—class of M contains at least one α—iedmpotent, say e, (i, e. $e\alpha e = e$) for all $\alpha \in \Gamma$. So every $\mathfrak{R}$—class of $M \times \Gamma$ contains at least one idempotent which belongs to the set $\{(x, \alpha) : x \in M\}$ for all $\alpha \in \Gamma\}$ by lemma 3.2. Now, suppose that for some (but fixed) $\delta \in \Gamma$., the related semigroup M_δ is a union of groups, and (e, α), (f, β) ∈ B as well as (e, α) $\mathscr{D}$ (f, β) in M × Γ. As above there exists (c, δ) ∈ B such that (c, δ) ∈ $R_{(f,\beta)}$, and so (c, δ)J^B (f, β) by Lemma 3.8 (ii). Since M_δ is a union of groups, there exists (d, δ) ∈ B such that (d, δ) ∈ R (e, α) and (d, δ) ∈ L (c, δ). Hence (e, α) J^B (d, δ) by Lemma 3.4 (ii), and (d, δ) J^B (c, δ), M_δ being a union of groups. So (e, α) J^B (f, β), because M is an orthodox Γ semigroup, M × Γ is an orthodox semigroup by lemma 1.3. Now notice we have proved that (e, α), (f, β) ∈ B and (e, α) $\mathscr{D}$ (f, β) imply (e, α) J^B(f, β), thus M × Γ is a union of groups by proposition vl. 3.3 of Howie J.M. [6].

A Γ—semigroup M is said to be a Γ—union of groups if its every related semigroups is a union of groups.

Lemma 3.5 A Γ—semigroup M is a Γ—union of groups if and only if its left (right, respectively) operator semigroups is a union of groups.

Proof. Omitted

Now, the following theorem can be directly obtained by Lemma 3.4 and Lemma 3.5.

Theorem 3.6. A Γ—semigroup M is an orthogroup Γ—semigroup if and only if it is both an orthodox Γ—semigroup and a Γ—union of groups.

Proof. Omitted.

Finally, We shall illustrate that not every othodox Γ—regular semigroup is an orthogroup Γ—semigroup by the following example.

Example 3.7. let M be the direct product T_EXG of the bicyclic semigroup and a group G with |G|> 1. Suppose, the Γ is the $\mathscr{H}$—class containing the identity e of M, and aαb is the product of three elements a, α, b of the semigroup M for all, a, b ∈ M and all α∈Γ, then M is an orthodox Γ regular semigroup by proposition 2.10 (ii). But if α denotes the identity e of M, then the related semigroup M_α with the identity is not union of groups, so M is not an orthogroup Γ—semigroup.

4. RECTANGULAR Γ—GROUPS

We consider here orthodox completely simple Γ—semigroup. As we shall see they are quite special. In the context, the following concept plays a basic role.

A Γ—rectangular band means a Γ—semigroup such that its every related semigroup is rectangular band. Also it is clear that for all a, b ∈ M and all α, β∈Γ, aαb = aβb holds in a Γ—rectangular band M. So it is often convenient to say rectangular band instead of Γ—rectangular band.

Definition 4.1. A Γ—semigroup is said to be a rectangular Γ—group if it is isomorphic to the direct product of a rectangular band and a Γ—group.

By Seth A. [5] every completely simple Γ—semigroup admits a Rees matrix representation with a set of sandwitch matrices.

Proposition 4.2 The following conditions on $M = M(G. I. \Lambda. P)$ are equivalent.

(i) M is an orthodox Γ—semigroup.

(ii) For any $i, j \in I$, $\lambda\ \mu \in \Lambda$, $P = (P_{\lambda i})$, $Q = (q_{\mu j}) \in P$.

$p_{\lambda i}^{-1} q_{\lambda i}^{-1} q_{\mu j}^{-1} p_{\mu i} = e$ and $q_{\mu j}^{-1} q_{\mu j} p_{\lambda i}^{-1} p_{\lambda i} = e$

where e is the identity of G.

Proof : Firstly, notice that $E_p = \{(p_{\lambda i}^{-1}, i, \lambda);\ i \in I, \lambda \in \Lambda\}$

and $E_Q = \{(q_{\mu i}^{-1}, j, u) : j \in I, \mu \in \Lambda\}$. For any $(q_{\lambda i}^{-1}, i, \lambda) \in E_p$ and any $(q_{\mu i}^{-1}, j, \mu) \in E_Q$, we have that

$(p_{\lambda i}^{-1}, i, \lambda)\ Q(q_{\mu i}^{-1}, j, \mu) = (p_{\lambda i}^{-1}\ q_{\lambda i}, (q_{\mu i}^{-1}, i, \mu) \in E_P$ and $(q_{\mu i}^{-1}, j, \mu)\ Q(q_{\lambda i}^{-1}, i, \lambda) = (q_{\mu j}^{-1}\ q_{\mu i}\ p_{\lambda i}^{-1}\ j, \lambda) \in E_Q$ if and only if $p_{\mu i}^{-1}\ q_{j \mu}^{-1}\ q_{\mu j}^{-1}\ p_{\mu j}^{-1} = q_{\mu j}^{-1}\ q_{\mu j}\ p_{\lambda j}^{-1}\ p_{\lambda j} = e$.

Thus (i) if and only if (ii) holds.

Theorem 4.3. A Γ—semigroup M is an orthodox completely simple Γ—semigroup if and only if it is a rectangular Γ—group.

Proof : A rectangular Γ—group is obviously orthodox completely simple Γ—semigroup by proposition 4.2. Suppose that M is an Orthodox completely simple Γ—semigroup. Given $\alpha \in \Gamma$, by theorem 2.5. of J. M. Howise [6] $\varnothing : H_n \times I \times \Lambda \to M$ given by $(a,i,\lambda)\varnothing = e_i \alpha a \alpha f_\lambda$ is an isomorphism from the Rees matrix semigroup $M[H_n ; I\Lambda ; P]$ onto M_α, where e_i and f_λ are α—identity of H_{i1} and $H_{1\lambda}$ respectively. Since M_α is a union of groups for all $\alpha \in \Gamma$, every $\mathscr{H}$—class in M is a sub Γ—group of M if we define $(i, \lambda)\ \beta(j, \mu) = (i, \mu)$ for all $(i, \lambda) \in I \times \Lambda$ and all $\beta \in \Gamma$, then $I \times \Lambda$ becomes a rectangular band. So $H_n \times I \times \Lambda$ is a rectangular Γ—group. Now we show that φ is an isomorphism from $H_n \times I \times \Lambda$ onto Γ—semigroup M. Since for any (a,j,λ), (b,i,μ), $\in H_n \times I \times \Lambda$ and, $\beta \in \Gamma$,

we have

$((a,i,\lambda)\ \beta\ (b,i,\mu)\varphi$

$= (a\beta b,\ i,\mu)\varphi$

$= e_j \alpha(a,\beta b)\alpha f_\mu$

$= (e_i \alpha a)\alpha(I_\alpha \beta I_\beta)\ \beta b \alpha f_\mu$ (I_α and 1_β being α—indentity and β—indentity in Γ—group H_n respectively)

$= (e_i \alpha a)\alpha(I_\alpha \alpha f_\lambda \beta I_\beta)\ \beta b \alpha f_\mu$

$= (e_i \alpha a)\alpha(f_\lambda \beta I_\beta)\ \beta b \alpha f_\mu$

$= (e_i \alpha a)\alpha(f_\lambda \beta I'_\beta\ \beta I_\beta)\ \beta b \alpha f_\mu$ (I'_β being β—indentity in H_n)

$= (e_i \alpha a \alpha f_\lambda)\beta(I'_\beta\ \beta I_\beta)\ \beta b \alpha f_\mu$

$= (e_i \alpha a \alpha f_\lambda)\beta(I'_\beta\ \beta e_i \alpha I_\beta)\ \beta b \alpha f_\mu$

$= (e_i \alpha a \alpha f_\lambda)\beta(e_i \alpha b \alpha f_\mu)$

$= (a, i, \lambda)\ \varphi\ \beta(b, j, u)\varphi$, as required.

Theorem 4.4. An orthogroup Γ—semigroup is a semilattice of rectangular Γ—groups.

Proof : An orthogroup Γ—semigroup M is a Γ—union of groups. It means its every related semigroup $M_\alpha(\alpha \in \Gamma)$ is a union of groups. So we can say the Green's relations J on M_α (for all $\alpha \in \Gamma$) are same, and M is a semilattice of completely simple Γ—semigroups. Moreover, note that every related semigroup M_α of M is orthodox. So we have M is a semilattice of Orthodox completely simple Γ—semigroup. Hence M is a semilattice of rectangular Γ—groups by theorem 4.3.

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REFERENCE

1. Sen M. K., Saha N. K., On Γ—semigroup I, Bull. Cal. Math. Soc. Vol 78 (1986): 180—186.
2. Sen M. K., Saha N. K., Orthodox Γ—semigroup, Internat J. Math. & Math. Sci, Vol 13 (1990) : 527—534.
3. Zhao Xian Zhong, Γ—orthodox semigroup. Pure and applied Math : Vol. 10 (1994) : 116—120.
4. Zhao Xian Zhong,, A remark on orthodox Γ—semigroup J. Northwest University, Vol. 24 (1) (1994) 101—109.
5. Seth A., Ree's theorem for Γ—semigroup. Bull. Cal. Math. Soc Vol. 81 (1989) 217—226.
6. Howie J. M., An introduction to semigroup theory, London—New York—San Francisco : Academic Pr. 1976.
7. Saha N. K., On Γ—semigroup II, Bull. Cal. Math. Soc. Vol. 81 (1989), 217—226.

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ON IRRESOLUTE FUZZY MULTIFUNCTIONS

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ABSTRACT : The paper deals with the study of irresolute fuzzy multifunctions. Several characterizations and certain properties of such multifuntions are established.

Key words : Fuzzy multifunctions, fuzzy upper and lower irresolute multifunctions, fuzzy graph multifunctions

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1. INTRODUCTION AND PRELIMINARIES

The study of functions taking points of an ordinary topological space X to fuzzy sets [21] in a fuzzy topological space [2] has so far been made by many researchers. Such functions have been termed as fuzzy multifunctions. Papageourgiou [15] took a leading role in this direction. Later on, Mukherjee and Malakar [11] suitably redefined the concepts of lower inverse and lower semicontinuity of such multifunctions in a natural way and obtained certain expected results. The sole objective of all these papers has been to generalize the existing results on some well known functions which are usually treated between two topological spaces.

The class of irresolute functions, a type of functions independent of continuity, was introduced by Crossley and Hildebrand [4], and subsequently studied by many others. In the present paper, an attempt has been made to introduce the idea of irresolute fuzzy multifunctions. In Section 2, we define, characterize and study such a multifunction in its two splitted forms viz. lower and upper irresolute fuzzy multifunctions. In the next section, our intention has been to apply these fuzzy multifunctions to a few situations to determine whether certain topological properties are transfered to the corresponding fuzzy topological properties under such multifunctions.

Throughout the paper, by (X,T) or simply by X we shall mean a topological space in the classical sense and (Y, T_1) or simply Y will stand for a fuzzy topological space (fts, for short) as defined by Chang [2]. A fuzzy point [16] with the singleton support $y \varepsilon Y$ and the value α $(0 < \alpha \leq 1)$ at y will be denoted by y_α. The closure and interior of a set A in X (or a fuzzy set A in Y) will be denoted by $\mathrm{cl}A$ and $\mathrm{int}\,A$ respectively. The support of a fuzzy set A in Y will be denoted by A_0, i.e. $A_0 = \{y \in Y : A(y) \neq 0\}$. For a fuzzy set A in Y, 1-A will stand for the complement of A in Y [21]. For two fuzzy sets A, B in Y, we write $A \leq B$ to mean that $A(y) \leq B(y)$, for each $y \in Y$; and write $A\ q\ B$ to mean that A is quasicoincident (q-coincident, for short) with B [16]. The negation of the last statement, i.e. when A is not q-coincident with B, is denoted by $A \not{q} B$. A fuzzy set B is called a quasi-neighbourhood (q-nbd, for short) of a fuzzy set A if there exists a fuzzy open set U such that $A\ q\ U \leq B$ [16]. A fuzzy set A (a set A) in an fts Y (in a topological space X) is called fuzzy semiopen [1] (resp. semiopen [7]) if there exists a fuzzy open set (resp. open set) B such that $B \leq A \leq \mathrm{cl}\,B$. The complements of semiopen (fuzzy semiopen) sets are called semiclosed (fuzzy semiclosed) sets. Union of semiopen (fuzzy semiopen) sets is known [7,1] to be respectively so. Also, clearly every open (fuzzy open, closed, fuzzy closed) set is semiopen (fuzzy semiopen, semiclosed, fuzzy semiclosed). A fuzzy set A in Y is said to be a semi-q-nbd of a fuzzy point y_α [6] if there exists a fuzzy semiopen set V in Y such that $y_\alpha\ q\ V \leq A$. The fuzzy semiclosure of a fuzzy set A in Y, to be denoted by $\mathrm{scl}\,A$, is the union of all fuzzy points y_α such that every fuzzy semiopen semi-q-nbd of y_α is q-coincident with A [6]. The semiclosure [3] of any set A in X will also be denoted by $\mathrm{scl}\,A$. It is known [6,3] that semiclosure (fuzzy semiclosure) of a set (fuzzy set) A in X (resp. Y) is the intersection of all semiclosed (fuzzy semiclosed) sets containing A. A (fuzzy) set A is (fuzzy) semiclosed iff $A = \mathrm{scl}\,A$ [6,3]. A (fuzzy) set B is called a (fuzzy) seminbd of a (fuzzy) set A if there is a (fuzzy) semiopen set U containing A and is contained in B. The union of all (fuzzy) semiopen sets contained in a (fuzzy) set A is called the (fuzzy) semi-interior of A, to be denoted by $\mathrm{sint}\,A$. It is clear that a (fuzzy) set A is (fuzzy) semiopen iff $A = \mathrm{sint}\,A$.

2. UPPER AND LOWER IRRESOLUTE FUZZY MULTIFUNCTIONS

Definition2.1. **[15]** Let (X,T) be a topological space in the classical sense and (Y,T_1) be an fts. We say that $F : X \to Y$ is a fuzzy multifunction, if for each $x \in X$ $F(x)$ is a fuzzy set in Y. Such A fuzzy multifunction will be called surjective if $F(X) = 1_Y$, where by 1_Y we mean the fuzzy set in Y given by $1_Y(y) = 1$, for all $y \in Y$.

Throughout the paper, by $F : X \to Y$ we shall mean that F is a fuzzy multifunction from a topological space (X,T) to an fts (Y,T_1).

Definition2.2. **[11]** For a fuzzy multifunction $F : X \to Y$, the upper inverse $F^+(A)$ and the lower inverse $F^-(A)$ of a fuzzy set A in Y are defined as follows :

$F^+(A) = \{ x \in X : F(x) \leq A\}$ and $F^-(A) = \{ x \in X : F(x)\ q\ A\}$.

The following theorem gives a relation between $F^+(A)$ and $F^-(A)$.

Theorem 2.3. **[11]** For a fuzzy multifunction $F : X \to Y$, we have $F^-(1 - G) = X - F^+(G)$, for any fuzzy set G in Y.

Definition 2.4. A Fuzzy multifunction $F : X \rightarrow Y$ is said to be fuzzy

(a) upper irresolute at a point $x_0 \varepsilon X$, if for every fuzzy semiopen set V in Y with $x_0 \varepsilon F^+(V)$ there exists a semiopen set U in X with $x_0 \varepsilon U$ such that $U \subset F^+(V)$,

(b) lower irresolute at a point $x_0 \varepsilon X$, if for every fuzzy semiopen set V in Y with $x_0 \varepsilon F^-(V)$ there exists a semiopen set U in X with $x_0 \varepsilon U$ such that $U \subset F^-(V)$,

(c) upper irresolute (lower irresolute) on X if it is upper irresolute (lower irresolute) at each point of X.

Theorem 2.5. A fuzzy multifunction $F : X \rightarrow Y$ is fuzzy upper irresolute iff $F^+(U)$ is semiopen for every fuzzy semiopen set U in Y.

Proof : Let $F : X \rightarrow Y$ be a fuzzy upper irresolute multifunction. For a fuzzy semiopen set U in Y, let $x \varepsilon F^+(U)$. Then there is a semiopen set V in X with $x \varepsilon V$ such that $V \subset F^+(V)$. Thus $F^+(U) \subset \text{sint } F^+(U)$ and hence $F^+(U)$ is semiopen in X. Conversely, let $F^+(U)$ be semiopen in X for each fuzzy semiopen set U in Y. Let $x_0 \varepsilon X$ and V be a fuzzy semiopen set in Y with $x_0 \varepsilon F^+(V)$. Now $F^+(V)$ (= U, say) is semiopen in X containing x_0 such that $x_0 \varepsilon U \subset F^+(V)$, proving F to be fuzzy upper irresolute at x_0 and hence on X.

Theorem 2.6. A fuzzy multifunction $F : X \rightarrow Y$ is fuzzy lower irresolute iff $F^-(U)$ is semiopen in X for each fuzzy semiopen set U in Y.

Proof : The proof is quite similar to that of Theorem 2.5, and is thus omitted.

Theorem 2.7. A fuzzy multifunction $F : X \rightarrow Y$ is fuzzy lower (upper) irresolute iff $F^+(W)$ (resp. $F^-(W)$) is semiclosed in X for every fuzzy semiclosed set W in Y.

Proof : Follows from Theorems 2.3, 2.5 and 2.6.

Lemma 2.8.[13] Let A be an open subset of a topological space (X, T). Then

(a) U ($\subset X$) is semiopen in X $\Rightarrow U \cap A$ is semiopen in A i. e. in (A, T_A)

(b) U ($\subset A$) is semiopen in A $\Rightarrow$ U is semiopen in X.

Theorem 2.9. Let $\{U_\alpha : \alpha \varepsilon \Lambda\}$ be an open cover of a topological space X. A fuzzy multifunction $F : X \rightarrow Y$ is fuzzy upper (lower) irresolute iff $F/u_\alpha : U_\alpha \rightarrow Y$ is fuzzy upper (lower) irresolute for each $\alpha \varepsilon \Lambda$.

Proof : Let $F : X \rightarrow Y$ be fuzzy upper (lower) irresolute. Let $x \varepsilon U_\alpha$, for some $\alpha \varepsilon \Lambda$. Suppose V is a fuzzy semiopen set in Y such that $x \varepsilon F^+(V)$ ($x \varepsilon F^-(V)$). Then there is a semiopen set U in X with $x \varepsilon U$ such that $U \subset F^+(V)$ (resp. $U \subset F^-(V)$). By Lemma 2.8. (a), $U \cap U_\alpha$ is semiopen in U_α containing x, and $U \cap U_\alpha \subset (F/u_\alpha)^+ (V)$ (resp. $U \cap U_\alpha \subset (F/u_\alpha)^- (V)$. Hence $F/u_\alpha : U_\alpha \rightarrow Y$ is fuzzy upper (lower) irresolute for each $\alpha \varepsilon \Lambda$. Conversely, let $x \varepsilon X$ and V be a fuzzy open set in Y such that $x \varepsilon F^+(V)$ ($x \varepsilon F^-(V)$). Now $x \varepsilon U_\alpha$ for some $\alpha \varepsilon \Lambda$, and $F/u_\alpha : U_\alpha \rightarrow Y$ is fuzzy upper (lower) irresolute. So there exists a semiopen set U in U_α such that $F/u_\alpha (U) \leq V$ ($U \subset (F/u_\alpha)^- (V)$). Now by Lemma 2.8 (b), U is semiopen in X and $U \subset U_\alpha$. Thus $F(U) = F/u_\alpha (U) \leq V$ (resp. $F(U) = (F/u_\alpha) (U)\ q\ V$), i. e. $U \subset F^+(V)$ (resp. $U \subset F^-(V)$). Hence F is fuzzy upper (lower) irresolute.

Definition 2.10. Let $F : X \rightarrow Y$ be a fuzzy multifunction. The fuzzy multifunction $\text{scl } F : X \rightarrow Y$ is defined as $(\text{scl } F)(x) = \text{scl } F(x)$, for each $x \varepsilon X$; and the multifunction $\text{cl}F : X \rightarrow Y$ is defined as $(\text{cl } F)(x) = \text{cl } F(x))$, for each $x \varepsilon X$.

Lemma 2.11. If $F : X \to Y$ is a fuzzy multifunction, then $(\text{scl } F)^-(V) = F^-(V)$, for every fuzzy semiopen set V in Y.

Proof : $x \varepsilon F^-(V) \Rightarrow F(x)\ q\ V \Rightarrow \text{scl}\,(F(x))\ q\ V \Rightarrow x\ \varepsilon (\text{scl } F)^-(V)$. Again, $x\ \varepsilon (\text{scl } F)^-(V) \Rightarrow V\ q\ \text{scl}\,(F(x)) \Rightarrow$ there exists $y\ \varepsilon\ V_0 \cap (\text{scl } F(x))_0$ such that $V(y) + (\text{scl } F(x))(y) > 1$. Let $(\text{scl } F(x))(y) = \alpha$. Then $y_\alpha \le \text{scl } F(x)$ and $\alpha + V(y) > 1$. Then V is a semiopen semi-q-nbd of y_α. Hence $v\ q\ F(x)$, i.e. $x\ \varepsilon\ F^-(v)$.

Theorem 2.12. A fuzzy multifunction $F : X \to Y$ is fuzzy lower irresolute iff $\text{scl } F : X \to Y$ is fuzzy lower irresolute.

Proof : Follows at once from Lemma 2. 11.

Lemma 2.13. If $F : X \to Y$ is a fuzzy multifunction, then $(\text{scl } F)^+(C) = F^+(C)$ for every fuzzy semiclosed set C in Y.

Proof : $x\ \varepsilon\ (\text{scl}F)^+(C) \Rightarrow (\text{scl } F)(x) \le C \Rightarrow F(x) \le C \Rightarrow x\ \varepsilon\ F^+(C)$. Again, $x\ \varepsilon\ F^+(C) \Rightarrow F(x) \le C \Rightarrow \text{scl } F(x) \le \text{scl } C = C \Rightarrow x\ \varepsilon\ (\text{scl}F)^+(x)$.

Theorem 2.14. A fuzzy multifunction $F : X \to Y$ is fuzzy upper irresolute iff $\text{scl } F : X \to Y$ is upper irresolute.

Proof : Follows immediately from Lemma 2.13.

Theorem 2.15. For A fuzzy multifunction $F : X \to Y$, the following are equivalent:

(a) F is fuzzy lower irresolute.

(b) For each $x\ \varepsilon\ X$ and each fuzzy semi q-nbd V of F(x) in Y, $F^-(v)$ is a semi-nbd of x in X.

(c) For each $p\ \varepsilon\ X$ and each fuzzy semi q-nbd V of F(p) in Y, there is a semiopen set U containing p in X such that $U \subset F^-(V)$.

(d) $F(\text{scl}A) \le \text{scl}\,(F(A))$, for any subset A of X.

(e) $\text{scl}\,(F^-(B)) \subset F^+(\text{scl}B)$, for any fuzzy set B in Y.

(f) $F(\text{int cl}A) \le \text{scl } F(A)$, for any subset A of X.

Proof : (a) $\Rightarrow$ (b) : Let $x\ \varepsilon\ X$ and V be a fuzzy semi-q-nbd of F (x) in Y, then there exists a fuzzy semiopen set U in Y such that $F(x)\ q\ U \le V$. Then $x\ \varepsilon\ F^-(U) \subset F^-(V)$. By (a), $F^-(U)$ is a semiopen set in X. Hence $F^-(V)$ is a semi-nbd of x in X.

(b) $\Rightarrow$ (c) : Let $p\ \varepsilon\ X$ and V be a fuzzy semi-q-nbd of F (p) in Y. By (b), $F^-(V)$ is a semi-nbd of p in X. Then there exists a semiopen set U in X such that $p\ \varepsilon\ U \subset F^-(V)$.

(c) $\Rightarrow$ (a) : Let $x\ \varepsilon\ X$ and V be a fuzzy semiopen set in Y with $x\ \varepsilon\ F^-(V)$. Then $f(x)\ q\ V$, i.e. V is a fuzzy semi-q-nbd of F (x) in Y. By (c), there is a semi-open set U containing x in X such that $U \subset F^-(V)$. Hence F is fuzzy lower irresolute.

(a) $\Rightarrow$ (d) : For any $A \subset X$, $\text{scl } F(A)$ is fuzzy semiclosed in Y. By (a), $F^+(\text{scl } F(A))$ is semiclosed in X and clearly contains A. Thus $\text{scl } A \subset F^+(\text{scl}\,(F(A)))$, i.e., $F(\text{scl } A) \le F\,F^+(\text{scl}\,(F(A))) \le \text{scl } F(A)$.

(d) $\Rightarrow$ (a) : Let V be a fuzzy semiclosed set in Y and let $A = F^+(V)$. By (d), $F(\text{scl } A) \le \text{scl}\,(F(A)) = \text{scl}\,(F\,F^+(V)) \le \text{scl } V = V$.

Thus $\text{scl } A \subset F^+(V) = A$, i.e. $A = \text{scl } A$. Hence $A = F^+(V)$ is semiclosed in X. Then by Theorem 2.7, F becomes fuzzy lower irresolute.

(d) $\Rightarrow$ (e) : Let B be a fuzzy set in Y and let $A = F^+(B)$. Then by (d), $F(\mathrm{scl}\ A) \leq \mathrm{scl}\ (F(A))$, i.e. $F(\mathrm{scl}\ (F^+(B))) \leq \mathrm{scl}\ (F F^+(B)) \leq \mathrm{scl}\ B$, i.e. $\mathrm{scl}\ (F^+(B)) \subset F^+(\mathrm{scl}\ B)$.

(e) $\Rightarrow$ (d) : Let $A \subset X$. Then $B = F(A)$ being a fuzzy set in Y, by (e) we have $\mathrm{scl}\ (F^+(B)) = \mathrm{scl}\ (F^+(F(A))) \subset F^+(\mathrm{scl}\ B) = F^+(\mathrm{scl}\ F(A))$, i.e. $\mathrm{scl}\ A \subset F^+(\mathrm{scl}\ (F(A)))$. Hence $F(\mathrm{scl}\ A) \leq \mathrm{scl}\ (F(A))$.

(a) $\Rightarrow$ (f) : Since F is fuzzy lower irresolute, by Theorem 2.7 we have, $F^+(\mathrm{scl}\ (F(A)))$ to be semiclosed in X, for each $A \subset X$. Then $\mathrm{int\ cl}\ F^+(\mathrm{scl}\ (F(A))) \subset F^+(\mathrm{scl}\ (F(A)))$. Thus $\mathrm{intcl}\ A \subset \mathrm{intcl}\ (F^+(\mathrm{scl}\ F(A))) \subset F^+(\mathrm{scl}\ F(A)))$, i.e. $F(\mathrm{intcl}\ A) \leq \mathrm{scl}\ F(A)$.

(f) $\Rightarrow$ (d) : For any $A \subset X$, we have $\mathrm{scl}A = A \cup \mathrm{intcl}A \Rightarrow F(\mathrm{scl}A) = F(A) \cup F(\mathrm{inlcl}\ A) \leq F(A) \cup \mathrm{scl}\ F(A)$ (by (f)) $= \mathrm{scl}\ F(A)$.

Theorem 2.16. For a fuzzy multifunction $F : X \to Y$, the following are equivalent:

(a) F is fuzzy upper irresolute.

(b) For each $x \in X$ and each fuzzy semi-nbd V of $F(x)$ in Y, $F^+(V)$ is a semi-nbd of x in X.

(c) For each $x \in X$ and each fuzzy semi-nbd V of $F(x)$, there exists a semi-nbd U of x in X such that $F(U) \leq V$.

(d) For any fuzzy set B in Y, $\mathrm{scl}\ (F^-(B)) \subset F^-(\mathrm{scl}\ B)$.

Proof. (a) $\Rightarrow$ (b) : Let $x \in X$ and V be a fuzzy semi-nbd of $F(x)$ in Y. Then there is a fuzzy semi-open set U in Y such that $F(x) \leq U \leq V$. As F is fuzzy upper irresolute, by Theorem 2.5, $F^+(U)$ is a semiopen set in X. Now, $x \in F^+(U) \subset F^+(V)$, which shows that $F^+(V)$ is a semi-nbd of x in X.

(b) $\Rightarrow$ (c) : Obvious.

(c) $\Rightarrow$ (a) : Let $x \in X$ and V be a fuzzy semiopen set in Y such that $F(x) \leq V$. By (c), there is a semi-nbd U of x in X such that $F(U) \leq V$. Then there is a semi-open set W in X such that $x \in W \subset U$, i.e., $F(x) \leq F(W) \leq F(U) \leq V$. Hence F is fuzzy upper irresolute.

(a) $\Rightarrow$ (d) : For any fuzzy set B in Y, scl B is fuzzy semiclosed and hence by (a), $F^-(\mathrm{scl}\ B)$ is semiclosed in X and obviously contains $F^-(B)$. Thus $\mathrm{scl}\ (F^-(B)) \subset F^-(\mathrm{scl}\ B)$.

(d) $\Rightarrow$ (a) : Let B be fuzzy semiclosed in Y. Then by (d), $\mathrm{scl}\ (F^-(B)) \subset F^-(\mathrm{scl}\ B)) = F^-(B)$. Thus $\mathrm{scl}\ (F^-(B))) = F^-(B)$, i.e. $F^-(B)$ is semiclosed in X. Hence F is fuzzy upper irresolute.

A subset A in a topological space (X, T) can be treated as a fuzzy set in X with characteristic function given by $A(x) = 1$, if $x \in A$, and $A(x) = 0$, if $x \in X \backslash A$. Thus with this understanding (X, T) becomes a fuzzy topological space too. We then consider the product fts $X \times Y$ in the usual way as defined by Wong [20]. Thus the fuzzy sets of the form $U \times V$ with $U \in T$ and $V \in T_1$ form an open basis for the product fuzzy topology $T \times T_1$, on $X \times Y$, where for any $(x, y) \in X \times Y$.

$$(U \times V)(x, y) = \begin{cases} \min(U(x), V(y)) = 0 \text{ if } x \notin U \\ V(y), \text{ if } x \in U \end{cases}$$

i.e. $(U \times V)(x, y) \leq V(y)$ in this situation.

In [1] Azad set forth a condition under which an fts X is called product related to another fts Y. In the same paper it was proved that if an fts X is product related to another fts Y, then for any fuzzy sets A, B in X and Y respectively, cl $(A \times B) =$ cl $A \times$ cl B.

Definition 2.17 [11] For a fuzzy multifunction $F : X \to Y$, the fuzzy graph multifunction $G_F : X \to X \times Y$ of F is defined as $G_F(x) =$ the fuzzy set $x_1 \times F(x)$ of $X \times Y$, where x_1 is the fuzzy set in X, whose value is 1 at $x \in X$ and O at other points of X. We shall write $\{x\} \times F(x)$ for $x_1 \times F(x)$.

Theorem 2·18. When X is product related to Y, a fuzzy multifunction $F : (X, T) \to (Y, T_1)$ is fuzzy lower irresolute if its fuzzy graph multifunction $G_F : X \to X \times Y$ is fuzzy lower irresolute.

Proof. Let $G_F : X \to X \times Y$ be fuzzy lower irresolute. Let $x \in X$ and V be a fuzzy semiopen set in Y such that $x \in F^-(V)$. i.e. $F(x)\, q.\, V$. Then there exists $y \in [F(x)]_0 \cap V_0$ such that $[F(x)](y) + V(y) > 1$. So $[G_F(x)](x, y) + (X \times V)(x,y) = [F(x)](y) + V(y) > 1$. Hence $G_F(x)\, q\, (X \times V)$, where $X \times V$ is clearly a fuzzy semiopen set in $X \times Y$. In fact, V being fuzzy semiopen in Y, $U \leq V \leq$ cl U, for some fuzzy open set U in Y ; then $X \times U$ is a basic open fuzzy set in $X \times Y$ such that $X \times U \leq X \times V \leq$ cl $X \times$ cl $U =$ cl $(X \times U)$, as X is product related to Y. Now, G_F being fuzzy lower irresolute, there is a semiopen set U in X such that $x \in U$, and $G_F(z)\, q\, (X \times V)$, for all $z \in U$. We are only to show that $F(z)\, q\, V$, for all $z \in U$. If possible, let there exist some $z_0 \in U$ such that $F(z_0) \not{q}\, V$. Then for all $y \in Y$, $[F(z_0)](y) + V(y) \leq 1$. Now, for any $(r,s) \in X \times Y$, $[G_F(z_0)](r,s) \leq [F(z_0)](s)$, and $(X \times V)(r, s) = V(s)$. Now, $[G_F(z_0)](r, s) + (X \times V)(r,s) \leq [F(z_0)](s) + V(s) \leq 1$. Thus $G_F(z_0) \not{q}\, (X \times V)$, where $z_0 \in U$. This is a contradiction and hence F is fuzzy lower irresolute.

Theorem 2·19. When X is product related to Y, a fuzzy multifunction $F : (X, T) \to (Y, T_1)$ is fuzzy upper irresolute if its fuzzy graph multifunction $G_F : X \to X \times Y$ is fuzzy upper irresolute.

Proof. Let G_F be fuzzy upper irresolute. We shall show that F is fuzzy upper irresolute. Let $x \in X$ and V be a fuzzy semiopen set in Y with $F(x) \leq V$. Then $G_F(x) \leq X \times V$, and $X \times V$ is easily seen to be fuzzy semiopen in $X \times Y$. Then by hypothesis, there is a semiopen set U in X with $x \in U$ such that $G_F(U) \leq X \times V$. Now for any $z \in U$ and for any $Y \in Y$, $[F(z)](y) = [G_F(z)](z,y) \leq (X \times V)(z, y) = V(y)$, i.e. $[F(z)](y) \leq V(y)$, for all $y \in Y$. Thus $F(z) \leq V$, for any $z \in U$, i.e. $F(U) \leq V$. Hence F s fuzzy upper irresolute.

3. APPLICATIONS

It is known [5] that a topological space X is semicompact if every semiopen cover of X has a finite subcover. The definitions of cover and semicompactness in fuzzy setting are now recalled below :

Definition 3.1. [7] A collection ϑ of fuzzy sets in an fts Y is said to be a fuzzy cover of a fuzzy set A in Y if $(\cup \vartheta)(x) = 1$, for each $x \in A_0$; ϑ is called a fuzzy cover of Y if $(\cup \vartheta) = 1_Y$.

Definition 3.2. **[8]** A fuzzy set A in an fts Y is said be a fuzzy semicompact set if every fuzzy cover ϑ of A fuzzy semiopen sets in Y has a finite subfamily ϑ_0 such that $A \leq \cup \vartheta_0$; if in addition, $A = 1_Y$, then Y is said to be fuzzy semicompact.

Theorem 3.3. Let $F : X \multimap Y$ be a surjective fuzzy multifunction and F(x) be a fuzzy semicompact set in Y for each $x \varepsilon X$. If F is fuzzy upper irresolute and X is semicompact, then Y is fuzzy semicompact.

Proof : Let $\{A_\alpha : \alpha \varepsilon \Lambda\}$ be a fuzzy cover of Y by fuzzy semiopen sets in Y. Now, for each $x \varepsilon X$, F(x) is a fuzzy semicompact set in Y, and so there is a finite subset Λ_x of Λ such that $F(x) \leq \cup \{A_\alpha : \alpha \varepsilon \Lambda_x\}$. Let $A_x = \{A_\alpha : \alpha \varepsilon \Lambda_x\}$. Then $F(x) \leq A_x$, and A_x is fuzzy semiopen in Y. Since F is fuzzy upper irresolute, there exists a semiopen set U_x in X containing x such that $U_x \subset F^+(A_x)$. The family $\{U_x : x \varepsilon X\}$ is then a semiopen cover of the semicompact space X. Thus there exist finitely many points $x_1, x_2, \ldots x_n$ in X such that $X = \cup\{U_{x_i} : i = 1,2\ldots, n\}$. As F is surjective, we have

$$1_Y = F(X) = \bigcup_{i=1}^{n} F(U_{x_i}) \leq \bigcup_{i=1}^{n} A_{x_i} \leq \bigcup_{i=1}^{n} \bigcup_{\alpha \varepsilon \Lambda_{x_i}} A_\alpha.$$ Hence Y is fuzzy semicompact.

We recall that a subset A of a topological space X is called an S-closed set [14] if every cover of A by semiopen sets of X has a finite proximate subcover ; if in addition, A = X, then X is called an S-closed space [19]. The corresponding definition for an fts runs as follows.

Definition 3.4. **[10]** A fuzzy set A in an fts Y is said be a fuzzy S-closed set if for every fuzzy cover ϑ of A by fuzzy semiopen sets in Y has a finite proximate subcover, i.e., has a finite subfamily ϑ_0 of ϑ such that $A \leq \cup\{\mathrm{cl}\ U : U \varepsilon \vartheta_0\}$: if $A = 1_Y$ then the fts Y is called a fuzzy S-closed space.

Lemma 3.5. **[9]** A topological space X is, S-closed iff for every cover ϑ of X by semiopen sets of X has a finite subcollection ϑ_0 such that $X = \mathrm{scl}\ (\cup\{U : U \varepsilon \vartheta_0\})$.

Theorem 3.6. Let $F : X \multimap Y$ be a fuzzy multifunction and F(x) be a fuzzy S-closed set in Y, for each $x \varepsilon X$. If F is fuzzy upper irresolute and lower irresolute, and X is S-closed, then Y is fuzzy S-closed.

Proof : Let $\{A_\alpha : \alpha \varepsilon \Lambda\}$ be any fuzzy semiopen cover of Y. For each $x \varepsilon X$, F(x) being a fuzzy S-closed set in Y, there is a finite subset Λ_x of Λ such that $F(x) \leq \cup \{\mathrm{cl}\ A_\alpha : \alpha \varepsilon \Lambda_x\} = B_x$(say). Then B_x is a fuzzy semiopen set in Y containing F(x). Since F is fuzzy upper irresolute, $F^+(B_x)$ becomes semiopen in X, by theorem 2.5. Now, $\{F^+(B_x) : x \varepsilon X\}$ is a semiopen cover of the S-closed space X. By Lemma 3.5, there exists a finite number of points $x_1, x_2, \ldots x_n$ in X such that $X = \mathrm{scl}\ (\bigcup_{i=1}^{n} F^+(B_{x_i}))$, i. e. $X = \mathrm{scl}\ (F^+(\bigcup_{i=1}^{n} (B_{x_i}))$. Now, F being fuzzy lower irresolute, from Theorem 2.15 we have

$1_Y = F(X) = F(\text{scl}(F^+(\bigcup_{i=1}^{n}(B_{x_i})))) \leq F(F^+(\text{scl}(\bigcup_{i=1}^{n}(B_{x_i})))) \leq \text{scl}(\bigcup_{i=1}^{n}(B_{x_i}))) = \text{scl}(\bigcup_{i=1}^{n}\bigcup_{\alpha \varepsilon \Lambda_{x_i}}(\text{cl } A_\alpha)) = \bigcup_{i=1}^{n}\bigcup_{\alpha \varepsilon \Lambda_{x_i}}(\text{cl } A_\alpha)$. Hence Y is fuzzy S-closed.

Sinha and Malakar [18] defined a subset A of a topological space (X,T) to be an s-closed set if every cover ϑ of A by semiopen sets of X has a finite subfamily ϑ_0 such that $A \subset \cup \{\text{scl } U : U \varepsilon \vartheta_0\}$.

Theorem 3.7. Let $F : X \to Y$ be a fuzzy multifunction such that F(x) is a fuzzy S-closed set in Y for each $x \varepsilon X$. If F is fuzzy upper as well as lower irresolute and a set A in X is an s-closed set, then F(A) is a fuzzy S-closed set in Y.

Proof : Let $\{A_\alpha : \alpha \varepsilon \Lambda\}$ be a fuzzy cover of F(A) by fuzzy semiopen sets in Y. For each $x \varepsilon A$, F(x) is a fuzzy S-closed set in Y. So there is a finite subset Λ_x of Λ such that $F(x) \leq \cup \{\text{cl } (A_\alpha) : \alpha \varepsilon \Lambda_x\}$. Let $B_x = \cup \{\text{cl } A_\alpha : \alpha \varepsilon \Lambda_x\}$. Then B_x is fuzzy semiopen in Y and $F(x) \leq B_x$. Since F is fuzzy upper irresolute, $F^+(B_x)$ is semiopen in X. The family $\{F^+(B_x) : x \varepsilon A\}$ is then a cover of the s-closed set A by semiopen sets of X. Consequently, there are finitely many points $x_1, \ldots x_n$ in A such that $A \subset \bigcup_{i=1}^{n} \{\text{scl } (F^+(B_{x_i}))\}$. Since F is fuzzy lower irresolute, by Theorem 2.15 we have $F(A) \leq F(\bigcup_{i=1}^{n} \{\text{scl } (F^+(B_{x_i}))\}) \leq F(\text{scl } (F^+(\bigcup_{i=1}^{n} B_{x_i}))) \leq F(F^+(\text{scl } (\bigcup_{i=1}^{n} (B_{x_i})))) \leq \text{scl } (\bigcup_{i=1}^{n} (B_{x_i}) = \text{scl } (\bigcup_{i=1}^{n} \bigcup_{\alpha \varepsilon \Lambda_{x_i}} \text{cl } A_\alpha) = \bigcup_{i=1}^{n} \bigcup_{\alpha \varepsilon \Lambda_{x_i}} \text{cl } (A_\alpha)$. Hence F(A) is a fuzzy S-closed set in Y.

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REFERENCES

1. K. K. Azad, On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl. 82(1981), 14-32.
2. C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24(1968), 182-190.

3. S. G. Crossley and S. K. Hildebrand, Semi-closure, Texas Jr. of Science 22 (1971), 99-112.
4. S. G. Crossley and S. K. Hildebrand, Semi-topological properties, Fund. Math. 74(1972), 233-254.
5. C. Dorsett, Semicompactness, semi separation axioms and product spaces, Bull. Malaysian Math. Soc. 4(2) (1981), 21-28.
6. B. Ghosh, Semi continuous and semi-closed mapping and semiconnectedness in fuzzy setting, Fuzzy Sets and Systems 35(1990), 345-355.
7. Norman Levine, Semiopen sets and semi-continuity in topological spaces, Amer. Math. Monthly 70 (1963), 36-41.
8. S. Malakar, On fuzzy semi irresolute and strongly irresolute functions, Fuzzy Sets and Systems 45 (1992), 239-244.
9. M. N. Mukherjee and C. K. Basu, on S-closed and s-closed spaces, Bull. Malaysian Math. Soc. (Second Series) 15(1992), 1-7.
10. M. N. Mukherjee and B. Ghosh, On fuzzy S-closed space and F.S.C. sets, Bull Malaysian Math. Soc. (Second Series) 12 (1989), 1-14.
11. M. N. Mukherjee and S. Malakar, On almost continuous and weakly continuous fuzzy multifunction, Fuzzy Sets and Systems 41 (1991), 113-125.
12. M. N. Mukherjee and S. P. Sinha ; Irresolute and almost open functions between fuzzy topological space, Fuzzy Sets and System 29 (1989), 381-388.
13. T. Noiri, On semi continuous mappings, Atti-Accad. Naz. Lincei Rend. cl. Sci. Fis. Mat. Natur. (8) 54 (1973), 210-214.
14. T Noiri, On S-closed spaces, Ann. Soc. Sci. Bruxelles 91 (1977), 189-194.
15. N. S. Papageorgiou, Fuzzy topology and fuzzy multifunction, J, Math. Anal. Appl. 109 (1985), 397-425.
16. Pao Ming Pu and Ying Ming Liu, Fuzzy topology I. Neighbourhood structure of a fuzzy point and Moore-Smith convergence, J. Math Anal. Appl. 76 (1980), 571-599.
17. S. P. Sinha, A note on fuzzy almost compactness, Soochow Journal of Mathematics 18 (1992), 205-209.
18. S. P. Sinha and S. Malakar, On s-closed fuzzy topological spaces, The Journal of Fuzzy Mathematics 2 (1994), 95-103.
19. T. Thompsom, S-closed spaces, Proc. Amer. Math. Soc. 60 (1976), 335-338.
20. C. K. Wong, Fuzzy topology, product and quotient theorems, J. Math. Anal. Appl. 45 (1974), 512-521.
21. L. A. Zadeh, Fuzzy sets, Inform. Control 8 (1965), 338-353.

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ON STRONGLY IRRESOLUTE AND γ-CONTINUOUS FUNCTIONS

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ABSTRACT. Two new classes of functions are introduced and investigated as to their characterizations, properties and interrelations. Relationships with some other allied types of functions are also studied.

1. INTRODUCTION AND PRELIMINARIES

The paper is meant for the initiation to the study of two new types of functions, each of which being independent of continuity. The deliberations are divided into the next two sections, the first one of which deals with the first type viz. strongly irresolute functions while the other type of newly introduced functions termed γ-continuity is taken up in the third section. Some allied classes of functions of the know types that come in the relevance are also treated with regard to their relationships with the newly introduced ones.

Throughout the paper, by X we shall mean a topplogical space (X, τ). A set A in X is said to be semi-open [9] if there exists an open set U such that $U \subset A \subset clU$. The complment of a semi-open set is called semi-closed. We denote the closure and interior of a subset A by clA and intA respectively. Semi-closure [3] of a subset A, which is the intersection of all semi-closed sets containing A, is donoted by sclA. A set A in X is said to be semi-regular [11] if A is semi-open as well as semi-closed. The family of all semi-open (semi-closed, semi-regular) sets of X will be denoted by SO(X) (resp. SC(X), SR(X)) while the collection of all members of SO(X) (SR(X)) each containing a point x of X will be denoted by SO(x) (resp. SR(x)). A point x of X is said to be a semi-θ-adherent point [11] of a subset A of X if $sclU \cap A \neq \emptyset$, for every $U \in SO(x)$. The set of all semi-θ-adherent points of A is called the semi-θ-closure of A, to be denoted by $[A]_{s\text{-}\theta}$. A is called semi-θ-closed (s-θ-closed, for short) if $A = [A]_{s\text{-}\theta}$. The complement of an s-θ-closed set is called semi-θ-open (s-θ-open, for short). also a set A in X is called regularly open if A = int clA.

It is clear that for any subset A in X, $A \subset sclA \subset [A]_{s\text{-}\theta}$ and therefore every s-θ-open set is semi-open, but converse is not true [12]. In [12] we have shown that the collections of semi-θ-open sets and open sets are non-comparable. Maio and Noiri

[11] proved that every semi-regular set is s-θ-clesed, but there exist s-θ-closed sets which are not semi-regular. The following example substantiates the claim.

Example 1.1. Consider the space (X, τ), where $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{b,c\}, \{b\}, \{c\}\}$. Then $SR(X) = \{\emptyset, X, \{b\}, \{c\}, \{a,b\}, \{a,c\}\}$, and the s-θ-closed sets of X are given by the collection $\{\emptyset, X, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{a\}\}$. Here $\{a\}$ is s-θ-closed but not semi-regular.

Although $[A]_{s\text{-}\theta}$ is the intersection of all s-θ-closed sets containing A [12], the s-θ-closure $[A]_{s\text{-}\theta}$ has been characterized in [12] in terms of semi-regular sets viz. as the intersection of all semi-regular sets containing A.

2. STRONGLY IRRESOLUTE FUNCTIONS

Crossley and Hilderbrand [4] defined a function $F : X \to Y$ to be irresolute if for each $V \in SO(Y)$, $f^{-1}(V)$ is in $SO(X)$. It is easy to see that $f : X \to Y$ is irresolute iff for each $x \in X$ and each $V \in SO(f(x))$, there is a $U \in SO(x)$ such that $f(U) \subset V$. A weaker form of such functions under the terminology quasi-irresolute function was defined by Maio and Noiri [11] as follows. A function $f : X \to Y$ is quasi-irresolute if for each $x \in X$ and each $V \in SO(f(x))$, there is a $U \in SO(x)$ such that $f(U) \subset \mathrm{scl} V$. We now introduce a new class of functions as follows.

Definition 2.1. A function $f : X \to Y$ is said to be strongly irresolute if for each point x of X and each $V \in SO(f(x))$, there exists a $U \in SO(f(x)$ such that $f(\mathrm{scl}\ U) \subset V$.

Remark 2.2. Clearly a strongly irresolute function is irresolute. we give examples below to show that an irresolute function may not be strongly irresolute and that strong irresoluteness does not imply continuity. Since irresoluteness and continuity are independent notions [4], it then following that so are strong irresoluteness and continuity.

Example 2.3. Let $X = \{a,b,c\}$ and $T = \{\emptyset, X, \{a\}\}$. The identily mapping $i : (X,T) \to (X, T)$ is an irresolute map. Now, $SO(X) = \{\emptyset, X, \{a\}, \{a,b\}, \{a,c\}\}$. Consider $a \in X$. Then $A = \{a\}$ is a semi-open set containing $i(a)$ (= a), but there is no semi-open set U such that $i(\mathrm{scl}\ U) \subset A$, because $\mathrm{scl} U = X$, for amy non-null semi-open set J. Hence i is not strongly irresolute.

Example 2.4. On the set $X = \{a,b,c\}$, consider the topology T_1 and T_2 given by $T_1 = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$ and $T_2 = \{\emptyset, X, \{b,c\}\}$. Then $SO(X,T_1) = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}. \{b,c\}\}$ and $SO(X, T_2) = T_2$. It is easy to check that the identity map from (X, T_1) to (X, T_2) is strongly irresolute but not continuous.

A space X is said to be semi-regular [5] if for each $x \in X$ and each semi-closed set F with $x \notin F$, there exist disjoint semi-open sets U and V such that $x \in U$ and $F \subset V$.

Theorem 2.5. Let X be semi-regular. Then an irresolute finction $f : X \to Y$ is strongly irresolute.

Proof : Let $x \in X$ and V be a semi-open set containing f(x). Since f is irresolute, there exists a semi-open set U containing x such that $f(U) \subset V$. Since X is semi-regular, there is a semi-open set W containing x such that $\mathrm{scl} W \subset U$. Then $f(\mathrm{scl}\ W) \subset V$ proving that f is strongly irresolute.

Theorem 2.6. For a function $f : X \to Y$, the following staements are equivalent:

(a) f is strongly irresolute.

(b) $[f^{-1}(B)]_{s\text{-}\theta} = f^{-1}(\mathrm{scl}B)$, for any $B \subset Y$.

(c) $f([A]_{s\text{-}\theta}) \subset \mathrm{scl}\ f(A)$, for any $A \subset X$.

(d) $f^{-1}(B)$ is s-θ-closed in X, for every semi-closed set B in Y.

(e) f^{-1} (D) is s-θ-open in X, for every semi-open set D in Y.

Proof. (a)→(b) : Let B be any set in Y. For any $x \notin f^{-1}(\mathrm{scl}B)$, there exists a semi-open set G in Y containing f(x) such that $G \cap B = \emptyset$. By (a), there is a semi-open set V in X containing x such that $f(\mathrm{scl}V) \subset G$ and hence $f(\mathrm{scl}V) \cap B = \emptyset$. This means that $\mathrm{scl}V \cap f^{-1}(B) = \emptyset$ and hence $x \notin [f^{-1}(B)]_{s\text{-}\theta}$.

(b)→(c) : For any set A in X we have by (b),

$[A]_{s\text{-}\theta} \subset [f^{-1}(f(A))]_{s\text{-}\theta} \subset f^{-1}(\mathrm{scl}\ f(A))$. and hence

$f([A]_{s\text{-}\theta}) \subset ff^{-1}(\mathrm{scl}\ f(A)) \subset \mathrm{scl}\ f(A)$.

(c)→(d) : For any semi-closed set B in Y, we get by (c), $f([f^{-1}(B)]_{s\text{-}\theta}) \subset \mathrm{scl}\ f(f^{-1}(B)) \subset \mathrm{scl}B = E$ so that $[f^{-1}(B)]_{s\text{-}\theta} \subset f^{-1}(B)$. Thus $f^{-1}(B)$ is s-θ-closed.

(d)→(e) : Clear.

(e)→(a) : Let x be a point in X and V be a semi-open set in Y containing f(x). Then $U = f^{-1}(V)$ is semi-θ-open in X, and $x \notin X - f^{-1}(V)$. Since $X - f^{-1}(V)$ is semi-θ-closed, there is a semi-open set U in X such that $x \in U$ and $\mathrm{scl}U \cap (X - f^{-1}(V)) = \emptyset$, i.e., $\mathrm{scl}\ U \subset f^{-1}(V)$. Hence $f(\mathrm{scl}U) \subset V$ proving that f is strongly irresolute.

The concepts of S-closed and s-closed spaces as introduced in [14] and [11] respectively have been extensively studied by renowned topologists. Genster and Reilly [8] proved that every infinite topological space is embeddable as a closed subspace in a connected S-closed space which is not s-closed.

Definition 2.7. A subset A of a topological space (X,T) is said to be semi-compact [2] (s-closed [11], S-closed [13]) relative to X or simply an s-compact set (resp. s-set, S-set) iff every cover ϑ of A by semi-open sets of X admits a finite subfamily ϑ_0 such that $A \subset \bigcup_{U \in \vartheta_0} U$ (resp. $A \subset \bigcup_{U \in \vartheta_0} \mathrm{scl}\ U$, $A \subset \bigcup_{U \in \vartheta_0} \mathrm{cl}U$).

It is clear that for any set A in a space X, A is s-compact→A is an s-set→ A is an S-set, it is well-known that the reverse implications are not true, in general.

Theorem 2.8. Let $f : X \to Y$ be strongly irresolute. If A is an s-set in X then f(A) is an s-compact set in Y.

Proof : Let $\{U_\alpha : \alpha \in I\}$ be a cover of f(A) by semi-open sets of Y. For each $x \in A$, there exists $\alpha_x \in I$ such that $f(x) \in U_{\alpha_x}$. Since f is strongly irresolute, there exists a semi-open set V_{α_x} containign x in X such that $f(\mathrm{scl}\ V_{\alpha_x}) \subset U_{\alpha_x}$. Since $\{V_{\alpha_x} : x \in A\}$ is a cover of A by semi-open sets of X, there exists a finite number of points $x_1, \ldots x_n$ in A such that $A \subset \bigcup_{i=1}^{n} \mathrm{scl}\ V_{\alpha_{x_i}}$ and hence $f(A) \subset f(\bigcup_{i=1}^{n} \mathrm{scl}\ V_{\alpha_{x_i}}) = \bigcup_{i=1}^{n} f(\mathrm{scl}\ V_{\alpha_{x_i}})$ $\subset \bigcup_{i=1}^{n} \mathrm{scl}\ U_{\alpha_{x_i}}$. Consequently, f(A) is an s-compact set in Y.

In view of Theorem 2·5 we now deduce :

Corollary 2.9. If a function f is irresolute from a semi-regular space X to another space Y, then the image of an s-set of X is an s-compact set of Y.

3. γ-CONTINUOUS FUNCTIONS

Definition 3.1. A function $f : X \to Y$ is said to by γ-continuous at $x \in X$ if for each $W \in SO(f(x))$, there is an open set V containing x such that $f(V) \subset \mathrm{scl}W$. We say that f is γ-continuous on X if f is γ-continuous at each point x of X.

Definition 3.2. Let $\mathscr{F}$ be a filterbase on X. Then ad $\mathscr{F}$ (s-θ-ad $\mathscr{F}$) is defined as the set $\cap \{\mathrm{cl}F : F \in \mathscr{F}\}$ (resp. $\cap \{[F]_{s\text{-}\theta} : F \in \mathscr{F}\}$).

Theorem 3.3. For a function $f : X \to Y$, where X and Y are topological spaces, the following statement are equivalent.

(a) f is γ-continuous.

(b) For each filterbase $\mathscr{F}$ on X, $f(\mathrm{ad}\ \mathscr{F}) \subset s\text{-}\theta\text{-ad}\ f(\mathscr{F})$.

(c) $f(\mathrm{cl}A) \subset s\text{-}\theta\text{-cl}f(A)$, for any $A \subset X$.

(d) $\mathrm{cl}(f^{-1}(A)) \subset f^{-1}(s\text{-}\theta\text{-cl}A)$, for any $A \subset Y$.

(e) $f^{-1}(F)$ is closed in X for each s-θ-closed set F of Y.

(f) For each semi-regular set R of Y, $f^{-1}(R)$ is clopen in X.

Proof. (a)$\to$(b) : Let $x \in \mathrm{ad}\,\mathscr{F}$ and U any semi-open set in Y containing f(x). Then by (a), there exists an open set V containing x such that $f(V) \subset \mathrm{scl}U$. Since $x \in \mathrm{ad}\,\mathscr{F}$. $V \cap F \neq \varnothing$, for each $F \in \mathscr{F}$. Thus $\mathrm{scl}U \cap f(F) \neq \varnothing$, for each $F \in \mathscr{F}$. Then $f(x) \in s\text{-}\theta\text{-cl}f(F)$, for each $F \in \mathscr{F}$. and hence $f(\mathrm{ad}\ \mathscr{F}) \subset s\text{-}\theta\text{-ad}\ f\ (\mathscr{F})$.

(b)$\to$(c) : Let $\mathscr{F}$ denote the collection of all sets containing A. Then $\mathscr{F}$ is a fillterbase on X. Now by (b), $f(\mathrm{ad}\ \mathscr{F}) \subset s\text{-}\theta\text{-ad}\ f\ (\mathscr{F})$, but $\mathrm{ad}\ \mathscr{F} = \mathrm{cl}A$ and $s\text{-}\theta\text{-ad}\ f\ (\mathscr{F}) = s\text{-}\theta\text{-cl}\ f\ (A)$. Hence $f(\mathrm{cl}A) \subset s\text{-}\theta\text{-cl}\ f\ (A)$.

(c)$\to$(d) : For $A \subset Y$, $f^{-1}(A) \subset X$ and hence the proof is immediate.

(d)$\to$(e) : Let A be an s-θ-closed set in Y. Now, $x \in \mathrm{cl}f^{-1}(A)$ implies by virtue of (d), $x \in f^{-1}\ (s\text{-}\theta\text{-cl}A)$. Then $f(x) \in s\text{-}\theta\text{-cl}A = A$, i.e., $x \in f^{-1}(A)$. Thus $f^{-1}(A)$ is closed in X.

(e)$\to$(f) : Since every semi-regular set is s-θ-closed as well as s-θ-open, (f) follows at once from (e).

(f)$\to$(a) : For any point $x \in X$, let W be a semi-open set containing f(x). Then sclW is a semi-regular set containing f(x). By (f), $f^{-1}(\mathrm{scl}W)$ is a clopen set in X. Taking $V = f^{-1}(\mathrm{scl}W)$ we get $f^{-1}(V) \subset \mathrm{scl}\ W$, where $x \in V$.

Remark 3.4. γ-continuity and continuity are independent notions ; also, so are γ-continuous and irresolute functions. We consider the following examples to this end.

Example 3.5. Let $X = \{a,b,c\}$ and $T = \{\varnothing, X, \{b\}, \{c\}, \{b,c\}\}$. Then $SO(X) = \{\varnothing, X, \{b\}, \{c\}, \{b,c\}, \{a,b\}, \{a,c\}\}$. The identity mapping $i : X \to Y$ is then continuous as well as irresolute, but not γ-continuous.

Example 3.6. Let $X = \{a,b,c\}$ and $T_1 = \{\varnothing, X, \{c\}, \{a,b\}\}$ and $T_2 = \{\varnothing, X, \{c\}, \{a, b\}, \{b\}, \{b,c\}\}$. Then $SO(X, T_1) = T_1$ and $SO(X, T_2) = T_2$. In this cace the identity mapping i, $(X, T_1) \to (X, T_2)$ is obviously neither continuous nor irresolute but it is γ-continuous.

Remark 3.7. γ-continuous and strongly irresolute functions are independent of sach other in view if the following examples.

Example 3.8. Let $X = \{a,b,c\}$ and $T = \{\varnothing, X, \{b\}, \{c\}, \{b, c\}\}$. Then $SO(X) = \{\varnothing, X, \{b\}, \{c\}, \{b,c\}, \{a,b\}, \{a,c\}\}$. The semi-closed and semi-θ-closed sets of (X, T) are given by the collections $\{\varnothing, X, \{a,c\}, \{a,b\}, \{a\}, \{c\}, \{b\}\}$ and $\{\varnothing, X, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{a\}\}$ respectively. Here $i : X \rightarrow X$ is obviously strongly irresolute. But i is not γ-continuous ; for, $B = \{b\}$ is an s-θ-closed set whereas $i^{-1}(B) = \{b\}$ is not closed.

Example 3.9. In Example 3.6. 1 is γ-continuous but not strongly irresolute.

Remark 3.10. Although each of γ-continuity and strong irresoluteness is independent of continuity, the latter property is achieved as the composition of the former two. That is, if $f : X \rightarrow Y$ is γ-continuous and $g : Y \rightarrow Z$ is strongly irresolute then $gof : X \rightarrow Z$ is continuous.

In the next few theorems we investigate for some properties of γ-continuous functions. Here certain similarities in the behaviours of such functions with those of continuous ones can be noticed, although these two types of functions are independent of each other.

Theorem 3.11. Let f $X \rightarrow Y$ be a γ-continuous surjection. If X is connected then so is Y.

Proof : If posible, let Y be not connected. Then there exists a non-void proper clopen set V (say) in Y. Obviously V is s-θ-closed as well as s-θ-open. Hence by γ-continuity of f, $f^{-1}(V)$ is a clopen set in X which is non-empty and proper, and this is a contdadiction. Thus Y is connected.

Theorem 3.12. Let $f : X \rightarrow Y$ be a γ-continuous onto map and X be S-closed. Then Y is s-closed.

Proof : Let $\vartheta = \{U_\alpha : \alpha \in I\}$ be a semi-open cover of Y. Then for each $\alpha, f^{-1}(\mathrm{scl}\ U_\alpha)$ is a closed set, since $\mathrm{scl}U_\alpha$ is an-s-θ-closed set in Y. Obviously $f^{-1}(\mathrm{scl}\ U_\alpha)$ is semi-open. Then by S-closedness of X, there exist $\alpha_1 \ldots\ldots \alpha_n \in I$ such that $X \subset \bigcup_{i=1}^{n} f^{-1}(\mathrm{scl}\ U_{\alpha_i})$ which show that $Y \subset \bigcup_{i=1}^{n} \mathrm{scl}\ U_{\alpha_i}$ and hence Y is s-closed.

Matheshwri and Prasad [10] defined semi-T_2 axiom in the usual manner by replacing open sets by semi-open sets in the definition of T_2-axiom.

Theorem 3.13. Let $f,g : X \rightarrow Y$ be two γ-continuous mappings and Y is semi-T_2. Then the set $\{x \in X : f(x) = g(x)\}$ is closed.

Proof : Let $A = \{x \in X : f(x) = g(x)\}$ and $x \notin A$ so that $f(x) \neq g(x)$. As Y is semi-T_2, there exist $U \in SO(f(x))$ and $V \in SO(g(x))$ such that $U \cap V = \varnothing$ and hence $\mathrm{scl}U \cap \mathrm{scl}V = \varnothing$. But $\mathrm{scl}U$ and $\mathrm{scl}V$ being semi-regular sets, $f^{-1}(\mathrm{scl}U)$ and $g^{-1}(\mathrm{scl}V)$ are clopen sets containing x (using γ-continuity of f and g). Taking $W = f^{-1}(\mathrm{scl}U) \cap g^{-1}(\mathrm{scl}V)$. We get an open set W containing x such that $W \cap A = \varnothing$. Therefore A is closed.

Corollary 3.14. Let $f, g : X \rightarrow Y$ be two γ-continuous mappings and Y be semi-T_2. If f and g coincide on a dense subset of X, then are identical on X.

Theorem 3.15. If Y is semi-T_2 and f : X→Y is a γ-continuous injection then X is Urysohn.

Proof : Let X_1, X_2, be distinct points of X. Then $f(x_1) \neq f(x_2)$ and hence there exist $U_1 \in SO(f(x_1))$ and $U_2 \in SO(f(x_2))$ such that $U_1 \cap U_2 = \varnothing$. This implies that $sclU_1 \cap sclU_2 = \varnothing$. But $sclU_i$ is semi-regular and hence $f^{-1}(scl\, U_i)$ is clopen, for i = 1, 2. But $f^{-1}(sclU_1) \cap f^{-1}(sclU_2) = \varnothing$. Thus X is Urysohn.

Finally, in order to ascertain the relative positions of the two types of functions introduced here among some of the various types of known functions, we recall the following definition.

Definition 3.16. A function f : X→Y is

(a) K-continuous [6] (completely irresolute [6]) if inverse image of each semi-open set in Y is open (resp. regularly open) in X.

(b) θ-continuous [7] if for each $x \in X$ and each open set V containing f(x), there is an open set U containing x such that $f(clU) \subset clV$.

(c) Completely continuous [1] if the inverse image of every open subset in Y is a regularly open subset of X.

Remark 3.17. Every K-continuous mapping is γ-continuous, that the converse is false follows from the example below.

Example 3.18. Let x = {a,b,c}, $\tau_1 = \{\varnothing, X, \{a\}, \{b,c\}\}$, Y = {1,m,n} and $\tau_2 = \{\varnothing, \{1\}, Y\}$. We define a function $f : (X, \tau_1) \to (Y, \tau_2)$ by f(a) = 1, f(b) = m and f(c) = n. Then f is γ-comtinuous but not K-continuous.

Remark 3.19. It is easy to see that every γ-continuous function is θ-continuous. But the converse is false, for, in Example 3.5 the identity function i is θ-continuous but not γ-continuous.

Theorem 3.20. Every completely irresolute function is strongly irresolute.

Proof : Let f : X→Y be completely irresolute. Let V be any semi-closed set in Y. Then $f^{-1}(V)$ is regularly closed, i.e., $f^{-1}(V) = \text{clint}\, f^{-1}(V)$. We claim that $f^{-1}(V)$ is s-θ-closed. Let $x \in X = f^{-1}(V)$. Then there exists an open set U_x containing x in X such that $U_x \cap f^{-1}(V) = \varnothing$, i.e., $U_x \cap \text{cl int}\, f^{-1}(V) = \varnothing$. But cl int $f^{-1}(V)$ is semi-open and hence $scl\, U_x \cap \text{cl int}\, f^{-1}(V) = \varnothing$ so that $scl\, U_x \cap f^{-1}(V) = \varnothing$. Then $x \in$ s-θ-cl $f^{-1}(V)$ and hence $f^{-1}(V)$ is s-θ-closed. Consequently, f is strongly irresolute.

Remark 3.21. We notice that a strongly irresolute maping is not completely irresolute. In fact, in Example 3.8, i : X→X is strongly irresolute, but {a,c} is semi-open without being regularly open.

The implicational aspects of different types of functions discussed so far, are thus finally summed up in the following diagram :

$$\begin{tikzcd}
\text{Strong irresolute} \arrow[r, Rightarrow] & \text{irresolute} \arrow[r, Rightarrow] & \text{quasi-irresolute} \\
\text{Completely irresolute} \arrow[r, Rightarrow] \arrow[u, Rightarrow] \arrow[d, Rightarrow] & \text{K-continuous} \arrow[r, Rightarrow] \arrow[u, Rightarrow] \arrow[d, Rightarrow] & \gamma\text{-continuous} \arrow[u, Rightarrow] \arrow[d, Rightarrow] \\
\text{Completely continuous} \arrow[r, Rightarrow] & \text{continuous} \arrow[r, Rightarrow] & \theta\text{-continuous}
\end{tikzcd}$$

REFERENCE

1. S. P. Arya and Ranjana Gupta, On strongly continuous mappings, Kyungpook Math. J. 14(1974),131-143.
2. D. A. Carnahan, Some properties related to Compactness in topological spaces, Ph. D. Thesis, Univ. of Arkansas 1973.
3. S. G. crossley and S. K. Hilderbrand, Semi-closure, Texas J. Sci. 22(1971), 99-119.
4. .... semi topological properties, Fund. Math., 74(1972), 233-254.
5. C. Dorset, Soochow J. Math. 8(1982), 45-53.
6. K. K. Dube, O. S. Panwar, R. K. Tiwari, K-continuous mappings, Acta Ciencia Indica, Vol XIVM, 4, 227-234 (1988).
7. S. V. Fomin, Extensions of topological spaces, Ann. of Math. (2) 44 (1943), 471-480.
8. M Ganster and I.L. Reilly, A note on S-closed spaces, Indian J. Pure Applied Math. 19(10) (1988), 1031-1033.
9. N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer Math. Monthly 70(1963), 36-41.
10. S. N. Maheswari and R. Prasad, some new separation axioms, Ann. Soc. Sci. Bruxelles I 89(1975), 227-234.
11. Maio and Noir, on s-closed spaces, Indian J. Pure Applied Math. 18(3) (1987), 226-233.
12. M. N. Mukherjee and C. K. Basu, Semi-θ-closed sets, semi-θ-connectedness and some associated mappings Bull, Cal. Math. Soc. 83(1991), 227-238.
13. T. Noiri, on S-closed spaces, Ann. Soc. Sci. Bruxelles T. SI, IV, (1977), 189-194.
14. T. Thompson, S-closed spaces, Proc. Amer. Math. Soc. 60(1976), 335-338.

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ON THE CURVATURE TENSOR OF A SEMI-SYMMETRIC SEMI-METRIC CONNECTION

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ABSTRACT. A semi-symmetric semi-mertic connection has been introduced by Barua and Mukhopadhyay and the curvature tensor of such a connection has been deduced by them. The nature of the Riemann curvature tensor has been studied after putting some restrictions on the curvature tensor of the semi-symmetric semi-metric connection.

1. INTRODUCTION

A semi-symmetric semi-metric connection $\overline{\nabla}$ on a Riemannian manifold is defined by [1].

$$(\overline{\nabla}_X g)(Y, Z) = 2\alpha(X) g(Y, Z) - \alpha(Y) g(X, Z) - \alpha(Z) g(Y, X) \quad \dots\dots (1.1)$$

where g is the Riemann metric on the manifold and α is a differentiable 1-form, called the associated 1-form. The torsion tensor of the connection $\overline{\nabla}$ is given by

$$T(y, z) = \alpha(z) y - \alpha(y) z \quad \dots\dots (1.2)$$

It is known that the connection $\overline{\nabla}$ is of the form [2].

$$\overline{\nabla}_Y Z = \nabla_Y Z - \alpha(Y) Z + g(Y, Z) A \quad \dots\dots (1.3)$$

where ∇ is the Riemann connection and A is the associated vectorfield defined by

$$g(A, X) = \alpha(X) \quad \dots\dots (1.4)$$

The curvature tensor $\overline{R}(X, Y) Z$ of $\overline{\nabla}$ is given by

$$\overline{R}(X, Y)Z = R(X, Y) Z - 2d\alpha(X, Y) Z + g(Y, Z) LX - g(X, Z) LY \dots\dots (1.5)$$

where $LX = \nabla_X A + \alpha(X)A$ and $R(X, Y) Z$ is the Riemann curvature tensor.

Defining $K(X, Y, Z, U) = g(R(X, Y) Z, U)$ and $\overline{K}(X, Y, Z, U) = g(\overline{R}(X, Y) Z, U)$, we get

$$\overline{K}(X, Y, Z, U) = K(X, Y, Z, U) - 2d\alpha(X, Y) g(Z, U) + g(Y, Z) \lambda(X, U) - g(X, Z) \lambda(Y, U) \dots\dots (1.6)$$

where

$\lambda(X, U) = g(LX, U) = (\nabla_X \alpha)(U) + \alpha(X)\,\alpha(U)$ ...... (1.7)

It can be seen from (1.6) that $\bar{K}(X, Y, Z, U) + \bar{K}(Y, X, Z, U) = 0$. But the other three identities, satisfied by $K(X, Y, Z, U)$ are not, in general, satisfied by $\bar{K}(X, Y, Z, U)$.

By direct calculation follows that—

$\bar{K}(X, Y, Z, U) + \bar{K}(Y, Z, X, U) + \bar{K}(Z, X, Y, U) = 0$

if and only if $d\alpha = 0$. Then (1.6) reduces to

$\bar{K}(X, Y, Z, U) = K(X, Y, Z, U) + g(Y, Z)\,\lambda(X, U) - g(X, Z)\,\lambda(Y, U)$...... (1.8)

and $\lambda(X, U) = \lambda(U, X)$.

Throughout this paper we have considered $d\alpha = 0$.

2. THE OTHER TWO IDENTITIES

It is known that the Riemann curvature tensor $K(X, Y, Z, U)$ satisfies. $K(X, Y, Z, U) + K(X, Y, U, Z) = 0$ and $K(X, Y, Z, U) = K(Z, U, X, Y)$...... (2.1)

which are, in general, not satisfied by $\bar{K}(X, Y, Z, U)$.

Let $\bar{K}(X, Y, Z, U) + \bar{K}(X, Y, U, Z) = 0$. Then,

$g(Y, Z)\,\lambda(X,U) - g(X, Z)\,\lambda(Y, U) + g(Y, U)\,\lambda(X, Z) - g(X, U)\,\lambda(Y, Z) = 0$

Summing for Y and Z, we find

$$\lambda(X, U) = \frac{tr\lambda}{n} g(X, U) \qquad \text{...... (2.2)}$$

The same result holds, if $\bar{K}(X, Y, Z, U) = \bar{K}(Z, U, X, Y)$

On the other hand, if (2. 2) holds, then the two identities in (2.1) are satisfied. In this case—

$$\bar{K}(X, Y, Z, U) = K(X, Y, Z, U) + \frac{tr\lambda}{n}\{g(Y, Z)\,g(X, U) - g(X, Z)\,g(Y, U)\} \qquad \text{......(2.3)}$$

If now, $\bar{K}(X, Y, Z, U)$ vanishes identicaly, then,

$$K(X, Y, Z, U) = \frac{tr\lambda}{n}\{g(X, Z)\,g(Y, U) - g(X, U)\,g(Y, Z)\} \qquad \text{..... (2.4)}$$

Theorem 2.1. If the curvature tensor of a semi-symmetric semi-metric connection vanishes identically, then the manifold is of constant curvature.

3. FIRST SET OF RESTRICTIONS ON $\bar{R}$(X, Y)Z AND T (Y,Z)

Let the curvature tensor $\bar{R}$ (X, Y) Z and the torsion tensor T (X, Z) of the semi-symmetric semi-metric connection satisfy the relations,

$$\alpha\ (\bar{R}(X, Y)Z) = O \tag{3.1}$$

and $\bar{R}\ (X, Y)\ oT = O$ ...... (3.2)

It is known that [3]

$$(\bar{R}(X, Y)T)\ (U, V) = \bar{R}(X, Y)\ T\ (U,V) - T(\bar{R}(X, Y)\ U, V) - T\ (U, \bar{R}(X, Y)V) - (\bar{\nabla}_{T(X,Y)}T)\ (U,V)$$

In view of (3.1) and (3.2), the above is equivalent to

$$(\bar{\nabla}_{T(X,Y)}T)\ (U,V) = O \tag{3.3}$$

Now, using (1.3), we get

$$(\bar{\nabla}_X T)\ (U,V) = \{\lambda(X, V) - ag(X, V)\}\ U - \{\lambda(X,U) - ag\ (X,U)\}\ V \tag{3.4}$$

where $a = g\ (A, A) = \alpha\ (A)$. Substituting from (3. 4) in (3. 3) we get

$$\alpha\ (X)\ [\{\ \lambda(Y, V) - ag\ (Y, V)\}\ U - \{\ \lambda\ (Y, U) - ag\ (Y, U)\}\ V]$$
$$= \alpha\ (Y)\ [\{\ \lambda(X, V) - ag\ (X, V)\}\ U - \{\ \lambda\ (X, U) - ag\ (X, U)\}\ V]$$

from which, we get

$$\alpha\ (X)\ [\{\ \lambda(Y, V) - ag\ (Y, V)\} = \alpha\ (Y) - \{\ \lambda\ (X, V) - ag\ (X, V)\}$$

Putting Y = A, we get form the last equation

$$\alpha(X)\ \{\ \lambda(A, V) - a\ \alpha\ (V)\ \} = a\ \{\ \lambda\ (X, V) - ag\ (X, V)\} \tag*{...... (3, 5)}$$

Again, putting V = A in (3.5), we get

$$\lambda\ (X, A) = \frac{\lambda(A, A)}{a}\alpha\ (X) \tag{3.6}$$

Substituting (3.6) in (3.5), we get

$$\lambda\ (X, V) = \beta\ \alpha\ (X)\ \alpha(V) + ag\ (X, V) \tag{3.7}$$

where

$$\beta = \frac{\lambda(A, A)}{a^2} - 1 \tag{3.8}$$

In view of (3. 7), we get from (1. 8)

$$\bar{K}\ (X, Y, Z, U) = K\ (X, Y, Z, U) + a\ \{\ g\ (Y, Z)\ g\ (X, U) - g\ (X, Z)\ g\ (Y, U)\} + \beta\ \{\ g\ (Y, Z)\ \alpha\ (X)\ \alpha\ (U) - g\ (X, Z)\ \alpha\ (Y)\ \alpha\ (U)\} \tag{3.9}$$

From (3.8) we find $\beta = O$ if and only if A is a geodesic congruence. If moreover, K (X, Y, Z, U) vanishes identically, then (3.9) reduced to—

$$\bar{K}\ (X, Y, Z, U) = a\ \{g\ (Y, Z)\ g\ (X, U) - g\ (X, Z)\ g\ (Y, U)\} \tag{3.10}$$

Theorem 3.1 If the curvature tensor of a semi-symmetric semi-metric connection satisfies (3.1) and (3.2), then it is given by (3.9).

Theorem 3.2 If on a flat manifold the associated vector field of a semi-symmetric semi-metric connection is a geodesic congruence then the manifold is of constant curvature with respect to this connection.

4. SECOND SET OF RESTRICTIONS ON $\overline{R}$ (X, Y) Z AND T (Y, Z)

Let the curvature tensor $\overline{K}$ (X, Y, Z, U) and the torsion tensor T (Y, Z) satisfy,

$$\overline{K}(X, Y, Z, U) = g(Y, U)\{\sigma(X)\,\alpha(Z) + \sigma(Z)\,\alpha(X)\}$$

$$- g(X, U)\{\sigma(Y)\,\alpha(Z) + \sigma(Z)\,\alpha(Y)\} \qquad \ldots\ldots (4.1)$$

and $(\nabla_X T)(Y, Z) = \sigma(X)\, T(Y, Z)$ ...... (4.2)

where σ is a differentiable 1-form.

From (4.2) we get

$$(\nabla_X \alpha)(Z) = \sigma(X)\,\alpha(Z) \qquad \ldots\ldots (4.3)$$

and $(\nabla_X A) = \sigma(X)\, A$ ......(4.5)

Therefore

$$\lambda(X, Z) = \sigma(X)\,\alpha(Z) + \alpha(X)\,\alpha(Z) \qquad \ldots\ldots (4.5)$$

Again—

$\lambda(X, Z) = \lambda(Z, X)$ implies

$$\sigma(X) = t\,\alpha(X),\ t = \frac{\sigma(A)}{\alpha(A)} \qquad \ldots\ldots (4.6)$$

Hence $\lambda(X, Z) = (t + 1)\,\alpha(X)\,\alpha(Z)$ ......(4.7)

Using (4.7) in (1.8) we get

$$2t\{g(Y, U)\,\alpha(X)\,\alpha(Z) - g(X, U)\,\alpha(Y)\,\alpha(Z)\}$$

$$= K(X, Y, Z, U) + (t + 1)\{g(Y, Z)\,\alpha(X)\,\alpha(U) - g(X, Z)\,\alpha(Y)\,\alpha(U)\} \ldots\ldots (4,8)$$

$t = 1$, we find $\sigma(X) = \alpha(X)$ and

$$K(X, Y, Z, U) = 2\{g(Y, U)\,\alpha(X)\,\alpha(Z) - g(X, U)\,\alpha(Y)\,\alpha(Z)$$

$$+ g(X, Z)\,\alpha(Y)\,\alpha(U) - g(Y, Z)\,\alpha(X)\,\alpha(U)\} \qquad \ldots\ldots (4.9)$$

Hence the manifold is of almost constant curvature [4].

Theorem 4.1. If the curvature tensor of a semi-symmetric semi-metric connection satisfies (4.1) and the torsion tensor is recurrent, then the 1-form of recurrence must be a scalar multiple of the associated 1-form. In this case the associated vectorfield is also recurrent.

Theorem 4.2. If the curvature tensor of a semi-symmeteric semi-metric connection satisfies (4.1) and the torsion tensor is recurrent, the 1-form of recurrence being the associated 1-form, then the manifold is of almost constant curvature.

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REFERENCES

(I) B.Barua and S. Mukhopadhyay, A sequence of semi-symmetric connections on a Riemannian manifold, Proc. of Seventh Nat. Seminar of Finsler, Langrange and Hamilton spaces, 1992. Brasov, Romania.

(2) S. Mukhopadhyay, Ph. D. Thesis, 1994.

(3) L.P. Eisenhart, Non Riemannian Geometry. American Mathematical Society, 1927.

(4) D. Smaranda., Pseudo Riemannian recurrent manifolds with almost constant curvature, The XVIII Int. Conf. on Geom. and Top (Oradea 1987) 4-7. Preprint 88-2 Univ. "Babes Balyai" Cluj-Napoca, 1988.

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CONVERGENCE OF THE EIGEN-FUNCTION EXPANSION CORRESPONDING TO A MIXED STURM-LIOUVILLE PROBLEM

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ABSTRACT : The convergence of the eigenfunction expansion of an arbitrary f ε $L^2[a,b]$, (the class of all functions square-integrable on [a,b]), associated with a mixed Sturm-Liouville problem is connected with the convergence of the eigenfunction expansion of f associated with a "suitable" separated Sturm-Liouville problem and thereby the congergence of the said expansion under "Fourier conditions" is established.

Key Words : Mixed and Separated Sturm-Liouville problem (SLP), Resolvent operator, Fourier conditions.

1990 AMS Subject Classification : 34B

1. INTRODUCTION

A study of boundary value problem associated with a second order differential equation plays an important role in the theory of differential equations because of its application to the physical problems. The differential equation

$$L[y] = -\ [p(x)\ y^{(1)}(x)]^{(1)} + q(x)y(x) = \lambda y(x),\ a \leq x \leq b. \ ....(1)$$

where p and q are real valued functions on [a,b], so that L[y] is well-defined on [a,b] for all sufficiently differentiable functions y, $p(x) \neq 0$ on [a,b] and λ is a complex parameter, is known as the Sturm-Liouville equation.

The boundary value problem consisting of this Sturm-Liouville equation and two boundary conditions, one at each end x = a and x = b is referred to as the separated SLP. In [3] Titchmarsh has discussed thoroughly this separated SLP and has shown that the eigenfunction expansion of an arbitrary function f ε $L^2[a,b]$ with respect to this separated SLP behaves as regards convergence in the same way as the ordinary Fourier series of f i.e. it converges if and only if the ordinary Fourier series converges and it converges to the same limit.

In this paper we deal with the convergence of the eigenfunction expansion of a function f ε $L^2[a,b]$ with respect to the mixed SLP consisting of the Sturm-Liouville equation (assuming p(x) = 1) and the two mixed boundary conditions.

$$\left.\begin{array}{l} a_1y(a) + a_2y^{(1)}(a) + a_3y(b) + a_4y^{(1)}(b) = 0, \\ b_1y(a) + b_2y^{(1)}(a) + b_3y(b) + b_4y^{(1)}(b) = 0. \end{array}\right\} \qquad \text{..................(2)}$$

where a_i, b_i (i = 1, 2, 3, 4) are real constants.

It is obvious that the a_i 's are not proportional to the b_i's (i = 1, 2, 3, 4), otherwise the boundary conditions in (2) will be linearly dependent.

Before we can write down the eigenfunction expansion of an f ε $L^2[a,b]$ with respect to a SLP, we are to assure that the corresponding SLP is self-adjoint, while a separated SLP is always self-adjoint. A mixed SLP is self-adjoint if and only if the co-efficients a_i, b_i of the boundary conditions (2) satisfy (vide [5])

$$a_1b_2 - a_2b_1 = a_3b_4 - a_4b_3. \qquad \text{........................(3)}$$

Following Eastham [2], we find that there exists a countable set of values of λ, called the eigenvalues, say $\{\lambda_n\}_n$, (vide [4]) so that the SLP consisting of (1) - (2) [under the restriction(3)], has non-trivial solution, say ψ_n ( ), called the eigenfunction, for $\lambda = \lambda_n$.

Given an arbitrary function f ε L^2 [a,b], the series $\sum\limits_n c_n\psi_n(x)$ where $c_n = \int_a^b f(t)\psi_n(t)\,dt$, is known as the eigenfunction expansion of f with regard to the mixed SLP(1) – (2) – (3).

Our aim is to establish the convergence of this eigenfunction expanision $\sum\limits_n c_n\psi_n(x)$ under suitable restrictions. In this direction we prove the following theorem :

Theorem : *Let p and q satisfy the following* :

(a) *p is absolutely continuous on [a,b] $\forall$ b > 0 and p(x) > 0.*

(b) *q is continuous and integrable on [a,b] $\forall$ b > 0. Then the eigenfunction expansion of an arbitrary function f ε L^2[a,b], associated with the mixed SLP consisting of (1) – (2) [under the resiriction (3)] behaves, as regards convergence, in the same way as the ordinary Fourier series of f does.*

That is, it converges to $\frac{1}{2}\{f(x + o) + f(x - o)\}$, if, for example f(x) is of bounded variation in the neighbourhood of x. Such conditions on f that ensure the convergence of the Fourier series of f are referred to as "Fourier conditions".

This is achieved by relating this eigenfunction expansion $\sum\limits_n c_n\psi_n(x)$ to the eigenfunction expansion of f, associated with a corresponding "suitable" separated SLP.

It may be pointed out that given a mixed SLP, we are to choose a separated SLP suitably so that our purpose is served. There do exist several such suitable separated SLPs, none of which has any special advantage over the others.

2. RESOLVENT OPERATORS OF THE BOUNDARY VALUE PROBLEM

To avoid cumbersome calculations, we assume that $p(x) = 1$ in (1). The case $p(x) \neq 1$ can be treated with suitable transformation reducing (1) to an equation of the same type but with $p(x) = 1$.

Given any $f \in L^2[a,b]$, we consider the inhomogeneous differential equation

$$-y^{(2)}(x) + q(x)y(x) = \lambda y(x) - f(x). \quad ....(4)$$

The solution of this differential equation that satisfies the boundary conditions of a SLP is known as the resolvent operator of the said SLP.

Let $\Phi_1(x, \lambda)$ denote the resolvent operator of the mixed SLP (1) – (2) – (3), while $\Phi_1(x, \lambda)$ denotes the resolvent operator of the separated SLP consisting of (ˉ) and the boundary conditions

$$\left.\begin{aligned} a_1y(a) + a_2y^{(1)}(a) &= 0, \\ b_3y(b) + b_4y^{(1)}(b) &= 0. \end{aligned}\right\} \quad ....(5)$$

The importance of this resolvent operator is clear from the following relation :

$$\Phi_1(x, \lambda) = \sum_{n=o}^{\infty} \frac{c_n\psi_n(x)}{\lambda - \lambda_n} \quad ....(6)$$

established by Titchmarsh [3]–p 16. So, if we consider the integral,

$$I_\Gamma = \frac{1}{2\pi i}\int_\Gamma \Phi_1(x, \lambda)\, d\lambda, \quad ....(7)$$

where Γ is a large closed contour in the λ–plane, then I_Γ is a finite sum of the Sturm-Liouville expansion $\sum_n c_n\psi_n(x)$.

Hence to prove the convergence of $\sum_n c_n\psi_n(x)$, it is sufficient to prove that I_Γ tends to a finite limit as Γ extends and becomes infinitely large.

If $\phi(x, \lambda)$ and $\chi(x, \lambda)$ are the solutions of (1) satisfying the conditions

$$\left.\begin{aligned} \phi(a, \lambda) &= a_2, & \phi^{(1)}(a, \lambda) &= -a_1, \\ \chi(b, \lambda) &= b_4, & \chi^{(1)}(b, \lambda) &= -b_3, \end{aligned}\right\} \quad ....(8)$$

Titchmarsh [3] has shown that the Wronskian $W(\phi, \chi)$ is independent of x so that we have

$$\begin{aligned} w(\lambda) &= W(\phi, \chi)(x) \\ &= \phi(x, \lambda)\chi^{(1)}(x, \lambda) - \phi^{(1)}(x, \lambda)\chi(x, \lambda) \\ &= a_1\chi(a, \lambda) + a_2\chi^{(1)}(a, \lambda) \\ &= -b_3\phi(b, \lambda) - b_4\phi^{(1)}(b, \lambda) \end{aligned} \quad ....(9)$$

and the resolvent operator $\Phi(x, \lambda)$ is given by

$$\Phi(x, \lambda) = \frac{1}{w(\lambda)}\left[\chi(x, \lambda)\int_a^x \phi(y, \lambda)\, f(y)\, dy + \phi(x, \lambda)\int_x^b \chi(y, \lambda)\, f(y)\, dy\right] \quad ......(10)$$

3. THE RELATION BETWEEN THE TWO RESOLVENT OPERATORS

As the resolvent operators $\Phi_1(x, \lambda)$ and $\Phi(x, \lambda)$ satisfy the inhomogeneous equation (4), their difference $\Phi_1(x, \lambda) - \Phi(x, \lambda)$ will satisfy the corresponding homogeneous equation (1) ; so it can be expressed as the linear combination of the two linearly independent solutions $\phi(x, \lambda)$ and $\chi(x, \lambda)$ of (1).

Let $\Phi_1(x, \lambda) - \Phi(x, \lambda) = c_1 \phi(x, \lambda) + c_2 \chi(x, \lambda)$ ... ... (11)

where c_1 and c_2 are independent of x.

Using (10) we then have

$$\Phi_1(x, \lambda) = c_1 \phi(x, \lambda) + c_2 \chi(x, \lambda) +$$

$$\frac{1}{W(\lambda)}\left[\chi(x, \lambda) \int_a^x \phi(y, \lambda)\, (f)\,(y)\, dy + \phi(x, \lambda) \int_x^b \chi(y, \lambda)\, f(y)\, dy\right]. \quad \ldots\ldots (12)$$

4. DETERMINATION OF THE CONSTANTS C_1 AND C_2

Since $\Phi_1(x, \lambda)$ satisfies the mixed boundary conditions (2), then

$$\begin{aligned} &a_1\Phi_1(a, \lambda) + a_2\Phi_1^{(1)}(a, \lambda) + a_3\Phi_1(b, \lambda) + a_4\,\Phi_1^{(1)}(b, \lambda) = 0, \\ &b_1\Phi_1(a, \lambda) + b_2\Phi_1^{(1)}(a, \lambda) + b_3\Phi_1(b, \lambda) + b_4\,\Phi_1^{(1)}(b, \lambda) = 0, \end{aligned} \qquad \ldots\ldots (13)$$

In the sequel, for brevity, we will use the notations

$$\int_a^b \phi(y, \lambda)\, f(y)\, dy = \{\phi, f\}.$$

$$\int_a^b \chi(y, \lambda)\, f(y)\, dy = \{\chi, f\}.$$

Substituting for $\Phi_1^{(r)}(a, \lambda)$, $\Phi_1^{(r)}(b, \lambda)$ $(r = 0, 1)$ from (12) in (13) we find the equations for the determination of c_1 and c_2 as

$$c_1\{a_3\phi(b, \lambda) + a_4\,\phi^{(1)}(b, \lambda)\} + c_2\{w(\lambda) + (a_3b_4 - a_4b_3)\}$$

$$= -\frac{1}{W(\lambda)}(a_3b_4 - a_4b_3)\{\phi, f\},$$

$$c_1\{-(a_1b_2 - a_2b_1) - w(\lambda)\} + c_2\{b_1\,\chi(a, \lambda) + b_2\chi^{(1)}(a, \lambda)\}$$

$$= -\frac{1}{W(\lambda)}(a_1b_2 - a_2b_1)\{\chi, f\}.$$

$$\text{Let } \Delta(\lambda) = \begin{vmatrix} a_3\phi(b, \lambda) + a_4\,\phi^{(1)}(b, \lambda) & w(\lambda) + (a_3b_4 - a_4b_3) \\ -(a_1b_2 - a_2b_1) - w(\lambda) & b_1\,\chi(a, \lambda) + b_2\chi^{(1)}(a, \lambda) \end{vmatrix}$$

By Cramer's rule [writing $\Delta(\lambda)\, w(\lambda) = \Delta w$] we get

$$\left.\begin{array}{l} c_1 = \dfrac{1}{\Delta W} [-(a_3b_4 - a_4b_3) \{b_1\, \chi(a, \lambda) + b_2\chi^{(1)} (a, \lambda)\} \{\phi, f\} \\ \qquad -(a_1b_2 - a_2b_1) \{w(\lambda) + (a_3b_4 - a_4b_3) \{\chi, f\}], \\ c_2 = \dfrac{1}{\Delta W} [-(a_3b_4 - a_4b_3) \{(a_1b_2 - a_2b_1) + w(\lambda)\} \{\phi, f\} \\ \qquad + (a_1b_2 - a_2b_1) \{a_3\, \phi(b, \lambda) + a_4\, \phi^{(1)} (b, \lambda)\} \{\chi, f\}]. \end{array}\right. \quad \ldots\ldots (15)$$

5. THE EIGENFUNCTION EXPANSION

In order to discuss the convergence of the eigenfunction expansion of $f \varepsilon L^2 [a,b]$ with respect to the mixed SLP, we shall consider $\frac{1}{2\pi i} \int_{\Gamma_N} \Phi_1(x, \lambda)\, d\lambda$,

where Γ_N is a closed contour in the λ-plane, which is symmetrical about the real λ-axis. The required eigenfunction expansion is obtained on extending the contour Γ_N indefinitely large.

Putting $\lambda = s^2 = (\sigma + it)^2$, we suppose that the contour Γ_N in the λ-plane corresponds to the contour γ_N in the s-plane, where γ_N consists of the lines

$$\left.\begin{array}{l} \sigma = \sigma_N = (N + \frac{1}{2})\, \pi / (b - a), \qquad 0 \le |t| \le (N + \frac{1}{2})\, \pi / (b - a), \\ t = \pm\, t_N = \pm (N + \frac{1}{2})\, \pi / (b - a),\ 0 \le \sigma \le (N + \frac{1}{2})\, \pi / (b - a). \end{array}\right\} \ldots\ldots (16)$$

We shall show that as $N \to \infty$

$$\frac{1}{2\pi i} \int_{\Gamma_N} \Phi_1 (x, \lambda)\, d\lambda \to \frac{1}{2}\{f(x + 0) + f(x - 0)\}, \qquad \ldots\ldots\ldots (17)$$

whenever the Fourier series of f converges to $\frac{1}{2}\{f(x + 0) + f(x - 0)\}$.

We note that

$$\int_{\Gamma_N} \Phi_1 (x, \lambda)\, d\lambda = \int_{\Gamma_N} c_1\phi (x, \lambda) + c_2\chi (x, \lambda)\}\, d\lambda + \int_{\Gamma_N} \Phi_1 (x, \lambda)\, d\lambda$$

$$= I_1 + \int_{\Gamma_N} \Phi_1 (x, \lambda)\, d\lambda \qquad \ldots\ldots\ldots (18)$$

Using the expressions (15) obtained for c_1 and c_2, we get

$$\begin{array}{l} I_1 = \displaystyle\int_{\Gamma_N} \frac{1}{\Delta w} (a_1 b_2 - a_2 b_1) [-w(\lambda)\, \chi (x, \lambda) - \{ b_1\chi (a, \lambda) + b_2\chi^{(1)} (a, \lambda)\}\, \phi (x, \lambda) \\ \qquad - (a_1 b_2 - a_2 b_1)\, \chi (x, \lambda)] \{\phi, f\} + [-w(\lambda)\, \phi (x, \lambda) + \{a_3\phi (b, \lambda) \\ \qquad + a_4\phi^{(1)} (b, \lambda)\}\, \chi (x, \lambda) - (a_3 b_4 - a_4 b_3)\, \phi (x, \lambda)] \{\chi, f\}.]\, d\lambda \end{array}$$

$$= \int_{\Gamma_N} \frac{1}{\Delta w} (a_1 b_2 - a_2 b_1) [\xi(x, \lambda) \{\phi, f\} + \eta (x, \lambda) \{\chi, f\}] \, d\lambda$$

$$+ \int_{\Gamma_N} \frac{1}{\Delta w} (a_1 b_2 - a_2 b_1) [\chi (x, \lambda) \{\phi, f\} + \phi (x, \lambda) \{\chi, f\}] \, d\lambda \text{......(19)}$$

where $\xi (x, \lambda) = - \{b_1 \chi (a, \lambda) + b_2 \chi^{(1)} (a, \lambda)\} \phi (x, \lambda) - (a_1 b_2 - a_2 b_1) \chi (x, \lambda)$,
$\eta (x, \lambda) = - (a_3 b_4 - a_4 b_3) \phi (x, \lambda) + \{a_3 \phi (b, \lambda) + a_4 \phi^{(1)} (b, \lambda)\} \chi (x, \lambda)$.

6. ESTIMATES OF $\xi (x, \lambda)$, $\eta (x, \lambda)$, $\phi (x, \lambda)$, $\chi (x, \lambda)$

It is clear that $\xi (x, \lambda)$ and $\eta (x, \lambda)$ are those solutions of (1), which satisfy the conditions

$$\left.\begin{aligned} &\xi (a, \lambda) = - b_2 w (\lambda), && \xi^{(1)} (a, \lambda) = b_1 w (\lambda), \\ &\eta (b, \lambda) = - a_4 w (\lambda), && \eta^{(1)} (b, \lambda) = a_3 w (\lambda), \end{aligned}\right\} \text{.........(20)}$$

where $w(\lambda)$ is given by (9).

Hence, by using the method of variation of parameters, we can prove that

$$\xi (x, \lambda) = - \frac{1}{s} b_1 w (\lambda) \sin \{s (x - a)\} - b_2 w (\lambda) \cos \{s (x - a)\}$$

$$+ \frac{1}{s} \int_a^x \sin \{s (x - y)\} q (y) \xi (y, \lambda) \, dy,$$

$$\eta (x, \lambda) = - \frac{1}{s} a_3 w (\lambda) \sin \{s (b - x)\} - a_4 w (\lambda) \cos \{s (b - x)\}$$

$$+ \frac{1}{s} \int_x^b \sin \{s (x - y)\} q (y) \eta (y, \lambda) \, dy.$$

Following Titchmarsh [3] we can then find the estimates of $\xi (x, \lambda)$ and $\eta (x, \lambda)$ as given below

$$\left.\begin{aligned} &\xi (x, \lambda) = - b_2 w (\lambda) \cos \{s (x - a)\} + 0 \{e^{|t| (x-a)}\}, \\ &\eta (x, \lambda) = - a_4 w (\lambda) \cos \{s (b - x)\} + 0 \{e^{|t| (b-x)}\}, \end{aligned}\right\} \text{.........(21)}$$

as $|\lambda| \to \infty$

The estimates of $\phi (x, \lambda)$, $\chi (x, \lambda)$ given below were obtained by Titchmarsh

$$\left.\begin{aligned} &\phi (x, \lambda) = a_2 \cos \{s (x - a)\} + 0 \{|s|^{-1} e^{|t| (x-a)}\}, \\ &\chi (x, \lambda) = b_4 \cos \{s (b - x)\} + 0 \{|s|^{-1} e^{|t| (b-x)}\}, \end{aligned}\right\} \text{ as } |\lambda| \to \infty \quad \text{.........(22)}$$

7. ESTIMATES OF $\int_{\Gamma_N}\{c_1\,\phi(x, \lambda) + c_2\,\chi(x, \lambda)\}\, d\lambda$.

Using (22) in (14) we can show that as $|\lambda| \to \infty$,

$$\left|\frac{1}{\Delta(\lambda)}\right| = 0\,(|s|^{-2} e^{-2|t|(b-a)}), \qquad \text{.......(23)}$$

$$\{\phi, f\} = \int_a^b \phi(y, \lambda)\, f(y)\, dy = 0\,(e^{|t|(b-a)}), \qquad \text{.........(24)}$$

$$\{\chi, f\} = \int_a^b \chi(y, \lambda)\, f(y)\, dy = 0\,(e^{|t|(b-a)}). \qquad \text{.........(25)}$$

Then using (23) – (25) in (19) we get

$$I_1 = \int_{\Gamma_N}\{c_1\,\phi(x, \lambda) + c_2\,\chi(x, \lambda)\}\, d\lambda.$$

$$= \int_{\Gamma_N} \frac{1}{\angle W}(a_1\, b_2 - a_2\, b_1)\,[\,\xi(x, \lambda).\ \{\phi, f\} + \eta(x, \lambda).\ \{\chi.\ f\}\,]\, d\lambda +$$

$$\int_{\Gamma_N} \frac{1}{\Delta W}(a_1\, b_2 - a_2\, b_1)\,[\,\chi(x, \lambda).\ \{\phi, f\} + \phi(x, \lambda).\ \{\chi.\ f\}\,]\, d\lambda.$$

$$= 0\,[\int_{\gamma_N} |s|^{-1}\{e^{-|t|(x-a)} + e^{-|t|(b-x)}\}\, ds], \text{ as } |\lambda| \to \infty. \qquad \text{.........(26)}$$

8. PROOF OF THE THEOREM

Writing $s = \sigma + it$ for the part of the contour Γ_N that corresponds to

$\sigma = (N + \frac{1}{2})\,\pi\,/\,(b - a) = \alpha_N$ (say), $-\,\alpha_N \le t \le \alpha_N$,

the contribution to I_1 is

$$0\,[\,\frac{1}{\alpha_N}\int_{-\alpha_N}^{\alpha_N}\{e^{-|t|(x-a)} + e^{-|t|(b-x)}\}\, dt] = 0\,(\frac{1}{\alpha_N}).$$

Similarly for the part of the contour Γ_N that corresponds to

$t = \pm\,(N + \frac{1}{2})\,\pi\,/\,(b - a) = \pm\,\alpha_N$ (say), $0 \le \sigma \le \alpha_N$,

the contribution to I_1 is

$$0\,[\,\frac{1}{\alpha_N}\int_0^{\alpha_N}\{e^{-|t|(x-a)} + e^{-|t|(b-x)}\}\, d\sigma] = 0\,(\frac{1}{\alpha_N}).$$

So, Γ_N being a contour symmetric about the real axis, we get

$$\left.\begin{aligned} I_1 &= \int_{\Gamma_N} \{c_1 \phi(x, \lambda) + c_2 \chi(x, \lambda)\}\, d\lambda. \\ &= 0\left(\frac{1}{\alpha_N}\right) = \underset{\sim}{O}(1) \text{ as } N \to \infty . \end{aligned}\right\} \qquad \text{.........(27)}$$

$$\text{Hence } \int_{\Gamma_N} \Phi_1(x, \lambda)\, d\lambda = \int_{\Gamma_N} \Phi(x, \lambda)\, d\lambda + \underset{\sim}{O}(1) \text{ as } N \to \infty . \qquad \text{.........(28)}$$

This proves that the eigenfunction expansion of an arbitrary f ε L^2 [a, b] associated with a mixed SLP is convergent if and only if the eigenfunction expansion of f associated with a suitable separated SLP is convergent.

Following Titchmarsh [3] we know that if f satisties "Fourier condition" then as $N \to \infty$

$$\frac{1}{2\pi i} \int_{\Gamma_N} \Phi_1(x, \lambda)\, d\lambda \to \frac{1}{2}\{f(x + 0) + f(x - 0)\}.$$

This, in conjunction with (6) and (28), proves that the eigenfunction expansion $\sum_n c_n \psi_n(x)$ of f converges to $\frac{1}{2}\{f(x + 0) + f(x - 0)\}$ whenever f satisties Fourier conditions. This establishes our theorem.

9. REMARK

Given any mixed SLP we can always select a corresponding separated SLP so that the convergence of the eigenfunction expansion of an arbitrary f ε L^2 [a, b], corresponding to the mixed SLP is guided by the convergence of the eigenfunction expansion corresponding to the separated SLP as shown in (28). It therefore becomes clear that as far as the convergence of the eigenfunction expansion is required, one need not go into the complication arising due to the mixed boundary conditions ; a suitable set of separated boundary conditions serves the purpose.

In this paper, the separated boundary conditions taken into consideration are given by (5) ; other types of separated boundary conditions might have been chosen but the resulting calculations might be too clumsy. It is obvious that (5) could be replaced by the set of boundary conditions

$$\left.\begin{aligned} b_1 y(a) + b_2 y^{(1)}(a) &= 0 \\ a_3 y(b) + a_4 y^{(1)}(b) &= 0, \end{aligned}\right\} \qquad \text{............(20)}$$

without any significant change in calculations.

REFERENCE

1. E. A. Coddirgton N. Levinson : Theory of Ordinary Differential equations. McGraw-Hill Book company Inc. 1955.
2. M. S. P. Eastham : Theory of Ordinary Differential equations. Van Nostrand Reinhold Company, London, 1970.
3. E. C. Titchmarsh : Eigen function expansions associated with second order differential equations. Part I, 2nd ed. Oxford University Press, London, 1962.
4. J. Weidmann : Linear Operators in Hilbert spaces. Graduate Texts in Mathematics, Vol. 68, Springer Verlag, New York, 1980.
5. Jyoti Das, Arnab Kumar Chakravorty : An alternative proof of self-adjointness condition of a mixed Sturm-Liouville problem. Mathematics Student vol 64, p. 9-14, 1995.
6. Jyoti Das, Arnab Kumar Chakravorty : Estimates of the eigenvalues of a mixed Sturm-Liouville problem from that of a corresponding separated Sturm-Liouville problem. Far East J. Math Sci 2(1), 9–16, 1994.

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CHARACTERIZATIONS OF BITOPOLOGICAL QHC SPACES VIA CERTAIN CLASSES OF SETS

J. N. NANDI AND M. N. MUKHERJEE

ABSTRACT : In the present article, certain new types of sets are introduced for a bitopological space. This provides a new approach towards the study of bitopological quasi H-closed spaces ; a few characterizations of such a space are established here to justify the contention.

Key Words : ij-θ^c-set, ij-θ^c-set, ij-QHC space, ij-α-open sets, ij-α-closure.

1990 AMS Subject Classification : 54 E 55

H-closed and quasi H-closed (QHC) spaces are extremely well known concepts of topology, and the same have so far been studied to a great extent. Extensions of these concepts in bitopological setting have also been investigated by many (e.g. see [3, 5, 7, 9]). The present paper is a continuation of the latter study, and we have introduced here certain types of sets which play key roles in characterizing a bitopological QHC space from an altogether new view point.

Throughout the paper, by a space (X, Q_1, Q_2) or simply by X we shall mean a bitopological space [4] X endowed with two arbitrary topologies Q_1, and Q_2. For a subset A of (X, Q_1, Q_2), Q_i-int A and Q_i-cl A will respectively stand for the interior and closure of A in (X, Q_i), where $i = 1, 2$. According to Kariofillis [3], a point x in (X, Q_1, Q_2) is said to be in the ij-θ-closure of a subset A of X, written as $x \in$ ij-θ-cl A iff for every Q_i-open set U containing x, $(Q_j\text{-cl } U) \cap A \neq \emptyset$, where and henceforth in every such sentence involving both i and j we assume $i, j = 1, 2$ and $i \neq j$. A set A in X is called ij-θ-closed iff A = ij-θ-cl A. The complement of an ij-θ-closed set is called an ij-θ-open set. The ij-θ-interior of a set A in X, denoted by ij-θ-int A, consists of those points x of A such that for some Q_i-open set U containing x, $Q_j\text{-cl}U \subset A$. A set A is called ij-θ-open iff A = ij-θ-int A, or iff $(X - A)$ is ij-θ-closed [1, 3]. A Q_i-open set U will be called an ij-regularly open set [12] or simply an ij-roset iff $U = Q_i\text{-int } Q_j\text{-cl } U$ and complements of ij-rosets are called ij-regularly closed sets or simply ij-rc set.

A set A in (X, Q_1, Q_2) is called ij-δ-closed [10] iff for each $x \in (X - A)$, there is a Q_i-open set U containing x such that $(Q_i\text{-int } Q_j\text{-cl } U) \cap A = \emptyset$. The set A is called ij-δ-open [10] iff $(X - A)$ is ij-δ-closed. The class of all ij-δ-open sets in X forms a topology Q_i^s, called the ij-semiregularization topology [8, 12] such that $Q_i^s \subset Q_i$, and the family of all ij-ro sets of (X, Q_1, Q_2) forms a base for Q_i. A property P of a bitopological space (X, Q_1, Q_2) is called a pairwise semiregular property [8] provided (X, Q_1, Q_2) has P iff (X, Q_1^s, Q_2^s) possesses P. We now state a few known results that we shall use in course of the subsequent deliberations.

Theorem 1. [1, 3, 8, 9, 10] For a bitopological space (X, Q_1, Q_2) the following hold :

(a) The collection of ij-θ-open sets in X forms a topology Q_i^θ on X such that $Q_i^\theta \subset Q_i^s \subset Q_i$.

(b) If $A \subset B \subset X$, then ij-θ-int $A \subset$ ij-θ-int B and ij-θ-cl $A \subset$ ij-θ-cl B.

(c) If A ($\subset X$) is Q_j-closed, then Q_i-int A = Q_i^s int A = ij-θ-int A.

(d) If A is Q_j-open in X, then Q_i-cl A = Q_i^s-cl A = ij-θ-cl A.

We now introduce the following definition.

Definition 2. A set A in a space (X, Q_1, Q_2) is called an

(i) ij-θ^o-set iff A = ij-θ-int B, for some $B \subset X$,

(ii) ij-θ^c-set iff A = ij-θ-cl B, for some $B \subset X$.

Note 3. It is easy to see that for any set A in X, ij-θ-cl $(X - A) = X -$ (ij-θ-int A). Thus a set A is an ij-θ^o-set iff $(X - A)$ is an ij-θ^c-set.

Proposition 4. For any two sets A and B in a space X,

(a) ij-θ-cl $(A \cup B)$ = (ij-θ-cl A) $\cup$ (ij-θ-cl B)

(b) ij-θ-int $(A \cap B)$ = (ij-θ-int A) $\cap$ (ij-θ-int B).

Proof. (a) Clearly, (ij-θ-cl A) $\cup$ (ij-θ-cl B) $\subset$ ij-θ-cl $(A \cup B)$. Now, $x \notin$ (ij-θ-cl A) $\cup$ (ij-θ-cl B) $\Rightarrow$ there exist Q_i-open sets U, V containing x such that $A \cap Q_j$-cl $U = B \cap Q_j$-cl $V = \emptyset$. Then $U \cap V$ is a Q_i-open set containing x such that $(A \cup B) \cap Q_j$-cl $(U \cap V) = \emptyset$. Hence $x \notin$ ij-θ-cl (AUB).

(b) Similar to (a) and is omitted.

Corollary 5. The family σ_{ij} = {ij-θ-int A : $A \subset X$} forms a base for some topology $Q_i(\sigma_{ij})$ on X.

Again, from Theorem 1 (c) it follows that

Proposition 6. Every ij-roset is an ij-θ^o-set.

It is known [9] that for any subset A of a space (X, Q_1, Q_2), ij-θ-cl A $= \cap\{Q_i$-cl $V : A \subset V \in Q_j\}$. It then follows that ij-θ-int A = U $\{Q_i$-int $V : V \subset A$ and V is Q_j-closed in X$\}$. Hence we obtain :

Theorem 7. Every ij-θ^o-set is the union of ij-rosets and hence is ij-δ-open as well as Q_i-open.

It now readily follows in view of corollary 5, Theorem 7 and proposition 6 that

Theorem 8. For a space (X, Q_1, Q_2), $Q_j(\sigma_{ij}) = Q_i^{\theta}$ and hence σ_{ij} forms an open base for the semiregularization topology Q_i^s on X.

Theorem 9. [5] A space (X, Q_1, Q_2) is said to be ij-QHC iff every Q_i-open cover of X has a finite Q_j-proximate subcover (i.e., a finite subfamily the union of whose members is Q_j-dense in X).

Theorem 10. A space (X, Q_1, Q_2) is ij-QHC iff whenever μ is a cover of X by ji-θ^c-sets such that for each point x of X some member of μ is a Q_i- neighbourhood of x, then μ has a finite subcover.

Proof. Let μ be a cover of an ij-QHC space X by ji-θ^c-sets with the stated property. For each $x \in X$, there exist a $U_x \in \mu$ and a Q_i-open set V_x such that $x \in V_x \subset U_x$. The collection $\{V_x : x \in X\}$ is then a Q_i-open cover of X, and consequently by ij-QHC property of X, there is a finite subset $\{x_1, x_2, \ldots\ldots\ldots, x_n\}$ of X such that $X = \bigcup\limits_{k=1}^{n} Q_j\text{-cl}\, V_{x_k} \subset \bigcup\limits_{k=1}^{n} Q_j\text{-cl}\, U_{x_k} = \bigcup\limits_{k=1}^{n} U_{x_k}$, since U_{x_k}'s are Q_j-closed sets.

Coversely, let the given condition hold for a space X and μ be a Q_i-open cover of X. For each $x \in X$, there exists $U_x \in \mu$ such that $x \in U_x$. By Theorem 1 (d) we have, $V_x = Q_j$ - cl U_x = ji-θ-cl U_x and hence V_x is a ji-θ^c-set for each $x \in X$. Then $\{V_x : x \in X\}$ is a cover of X by ji-θ^c-sets with the stipulated property. Thus we obtain,

$X = \bigcup\limits_{k=1}^{n} V_{x_k} = \bigcup\limits_{k=1}^{n} Q_j\text{-cl}\, U_{x_k}$ for a finite subset $(x_1, x_2, \ldots, x_n\}$ of X, and this proves that X is ij-QHC.

Definition 11. A family $\mathscr{F}$ of sets in a space (X, Q_1, Q_2) is said to possess ij-θ^o-FIP iff the ij-θ-interior of every finite intersection of members of $\mathscr{F}$ is non-void.

Theorem 12. A space (X, Q_1, Q_2) is ij-QHC iff every family of ij-θ^c-sets with ji-θ^o-FIP has non-null intersection.

Proof. Let X be an ij-QHC space and $\{F_\alpha : \alpha \in \Lambda\}$ be a family of ij-θ^c-sets in X with ji-θ^o-FIP. If $\bigcap\limits_{\alpha \in \Lambda} F_\alpha = \emptyset$, then $\mu = \{X - F_\alpha : \alpha \in \Lambda\}$ is a cover of X by ij-θ^o-sets and hence μ is also a Q_i-open cover of X. Since X is ij-QHC, we have :

$X = \bigcup\limits_{k=1}^{n} Q_j\text{-cl}\left(X - F_{\alpha_k}\right)$ for a finite subcollection $\{X - F_{\alpha_1}, X - F_{\alpha_2}, \ldots X - F_{\alpha_n}\}$ of μ.

Thus $\varnothing = X - \bigcup_{k=1}^{n} Q_j\text{-cl}\left(X - F_{\alpha_k}\right) = \bigcap_{k=1}^{n} Q_j\text{-int } F_{\alpha_k} = \bigcap_{k=1}^{n}$ ji-θ-int F_{α_k}(by therorem 1 (c)) = ji-θ-int $\left(\bigcap_{k=1}^{n} F_{\alpha_k}\right)$ [by Proposition 4 (b)], which contradicts that $\{F_\alpha : \alpha \in \Lambda\}$ has ji-θ°-FIP.

Conversely, let the given condition hold in a space X. We first show that (X, Q_1^s, Q_2^s) is ij-QHC. Since the family of all ij-θ°-sets of X forms a base for Q_i^s, it suffices to show that every cover of X by ij-θ°-sets has a finite Q_j^s-proximate subcover. So let $\mu = \{U_\alpha : \alpha \in \Lambda\}$ be a cover of X by ij-θ°-sets. Then $\{X - U_\alpha : \alpha \in \Lambda\}$ ($= \mathscr{F}$, say) is a family of ij-θ^c-sets with $\cap\mathscr{F} = \varnothing$, so that $\mathscr{F}$ cannot have ji-θ°-FIP. Thus there exists a finite subset $\{X - U_{\alpha_1}, \ldots, X - U_{\alpha_n}\}$ of $\mathscr{F}$ such that ji-θ-int $\left[\bigcap_{k=1}^{n}\left(X - U_{\alpha_r}\right)\right] = \varnothing$.

Then X = X—ji-θ-int $\left[\bigcap_{k=1}^{n}\left(X - U_{\alpha_k}\right)\right]$ = ji-θ-cl $\left[\bigcup_{k=1}^{n} u_{\alpha_k}\right] = \bigcup_{k=1}^{n}$ ji-θ-cl U_{α_k} (by proposition 4 (a)) $= \bigcup_{k=1}^{n} Q_j^s\text{-cl } U_{\alpha_k}$ (Since U_{α_k}'s are Q_i-open). Hence (X, Q_1^s, Q_2^s) is ij-QHC. Since the property of a space being ij-QHC is known [8] to be a pairwise semiregular property, (X, Q_1, Q_2) is ij-QHC.

Definition 13. Let $\{U_\alpha : \alpha \in D\}$ be a net of ij-θ°-sets in a space X with the directed set $(D, \geq)$ as its domain. A point x of X is said to be an ij-θ-adherent point of the net if for each $\alpha \in D$ and each Q_i-open set V containing x, there exists $\beta \in D$ with $\beta \geq \alpha$ such that $U_\beta \cap Q_j\text{-cl } V \neq \varnothing$.

Theorem 14. A space X is ij-QHC iff every net of non-null ji-θ°-sets has an ij-θ-adherent point.

Proof. Let $\{U_\alpha : \alpha \in D\}$ be a net of non-null ji-θ°-sets in the ij-QHC space X. For each $\alpha \in D$, let F_α = ij-θ-cl $[\cup \{U_\beta : \beta \in D \text{ and } \beta \geq \alpha\}]$. Then $\mathscr{F} = \{F_\alpha : \alpha \in D\}$ is a family of ij-θ^c-sets with ji-θ°-FIP. By Theorem 12 there is an $x \in \bigcap_{\alpha \in D} F_\alpha$. Then for any Q_i-open set V containing x and $\alpha \in D$ $(Q_j - \text{cl } V) \cap [\cup \{U_\beta : \beta \in D \text{ and } \beta \geq \alpha\}] \neq \varnothing$. Thus there is some $\beta \in D$ with $\beta \geq \alpha$ such that $(Q_j.\text{cl}V) \cap U_\beta \neq \phi$. Hence the net $\{U_\alpha : \alpha \in D\}$ of ji-θ^c-sets in X has an ij-θ-adherent point in X.

Conversely, let $\mathscr{F}$ be a collection of ij-θ^c-sets in X with ji-θ°-FIP. Then the family $\mathscr{F}^*$ of all finite intersections of members of $\mathscr{F}$ becomes a directed set under the relation $\geq$, where $F_1 \geq F_2$ iff $F_1 \subset F_2$ ($F_1, F_2 \in \mathscr{F}^*$). For each $F \in \mathscr{F}^*$, we assign the set ji-θ-int F which is non-null, as $\mathscr{F}$ has ji-θ°-FIP. Then {ji-θ-int F : $F \in (\mathscr{F}^*, \geq)$} is a net of non empty ji-θ°-sets in X. By hypothesis, some point x of X is an ij-θ-adherent point of this net. We only show that $x \in \cap\, \mathscr{F}$, the rest follows from Theorem 12. Let $F \in \mathscr{F}$ and V be a Q_i-open nbd of x. Since $F \in \mathscr{F}^*$, there is some $G \in \mathscr{F}^*$ with $G \geq F$ (i.e., $G \subset F$) such that (ji-θ-int G) $\cap$ Q_j - cl V $\neq \emptyset$. Then (ji-θ-int F) $\cap$ Q_j - cl V $\neq \emptyset$. Thus $x \in$ ij-θ-cl (ji-θ-int F) = Q_i-cl Q_j-int F (by Theorem 1, since F being an ij-θ^c-set, is Q_i-closed) $\subset$ F. Thus $x \in \cap\mathscr{F}$.

Let us now set the following definition of another class of sets in a bitopological space.

Definition 15. A subset A of a space (X, Q_1, Q_2) is said to be ij–α-open if A $\subset$ Q_i-int Q_j-Cl Q_i-int A.

Remark 16. Clearly, a Q_i-open set is ij-α-open and an ij-α-open set is ij-semiopen (a set A in a space (X, Q_1, Q_2) is called ij-semiopen [2], iff there exists a Q_i-open set U such that U $\subset$A $\subset$ Q_j-cl U, or equivalently, A$\subset Q_j$ cl Q_1-int A).

Definition 17. A point x of a space (X, Q_1, Q_2) is said to be an ij-α-adherent point of a set A ($\subset$ X) if every ij-α-open set U containing x interesects A. The set of all ij-α-adherent points of A will be called the ij-α-closure of A, to be denoted by ij-α-cl A.

Lemma 18. For any ji-semiopen set A in a space X, ij-α-cl A = Q_i - cl A.

Proof. Clearly, ij-α-cl A $\subset Q_i$-cl A, as Q_i-open sets are ij-α-open. Now, $x \notin$ ij-α-cl A $\Rightarrow$ there is an ij-α-open set V with $x \in V$ such that A $\cap$ V = $\emptyset$ $\Rightarrow$ Q_j-int A $\cap$ Q_i-int V = $\emptyset$ $\Rightarrow$ Q_j-int A $\cap$ Q_j-cl Q_i-int V = $\emptyset$ $\Rightarrow$ Q_j -int A $\cap Q_i$-int Q_j-cl Q_i-int V= $\emptyset$ $\Rightarrow$ Q_i-cl Q_j-int A $\cap$ Q_i-int Q_j-cl Q_i-int V =$\emptyset$. Now, $x \in V \subset$ Q_i-int Q_j-cl Q_i-int V = W (say) $\in Q_i$ such that W $\cap$ A $\subset$ W $\cap$ Q_i-cl Q_j-int A (as A is ji-semiopen) = $\emptyset$. Hence $x \notin$ Q_i-cl A. This proves the lemma.

Theorem 19. A space (X, Q_1, Q_2) is ij-QHC iff every cover μ of X by ij-α-open sets has α finite subcollection μ_0 such that X = $\cup${ji-α-cl U : U $\in \mu_0$}.

Proof : Let μ be a cover of an ij-QHC space X by ij-α-open sets. For each U $\in \mu$, Q_i-int U $\subset$ U $\subset$ Q_i-int Q_j-cl Q_i-int U = V(U) (say).

$\Rightarrow$ Q_j-cl Q_i-int U $\subset$ Q_j-cl U $\subset$ Q_j-cl V(U) $\subset$ Q_j-cl Q_i-int U

$\Rightarrow$ Q_j-cl V(U) = Q_j-cl Q_i-int U = Q_j-cl U. ....(1)

By Remark 16, U is ij-semiopen and hence by Lemma 18, ji-α-cl U = Q_j-cl U. Then ji-α-cl U = Q_j-cl V(U) (by (1)) ....(2)

Since U $\subset$ V (U) $\in Q_i$, for all U$\in\mu$, {V(U) : U$\in\mu$} is a Q_i-open cover of X. The ij-QHC property of X then implies that for a finite subcollection μ_0 of μ, X = $\cup$ {Q_j-cl V(U) : U$\in$ μ_0} = $\cup$ {ji-α-cl U : U$\in$ μ_0}. (by (2)).

Conversely, suppose μ is a Q_i-open cover of a space X for which the given condition holds. Then by Remark 16, μ is also a cover of X by ij-α-open sets. Thus for a finite subcollection μ_0 of μ, $X = \cup \{ji\text{-}\alpha\text{-cl } U : U \in \mu_0\}$. By Remark 16 and Lemma 18, $X = \cup \{Q_j\text{-cl } U : U \in \mu_0\}$ proving that X is ij-QHC.

REFERENCES

1. G. K. Banerjee, On pairwise almost strongly θ-continuous mappings, Bull. Cal. Math. Soc. 79 (1987), 314-320.
2. S. Bose, Semi-open sets, semi-continuity and semi-open mappings in bitopological spaces, Bull. Cal. Math. Soc. 73 (1981), 237-246.
3. C. G. Kariofillis, On pairwise almost compactness, Ann. Soc. Sci. Bruxelles 100 (1986), 129-137.
4. J. C. Kelly, Bitopological spaces, Proc. London Math. Soc. 3 (13) (1963), 71-89.
5. M. N. Mukherjee, On pairwise almost compactness and pairwise H-closedness in a bitopological space, Ann. Soc. Sci. Bruxelles 96 (1982), 98-106.
6. M. N. Mukherjee, On pairwise S-closed bitopological spaces, Internat. J Math. & Math. Sci. 8 (4) (1985), 729-745.
7. M. N. Mukherjee and G. K. Banerjee, A note on QHC and interiorly QHC bitopological spaces, Ann. Stiin. AL. Univ. "AL-I-CUZA" Din IASI 33 (1987), 93-98.
8. M. N. Mukherjee and G. K. Banerjee, Pairwise semiregularization and submaximality of a bitopological space, Ann. Stiin. AL. Univ. "AL-I-CUZA" Din IASI 34 (1988), 19-24.
9. M. N. Mukherjee, G. K. Banerjee and S. Malakar, Bitopological QHC spaces, Indian Jour. Pure Appl. Math. 21 (7) (1990), 639-648.
10. J. N. Nandi, Bitopological near compactness I : Characterizations, Bull. Cal. Math. Soc. 85 (1993), 337-344.
11. S. K. Sen, J. N. Nandi and M. N. Mukherjee, characterizations of some bitopological separation axioms in terms of ij-θ-closure operator, Mat. Vesnik 44 (1992), 75-82.
12. A. R. Singal and S. P. Arya, On pairwise almost regular spaces, Glasnik Mat. 6 (26) (1971), 335-343.

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2. E. Hewitt, Rings of continuous functions I, Amer. Math. Soc. 64(1948), 54-99.

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ALGEBRAIC L-VARIETIES AND INTERSECTION EQUATIONS

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ABSTRACT : We give an up to date account of the theory of nonlinear systems of intersection equations of L-singletons. The concept of an algebraic L-variety is presented in order to bring the current knowledge of fuzzy commutative ring theory to bear on the solution of such intersection equations.

Key words : Algebraic *L*-varietis ; intersection equations ; *L*-ideal ; *R*-primary *L*-representations.

1. INTRODUCTION

Unless otherwise specified, L is a complete chain throughout this paper. We let 1 and 0 denote the maximal and minimal elements of L, respectively.

The purpose of this expository paper is to give an up to date account of the theory of nonlinear systems of intersection equations of L-singletons. We bring fuzzy commutative ring theory to bear on a natural application area, namely, the solution of such intersection equations. Let R denote the polynomial ring $F[x_1,...,x_n]$ where F is a field and $x_1,...x_n$ are algebraically independent indeterminants over F. Let C be a field containing F. C may be taken to be the algebraic closure of F or an algebraically closed field with infinite transcendence degree over F. Let C^k denote the set of all ordered k-tuples with entries from C, k a positive integer. Our approach is to consider those L-ideals μ of R which are finite-valued and are such that $\mu(0) = 1$ since these are precisely the L-ideals of R which have R-primary L-representations [6, 8]. We define the algebraic L-variety $M(\mu)$ of μ and show that from an irredundant R-primary L-representation of μ, $M(\mu)$ is a finite union of irreducible algebraic L-varieties, no one of which is contained in the union of the others. We then apply this result to the solution of a nonlinear system of equations of L-singletons. We show that there exists an L-ideal μ of R which represents this system and the irredundant R-primary L-representation of $\sqrt{\mu}$ displays the solution of the system in a manner similar to that of the crisp situation.

2. ALGEBRAIC *L*-VARIETIES

If I is an ideal of R, we let $M(I)$ denote the algebraic variety of I, [9, p. 203]. If Z is a subset of C^k, we let $\mathscr{I}(Z)$ denote the set of all $f \in R$ which vanish at all points of Z. Then $\mathscr{I}(Z)$ is an ideal of R, [9, p. 203]. We now give definitions for the fuzzy counterparts of M *and* $\mathscr{I}$. Let c be a strictly decreasing function of L into itself such that $c(0) = 1$, $c(1) = 0$, and $\forall\ a \in L$, $c(c(a)) = a$. The following approach has the advantage that c may be changed to fit the application. The proofs of this sections can be found in [10, 11].

Definition 2.1. Let X be a finite-valved L-subset of C^k, say Im(X) = $\{a_0, a_1, \ldots, a_n\}$ where $a_0 < a_1 < \ldots < a_n$. Define the L-subset $\mathscr{I}(X)$ of R as follows :

$$\mathscr{I}(X)(f) = \begin{cases} c(a_n) \text{ if } f \in R \setminus \mathscr{I}(X_{a_n}) ; \\ c(a_i) \text{ if } f \in \mathscr{I}(X_{a_{i+1}}) \setminus \mathscr{I}(X_{a_i}),\ i = 1, \ldots, n-1 ; \\ c(a_0) \text{ if } f \in \mathscr{I}(X_{a_1}) \end{cases}$$

If $n = 0$, then we define $\mathscr{I}(X)(0) = 1$.

Definition 2.2. Let μ be a finite-valved L-ideal of R, say $Im(\mu) = \{b_0, b_1, \ldots, b_m\}$ where $b_0 < b_1 < \ldots < b_m$. Define the L-subset $M(\mu)$ of C^k as follows :

$$M(\mu)(z) = \begin{cases} c(b_m) \text{ if } z \in C^k \setminus M(\mu_{b_m}) ; \\ c(b_i) \text{ if } z \in M(\mu_{b_{i+1}}) \setminus M(\mu_{b_i}),\ \mathrm{i} = 1, \ldots, \mathrm{m}-1 ; \\ c(b_0) \text{ if } z \in M(\mu_{b_1}) \end{cases}$$

$M(\mu)$ is called an algebraic L-variety (of μ).

In Definition 2.1, it is possible for $\mathscr{I}(X_{a_{i+1}}) = \mathscr{I}(X_{a_i})$ or $R = \mathscr{I}(X_{a_n})$. In this case $c(a_i) \notin$ Im $(\mathscr{I}(X))$, $i = 1, \ldots, n$. Similarly, it is possible for $c(b_i) \notin$ Im $(M(\mu))$ for some $i = 1, \ldots, m$.

Proposition 2.3. Let X be defined as in Definition 2.1. Then

(1) $\mathscr{I}(X)_{c(a_i)} = \mathscr{I}(X_{a_{i+1}})$ for $i = 0, 1, \ldots, n-1$;

(2) if $0 \leq \mathrm{b} \leq c(a_n)$, then $\mathscr{I}(X)_b = R$;

(3) if $c(a_{i+1}) < \mathrm{b} < c(a_i)$, then $\mathscr{I}(X)_b = \mathscr{I}(X_{c(b)})$ for $i = 0, 1, \ldots, n-1$;

(4) if $c(a_0) < \mathrm{b} \leq 1$, then $\mathscr{I}(X)_b = \emptyset$.

For X as defined in Definition 2.1, $\mathscr{I}(X)_{c(a_i)} = \mathscr{I}(X_{a_{i+1}})$ is an ideal of R for $i = 0, 1, \ldots, n-1$. Thus since $\mathrm{Im}(X) \subseteq \{c(a_i) \mid i = 0, 1, \ldots, n\}$, $\mathscr{I}(X)$ is a L-ideal of R.

Proposition 2.4. Let μ be defined as in Definition 2.2. Then

(1) $M(\mu)_{c(b_i)} = M(\mu_{b_{i+1}})$ for $i = 0, 1, \ldots, m-1$;

(2) if $0 \leq a \leq c(b_n)$, then $M(\mu)_a = C^k$;

(3) if $c(b_{i+1}) < a < c(b_i)$, then $M(\mu)_a = M(\mu_{c(a)})$ for $i = 0, 1, \ldots, m-1$;

(4) if $c(b_0) < a \leq 1$, then $M(\mu) = \varnothing$.

Proposition 2.5. Let X and μ be as defined on Definition 2.1 and 2.2, respectively. Then

(1) $|\mathrm{Im}(M(\mathscr{I}(X))| = |\mathrm{Im}(\mathscr{I}(X))|$;

(2) $|\mathrm{Im}(\mathscr{I}(M(\mu))| = |\mathrm{Im}(M(\mu))|$;

Proposition 2.6. Let X and μ be defined as in Definition 2.1 and 2.2, respectively. Then

(1) $\forall\, a \in L,\ M(\mathscr{I}(X))_a = M(\mathscr{I}(X_a))$;

(2) $\forall\, b \in L,\ \mathscr{I}(M(\mu))_b = \mathscr{I}(M(\mu_b))$;

Proposition 2.7. Let X and μ be defined as in Definition 2.1 and 2.2, respectively. Then

(1) $\mathscr{I}(M(\mathscr{I}(X)) = \mathscr{I}(X)$;

(2) $M(\mathscr{I}(M(\mu))) = (M(\mu)$.

Proposition 2.8. Let α be a L-subset of C^k. Then α is an algebraic L-variety if and only if α is finite-valued and $\forall\, \alpha \in \mathrm{Im}(\alpha)$, α_a is an algebraic variety.

Proposition 2.9. If μ is a nonconstant prime L-ideal of R, then

$$\mu = \mathscr{I}(M(\mu).$$

Theorem 2.10. Suppose that μ is defined as in Definition 2.2 and $\mu(0) = 1$ Then $M(\mu) = M(\sqrt{\mu}) = M(R(\mu))$.

Corollary 2.11. Let L be a dense chain. If π is a prime L-ideal of R belonging to the primary L-ideal ψ of R, then $M(\psi) = M(\pi)$.

Lemma 2.12. Suppose that μ and γ are L-ideals of R such that $\mathrm{Im}(\mu) = \{b_0, b_1, \ldots, b_m\}$ where $b_0 < b_1 \ldots < b_m = 1$ and $\mathrm{Im}(\gamma) = \{b, 1\}$ where $b < 1$. Then $M(\mu \cap \gamma) = M(\mu) \cup M(\gamma)$.

Theorem 2.13. if μ and γ are finite-valued L-ideals of R such that $\mu(0) = \gamma(0) = 1$, then $M(\mu \cap \gamma) = M(\mu) \cup M(\gamma)$.

Proof.

Lemma 2.14. Let X and η be finite-valued L-subsets of a set S. If $X_b = \eta_b$ $\forall$ $b \in [0,1)$, then $X_1 = \eta_1$.

Lemma 2.15. Suppose that α and β are algebraic L-varieties such that $\mathrm{Im}(\alpha) = \{a_0, a_1, \ldots, a_n\}$ where $0 = a_0 < a_1 < \ldots < a_n$ and $\mathrm{Im}(\beta) = \{0, a\}$ where $0 < a \leq a_n$. Then $\mathscr{I}(\alpha \cup \beta) = \mathscr{I}(\alpha) \cap \mathscr{I}(\beta)$.

Lemma 2.16. If α and β are algebraic L-varieties such that $0 \in \mathrm{Im}(\alpha) \cap \mathrm{Im}(\beta)$, then $\mathscr{I}(\alpha \cup \beta) = \mathscr{I}(\alpha) \cap \mathscr{I}(\beta)$.

3. IRREDUCIBLE ALGEBRAIC *L*-VARIETIES

Definition 3.1. Let α be an algebraic L-variety. Then α is **irreducible** if $\forall$ algebraic L-varieties ξ and ζ such that $\alpha = \xi \cup \zeta$ either $\alpha = \xi$ or $\alpha = \zeta$; otherwise α is called **reducible.**

Theorem 3.2. Let α be an algebraic L-variety. Then α is irreducible and nonconstant if and only if $\mathrm{Im}(\alpha) = \{0,a\}$, $0 < a$, and α_a is irreducible.

Example 3.3. Let $L = \{1,a,b,0\}$ where $1 > a, b > 0$ and a and b are not comparable. Let G be an irreducible variety. Define the L-subset α of C^k by $\alpha(z) = 1$ if $z \in G$ and $\alpha(z) = 0$ otherwise. Define the L-subsets ξ and ζ of C^k by $\xi(z) = a$ if $z \in G$ and $\xi(z) = 0$ if $z \notin G$; $\zeta(z) = b$ if $z \in G$; $\zeta(z) = 0$ if $z \notin G$. Then ξ and ζ are algebraic L-varieties. Let $z \in C^K$. If $z \in G$, then $(\xi \cup \zeta)(z) = \xi(z) \vee \zeta(z) = a \vee b = 1 = \alpha(z)$. If $z \notin G$, then $(\xi \cup \zeta)(z) = \xi(z) \vee \zeta(z) = 0 \vee 0 = 0 = \alpha(z)$. Hence $\alpha = (\xi \cup \zeta)$ and $\alpha \supset \xi$ and $\alpha \supset \zeta$. Thus α is not irreducible. Hence Theorem 3.2 is not true if L is not a chain.

Theorem 3.4. Let μ be a nonconstant finite-valued L-ideal of R. Then $\mathscr{I}(M(\mu))$ is prime if and only if $M(\mu)$ is irreducible.

Theorem 3.5. Let μ be a finite-valued L-ideal of R with $\mu(0) = 1$. Then $\mathscr{I}(M(\mu)) = \sqrt{\mu} = R(\mu)$.

Since this section deals with L-ideals μ which are finite valued and have the porperty that $\mu(0) = 1$, it is evident by now that the results in this section hold interchangeably with $\sqrt{\mu}$ and $R(\mu)$.

Corollary 3.6. Let μ and γ be finite-valued L-ideals of R such that $\mu(0) = \gamma(0) = 1$. Then $M(\mu) \subset M(\gamma)$ if and only if $\sqrt{\mu} \supset \sqrt{\gamma}$.

Theorem 3.7. There exists a one to one correspondence between algebraic L-varieties α with $0 \in \mathrm{Im}\ (\alpha)$ and radical L-ideals.

Theorem 3.8. Every algebraic L-variety α with $0 \in \mathrm{Im}\ (\alpha)$ can be uniquely expressed as the union of a finite number of irreducible algebraic L-varieties no one of which is contained in the union of the others.

Given a finite-valued L-ideal μ of R with $\mu\ (0) = 1$. Let $\sqrt{\mu} = \pi_1 \cap \ldots \cap \pi$ be an irredundant R-primary L-representation. Then the π_i are the minimal prime L-ideals belonging to μ. We thus may obtain $M\ (\mu)$ as the union of algebraic L-varieties of the minimal prime L-ideals among the prime L-ideals belonging to the R-primary L-ideals in an irredundant R-primary L-representation of μ..

Example 3.9. Let $L = [0,1]$. Let $R = F\ [x, y, z]$ where F is the field of complex numbers and x, y, z are algebraically independent indeterminants over F. Define the fuzzy subset μ of R by

$$\mu\ (f) = \begin{cases} 1 & \text{if } f = 0 \\ \frac{1}{2} & \text{if } f \in \langle x^2 z\rangle \setminus (0) \\ \frac{1}{4} & \text{if } \langle x^2 + z^2 - 1,\ x^2 z\rangle \setminus \langle x^2 z\rangle \\ 0 & \text{if } f \in R \setminus \langle x^2 + z^2 - 1,\ x^2 z\rangle. \end{cases}$$

Then μ *is a* fuzzy ideal of R. Now $\sqrt{\mu}$ is such that

$$\sqrt{\mu}\ (f) = \begin{cases} 1 & \text{if } f = 0 \\ \frac{1}{2} & \text{if } f \in \langle xz\rangle \setminus (0) \\ \frac{1}{4} & \text{if } \langle x^2 + y^2 - 1,\ xz\rangle \setminus \langle xz\rangle \\ 0 & \text{if } f \in R \setminus \langle x^2 + y^2 - 1,\ xz\rangle. \end{cases}$$

Hence

$$\begin{array}{lll@{\qquad\qquad}lll} \mu_0 & = & R & \sqrt{\mu_0} & = & R \\ \mu_{\frac{1}{4}} & = & \langle x^2 + z^2 - 1,\ x^2 z\rangle & \sqrt{\mu_{\frac{1}{4}}} & = & \langle x^2 + z^2 - 1,\ xz\rangle \\ \mu_{\frac{1}{2}} & = & \langle x^2 z\rangle & \sqrt{\mu_{\frac{1}{2}}} & = & \langle xz\rangle \\ \mu_1 & = & (0) & \sqrt{\mu_1} & = & (0). \end{array}$$

Since $F^k = M((0))$,

$$M(\mu)(\omega) = \begin{cases} c(\frac{1}{2}) & \text{if } \omega \in M((0)) \setminus M(\langle xz\rangle) \\ c(\frac{1}{4}) & \text{if } \omega \in M(\langle xz\rangle) \setminus M(\langle x^2 + z^2 - 1, xz\rangle) \\ c(0) = 1 & \text{if } \omega \in M(\langle x^2 + z^2 - 1, xz\rangle) \end{cases}$$

Consider the fuzzy subsets ω, χ, η of R defined by

$$\omega(f) = \begin{cases} 1 & \text{if } f \in \langle x^2 + z^2 - 1, x^2 z\rangle \\ 0 & \text{otherwise} \end{cases}$$

$$\chi(f) = \begin{cases} 1 & \text{if } f \in \langle x^2 z\rangle \\ \frac{1}{4} & \text{otherwise} \end{cases}$$

$$\eta(f) = \begin{cases} 1 & \text{if } f \in (0) \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

Then ω, χ, η are fuzzy ideals of R and $\mu = \omega \cap \chi \cap \eta$. Define the fuzzy subsets $\psi^{(i)}$ of R, $i = 1, \ldots, 6$ by

$$\psi^{(1)}(f) = \begin{cases} 1 & \text{if } f \in \langle x^2, y - 1\rangle \\ 0 & \text{otherwise} \end{cases}$$

$$\psi^{(2)}(f) = \begin{cases} 1 & \text{if } f \in \langle x^2, y + 1\rangle \\ 0 & \text{otherwise} \end{cases}$$

$$\psi^{(3)}(f) = \begin{cases} 1 & \text{if } f \in \langle x^2 + y^2 - 1, z\rangle \\ 0 & \text{otherwise} \end{cases}$$

$$\psi^{(4)}(f) = \begin{cases} 1 & \text{if } f \in \langle x^2\rangle \\ \frac{1}{4} & \text{otherwise} \end{cases}$$

$$\psi^{(5)}(f) = \begin{cases} 1 & \text{if } f \in \langle z\rangle \\ \frac{1}{4}, & \text{otherwise} \end{cases}$$

$$\psi^{(6)}(f) = \begin{cases} 1 & \text{if } f \in (0) \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

Then $\psi^{(i)}$ is a fuzzy ideal of R, $i = 1,\ldots, 6$, such that $\omega = \psi^{(1)} \cap \psi^{(2)} \cap \psi^{(3)}$ since $\langle(x^2 + z^2 - 1,\ x^2z\rangle = \langle x^2, y^2 - 1\rangle \cap \langle x^2, y^2 + 1\rangle \cap \langle x^2 + z^2 - 1,\ z\rangle$, $\chi = \psi^{(4)} \cap \psi^{(5)}$ since

$$\langle x^2z\rangle = \langle x^2\rangle \cap \langle z\rangle,$$

$\eta = \psi^{(6)}$. Thus $\mu = \cap_{i=1}^{6} \psi^{(i)}$ and in fact this is an irredundant fuzzy primary represntation of μ. Now

$$\sqrt{\psi^{(1)}}(f) = \begin{cases} 1 \text{ if } f \in \langle x,\ y - 1\rangle \\ 0 \text{ otherwise} \end{cases}$$

$$\sqrt{\psi^{(2)}}(f) = \begin{cases} 1 \text{ if } f \in \langle x^2,\ y + 1\rangle \\ 0 \text{ otherwise} \end{cases}$$

$$\sqrt{\psi^{(3)}} = \psi^{(3)}$$

$$\sqrt{\psi^{(3)}}(f) = \begin{cases} 1 \text{ if } f \in (x) \\ \frac{1}{4} \text{ otherwise} \end{cases}$$

$$\sqrt{\psi^{(5)}} = \psi^{(5)}$$

$$\sqrt{\psi^{(6)}} = \psi^{(6)}$$

Hence $\sqrt{\mu} = \cap_{i=1}^{6} \pi^{(i)}$ where $\pi^{(i)} = \sqrt{\psi^{(i)}}$ is fuzzy prime ideal of R, $i = 1,\ldots,6$. We have the following fuzzy algebraic varieties :

$$M(\pi^{(1)})(\omega) = \begin{cases} 1 \text{ if } \omega \in M(\langle x,\ y - 1\rangle) \\ 0 \text{ otherwise} \end{cases}$$

$$M(\pi^{(2)})(\omega) = \begin{cases} 1 \text{ if } \omega \in M(\langle x,\ y + 1\rangle) \\ 0 \text{ otherwise} \end{cases}$$

$$M(\pi^{(3)})(\omega) = \begin{cases} 1 \text{ if } \omega \in M(\langle x^2 + y^2 - 1,\ z\rangle) \\ 0 \text{ otherwise} \end{cases}$$

$$M(\pi^{(4)})(\omega) = \begin{cases} c(\frac{1}{4}) \text{ if } \omega \in M(\langle x\rangle) \\ 0 \text{ otherwise} \end{cases}$$

$$M\ (\pi^{(5)})(\omega) = \begin{cases} c(\frac{1}{4}) \text{ if } \omega \in M\ (\langle z \rangle) \\ 0 \text{ otherwise} \end{cases}$$

$$M\ (\pi^{(6)})(\omega) = c(\frac{1}{2}) \text{ if } \forall\omega \in C^k$$

Then $M\ (\mu) = \cup_{i=1}^{6} M\ (\pi^{(i)})$ and in fact $M\ (\pi^{(i)})$ is irreducible and no $M\ (\pi^{(i)})$ is contained in the union of the others, $i = 1,\ldots, 6$. Consider the nonlinear system of equations of fuzzy singletons,

$$(x_b)^2 + (y_a)^2 - 1_{\frac{1}{4}} = 0_{\frac{1}{4}}$$

$$(x_b)^2\, z_u = 0_{\frac{1}{2}}.$$

Then a solution is given by $a \geq \frac{1}{4}$ and $b \wedge u = \frac{1}{2}$ and the solution of $(x^2 + y^2 - 1)$ and $x^2z = 0$. Note also that $\mu = \langle (x^2 + y^2 - 1)_{\frac{1}{4}}, (x^2z)_{\frac{1}{2}} \rangle$. If we let

$$c(0) = 1,\ c(\frac{1}{4}) = \frac{1}{2},\ c(\frac{1}{2}) = \frac{1}{4},\ c(1) = 0,$$

Then the above representation of $M\ (\mu)$ seems to better represent the solution of the above nonlinear system of equations of fuzzy singletons. The $M\ (\pi^{(i)})$ for i = 1,2,3, yield the crisp part of the solution while the $M\ (\pi^{(i)})$ for i = 4,5,6 yield the fuzzy part.

In Example 3.9 it was shown how a solution to a system of fuzzy intersection equations could be displayed by a primary representation of the L-ideal generated by the defining polynomials of the intersection equations. We now show this holds in general. The proofs of the results are in [13].

Theorem 3.10. Let $\mu = \langle (f_1)_{a_1},\ldots, ((f_q)_{a_q} \rangle \cup 1_{\{0\}}$ where $f_1,\ldots, f_q \in R$, $1 \geq a_1 \geq \ldots \geq a_1 > 0$ and $a \neq a_q$. Suppose that $\langle f_1,\ldots, f_q \rangle \neq R$. Let $\{a_{i_1} \ldots, a_{i_m}\} = \{a_1 \ldots, a_q\}$ be such that $a_{i_1} > \ldots > a_{i_m}$. Let

$$F_{a_{m-u-1}} = \{f_k \backslash a_k > a_{i_{m-u}}\},$$

u = 0,1,..., m − 1, and let $F_{a_{i_m}} = \{f_1,\dots, f_q\}$. Define the L-subsets ω, $\omega_1,\dots, \omega_m$ of R as follows :

$$\omega(r) = \begin{cases} 1 \text{ if } r \in \langle F_{a_{i_m}} \rangle \\ 0 \text{ if } r \notin \langle F_{a_{i_m}} \rangle \end{cases}$$

$$\omega_u(r) = \begin{cases} 1 \text{ if } r \in \langle F_{a_{i_{m-u}}} \rangle \\ a_{i_{m-u+1}} \text{ if } r \notin \langle F_{a_{i_{m-u}}} \rangle \end{cases}$$

u = 1,..., m − 1.

$$\omega_u(r) = \begin{cases} 1 \text{ if } r \in \langle \emptyset \rangle \\ a_{i_1} \text{ if } r \in \langle \emptyset \rangle \end{cases}$$

Then ω, $\omega_1,\dots,\omega_m$ are L-ideals of R and $\mu = \omega \cap \omega_1 \cap \dots \omega_m$.

Theorem 3.11. let $\mu = \langle (f_1)a_1,\dots, (f_q)_{a_q}\rangle \cup 1_{\{0\}}$ where $f_1 \dots, f_q \in R$, $1 \geq a_1 \geq \dots \geq a_q > 0$ and $a_{-} \neq a_1$. Suppose that $\langle f_1 \dots, f_q \rangle \neq R$. Let ω, $\omega_1,\dots,\omega_m$ be defined as in Theorem 3.10.

Let $1_\omega = Q_{01} \cap \dots \cap Q_{0k_0}$

and $1_{\omega_u} = Q_{u1} \cap \dots \cap Q_{uk_u}$

be R-primary representations of 1_ω and 1_{ω_u}, respectively, u = 1, ..., m. For ech u = 0, 1, ..., m, define the L-subsets $\mu_{u1}, \dots, \mu_{uk_u}$ of R as follows : ∀ r ∈ R,

$$\mu_{0_j}(r) = \begin{cases} 1 \text{ if } r \in Q_{0_j} \\ 0 \text{ if } r \notin Q_{0_j} \end{cases}$$

j = 1, ..., k_σ

$$\mu_{uj}(r) = \begin{cases} 1 \text{ if } r \in Q_{uj} \\ 0 \text{ if } r \notin Q_{uj} \end{cases}$$

$j = 1, \ldots, k_u$; $u = 1, \ldots, m$. Then the following asertions hold :

(1) $\mu_{u1}, \ldots, \mu_{uk_u}$ are L-ideals of R, $u = 0, 1, \ldots, m$.

(2) $\omega_\mu = \mu_{ul} \cap \ldots \cap \mu_{uk_u}$, $u = 0, 1, \ldots, m$.

(3) $\mu = (\mu_{01} \cap \ldots \cap \mu_{0k_0}) \cap (\mu_{11} \cap \ldots \cap \mu_{1k_1}) \cap \ldots \cap (\mu_{m1} \cap \ldots \cap \mu_{mk_m})$ is an R-primary L-representation of μ.

Let R denote the polynomial ring $F[x_1, \ldots, x_2]$ in n indeterminates over the field F. Then every ideal of R has a primary representation. Let

$$\sum_{i_1=1}^{k_{1j}} \ldots \sum_{i_1=1}^{k_{nj}} (r_{i_1} \ldots {}_{i_n j})_1 ((x_1) b_{1_j})^{i_1} \ldots ((x_n) b_{nj})^{i_1} = (r_j)_{a_j},\ j = 1, \ldots, q, \tag{1}$$

donote q nonlinear equations in the L-singletons $(x_1)_{b_1}, \ldots, (x_n)_{b_n}$ where $b_{ij} = b_i$ if x_i appears in equation j and 1 otherwise, $i = 1 \ldots, n$; $j = 1, \ldots, q$, and where the $(r_{i_1} \, . \, {}_{i_{nj}})_1$ and the $(r_j)_{aj}$ are L-singletons and the r_j and the $r_{i_1} \quad {}_{i_n}$ are in F. Let

$$f_j = \sum_{i_1=1}^{k_{1j}} \ldots \sum_{i_1=1}^{k_{nj}} r_{i_1} \ldots {}_{i_n j} (x_1)^{i_1} \ldots (x_n)^{i_1},\ j = 1, \ldots, q.$$

Then the system of equations (1) is equivalent to the following two systems of equations :

$$fj = r_j,\ j = 1, \ldots, q \tag{2}$$

and

$$b_{1_j} \wedge \ldots \wedge b_{nj} = a_j,\ j = 1, \ldots, q. \tag{3}$$

let $\mu = \langle (f_1)_{a_1}, \ldots, ((f_q)_{a_q} \rangle \cup 1_{\{0\}}$. It is clear that in (3) of Theorem 3.1,

$$\sqrt{\mu_{01}} \cap \ldots \cap \sqrt{\mu_{0k_0}}$$

gives, via unions of the corresponding irreducible algebraic L-varieties, the crisp part, (2), of the solution to the L-intersection equations, (1), while

$$\left(\sqrt{\mu_{11}} \cap \ldots \cap \sqrt{\mu_{1k_1}}\right) \cap \ldots \cap \left(\sqrt{\mu_{m1}} \cap \ldots \cap \sqrt{\mu_{mk_m}}\right)$$

gives the fuzzy part, (3).

4. LOCALIZED *L*-SUBRINGS

The notion of algebric *L*-varieties was introduced in order to use primary representation theory of *L*-ideals to examine the solution of fuzzy intersection equations. The concepts of quasi-local *L*-subrings and complete local *L*-subrings were developed in [7] and [4, 12], respectively, in order to lay the ground work for the examination of fuzzy intersection equations locally. In this section, we characterize local rings in terms of certain *L*-ideals. We also characterize rings of fractions at a prime ideal in terms of *L*-ideals. We apply our results to fuzzy intersection equations. In particular, we show that the *L*-ideal which represents a system of fuzzy intersection equations in a polynomial ring is such that its extension in a ring of fractions represents the same system of fuzzy intersection equations. The results in this section are taken from [1].

Throughout this section *R* denotes a commutative ring with identity and *L*- is a dense chain. Let $\mu_* = \{x \in R \setminus \mu(x) = \mu(0)\}$ and $\mu_\# = \{x \in R \setminus \mu(x) > \mu(1)\}$. Then μ_* and $\mu_\#$ are ideals of *R*. Let S be a set of *L*-singletons of *R* such that if $x_a, x_b \in$ S, then $a = b > 0$. Let foot (S) = $\{x \mid x_a \in \text{S}\}$. If μ is an *L*- ideal of *R* such that $\mu = \langle \text{S} \rangle \cup 0_{\mu(0)}$ for some S, then S is called a generating set for μ. If S is a generating set for μ, and $\langle \text{S} \setminus \{x_a\} \rangle \cup 0_{\mu(0)} \subset \mu \ \forall \ x_a \in$ S, then S is called a minimal generating set for μ. If S is a subset *R*, we let ⟨S⟩ denote the ideal of *R* generated by S.

A commutative ring with identity, but not necessarily Noetherian, is said to be local if it has a unique maximal ideal. (Such a ring is called quasi-local in [7]). In [7] the definition of a quasi-local *L*-subring of *R* was given when *R* was assumed to be local. That is, an *L*-subring μ of a local ring *R* was called quasi-local if $\mu(x) = \mu(x^{-1})$ for all units x of *R*. If μ is an *L*-ideal of *R*, then $\mu(x) = \mu(1)$ for all units x of *R*. Hence if *R* is a local ring and μ is an *L*-ideal of *R*, then μ is a quasi-local *L*-subring of *R*. We also know that if μ is an *L*-ideal of *R*, then $\mu(y) \geq \mu(1) \ \forall y \in R$, if μ is a nonconstant *L*-ideal of *R*, then $\mu(0) > \mu(1)$.

Definition 4.1. An *L*-ideal μ of *R* is called **local** if $\forall x \in R$, $\mu(x) = \mu(1)$ is equivalent to x being a unit in *R*.

Note that if μ is an *L*-ideal of *R* which is local, then μ is not constant since 0 is not a unit of *R*. Let *R* denote the polynomial ring $F[x]$ over the field *F*. Define the *L*-subring μ of *R* by $\mu(z) = 1$ if $z = 0$, $\mu(z) = \frac{1}{2}$ if $z \in F \setminus \{0\}$, and $\mu(z) = \frac{1}{4}$ if $z \in R \setminus F$. Then μ is an *L*-subring of *R*. Also, $\mu(z) = \mu(1)$ if and only if z is a unit. However μ is not an *L*-ideal of *R*. We also note that *R* is not a local ring.

Lemma 4.2. Let μ be a nonconstant *L*-ideal of *R*. Then μ is local if and only if $\mu_\#$ is the unique maximal ideal of *R*.

Example 4.3. Let $L = [0, 1]$. Let μ be the fuzzy subest of the ring Z_9 defined as follows, where $0 \le a_0 \le a_1 < a_2 \le 1$:

$$\mu(x) = \begin{cases} a_0 \text{ if } x \in Z_9 \setminus \langle 3 \rangle \\ a_1 \text{ if } x \in \langle 3 \rangle \setminus \{0\} \\ a_2 \text{ if } x = 0. \end{cases}$$

The μ is an L-ideal of R. Suppose that $a_0 < a_1$. Then $\mu_\# = \langle 3 \rangle$ is the unique maximal ideal of Z_9 and so μ is local. Suppose that $a_0 = a_1$. Then μ is not local even thought μ us a quasi-local L-subring of R.

The ring Z_6 has no L-ideals which are local since Z_6 is not local.

Recall that an L-ideal μ of R is a generalized maximal L-ideal if μ is not constant and for any L-ideal γ of R, if $\mu \subseteq \gamma$, then either $\mu_* = \gamma_* =$ or $\gamma = 1_R$. Then an L-ideal μ of R is maximal if and only if $|\text{ Im }(\mu)| = 2$, $\mu(0) = 1$, and μ_* is a maximal ideal of R. We see that the L-ideal μ of Example 4.3 with $a_0 < a_1$ and $a_2 = 1$ is not maximal even though μ is local.

Let μ and λ be L-ideals of R. Then μ and λ are said to be equivalent if

$$\{\mu_a \mid a \in \text{Im }(\mu)\} = \{\lambda_a \mid a \in \text{Im }(\lambda)\}.$$

Theorem 4.4. The

(1) R is local ;

(2) R has an L-ideal which is local ;

(3) all generalized maximal L-ideals of R are local ;

(4) all generalized maximal L-ideals of R are equivalent.

If R is Artinian, we say that an L-ideal μ of R is of maximal chain if the level ideals of μ form a composition series.

Theorem 4.5. Let R be Artinian. Then R is local if and only if every L-ideal of R of maximal chain is local.

An L-ideal μ of R is called normalized if $\mu(0) = 1$.

Theorem 4.6. R is a field if and only if the set of all normalized L-ideals of R which are local coincides with the set of all generalized maximal L-ideals of R.

Throughout the remainder of the section, S denotes a closed multiplicative system in R such that $0 \notin S$ and which is saturated, i.e., $\forall x,y \in R$, $xy \in S$ implies $x,y \in S$, [3]. Let RS^{-1} denote the corresponding ring of fractions. Then $RS^{-1} = \{\phi(r)/\phi(s) \mid r \in R, s \in R\}$, where ϕ is a homomorphism of R into RS^{-1} such that Ker $\phi = x \in R \mid xs = 0$ for some s in $S\}$ and the elements of $\phi(S)$ are units in RS^{-1}, [16, p. 222]. If I is an ideal of R, we use the notation IS^{-1} for the ideal of RS^{-1} generated by $\phi(I)$.

Definition 4.7. Assume μ and μ' are L-ideals of R and RS^{-1}, respectively. Then μ' is called the localized L-subring of μ in RS^{-1} if Im (μ) = Im (μ') and $\mu'_a = \mu_a S^{-1}$ $\forall a$ Im (μ).

In the following example, we show that not every L-ideal of R has localized L-subring in RS^{-1}.

Example 4.8. Consider ring of integers Z. Let $L = [0,1]$. Let $\mu = \langle 6_1, 2_{\frac{1}{2}} \rangle$. Then Im $(\mu) = \{0, \frac{1}{2}, 1\}$, $\mu_1 = \langle 6 \rangle$, and $\mu_{\frac{1}{2}} = \langle 2 \rangle$. Let S denote the complement of $\langle 2 \rangle$ in Z. Now $2 = 6.3^{-1} \in \langle 6 \rangle S^{-1}$ and so $\mu_1 S^{-1} = \langle 6 \rangle S^{-1} = \langle 2 \rangle S^{-1} = \mu_{\frac{1}{2}} S^{-1}$. Thus μ does not have a localized fuzzy subring μ' in ZS^{-1} since if μ' is a fuzzy ideal of ZS^{-1} such that $\mu'_a = \mu'_a S^{-1}$ $\forall a \in$ Im (μ), then Im $(\mu') = \{1,0\} \neq$ Im (μ).

Example 4.9. Let $L = [0,1]$ Let $Z_{(p)}$ be the localized ring of the ring of integers Z for a prime number p. Let μ and μ' be fuzzy subsets of R = Z and $RS^{-1} = Z_{(p)}$, respectively, defined as folloqs :

$$\mu\ (x) = a_i \text{ if } x \in p^i Z \setminus p^{i+1} Z,$$

$$\mu'\ (x) = a_i \text{ if } x \in p^i Z_{(p)} \setminus p^{i+1} Z_{(p)},$$

where $i = 0, 1, \ldots$ and $0 < a_0 < a_1 < \ldots < 1$. Then clearly μ, μ' are fuzzy ideals of Z, $Z_{(p)}$, respectively, and μ' is a localized fuzzy subring of μ in $Z_{(p)}$.

We say that the ring of fractions RS^{-1} is a localized ring of R at a prime ideal, if there exists a prime ideal P of R such that $S = cP$, the complement of P in R.

Theorem 4.10. The ring of fractions RS^{-1} is a localized ring of R at a prime ideal of R if and only if there exists an L-ideal μ of R which has a localized L-subring μ' in RS^{-1} and $S \supseteq R \setminus \mu_{\#}$. In such a case, $\mu_{\#}$ is a prime idel of R and RS^{-1} is a localized ring of R at $\mu_{\#}$.

Let S be a set of L-singletons. Define the L-subset σ_s of R by $\forall x \in P$,

$$\sigma_s\ (x) = \vee\{a \mid x_a\, S\}.$$

If $r \in R$ and x_a is an L-singleton, we let rx_a denote the L-singleton $(rx)_a$.

Theorem 4.11. Let S be a set of L-singletons of R. Let σ be the L-subset of R defined by $\forall x \in R$,

$$\sigma\ (x) = \vee\{(\sum_{i=1}^{k} r_i (x_i)_{a_i})(x) \mid r_i \in R,\ x_{a_i} \in S,\ i = 1, \ldots, k\ ;\ k \in \mathbb{N}\}.$$

Then $\sigma = \langle S \rangle$, where $\langle S \rangle = \langle \sigma_s \rangle$.

Lemma 4.12. Suppose that μ is an L-ideal of R such that μ has the sup property. Let $S = \cup_{a \in [0,1]} S_a$, where $S_a \subseteq \{x_a \mid x \in R, \mu(x) = a\}$ if $a \in \text{Im}(\mu)$ and $S_a = \emptyset$ if $a \in L \setminus \text{Im}(\mu)$. Then $\mu = \langle S \rangle \cup 0_{\mu(0)}$ if and only if μ_a = (foot $(\text{U}_{b \geq a} S_b)$) $\forall_a \in \text{Im}(\mu)$.

Example 4.13. Let $L = [0,1]$. Let $R = Z$. Define the fuzzy subset μ of R by

$$\mu(r) = \begin{cases} 1 \text{ if } r \in \langle 4 \rangle \\ \frac{1}{2} \text{ if } r \in \langle 2 \rangle \setminus \langle 4 \rangle \\ 0 \text{ if } r \in R \setminus \langle 2 \rangle. \end{cases}$$

Then μ is a fuzzy ideal of R, $S = \{4_1, 2_{\frac{1}{2}}\}$ is a minimal geneating set for μ, μ_1 $\langle 4 \rangle$, and $\mu_{\frac{1}{2}} = \langle 4,2 \rangle = \langle 2 \rangle$. Thus $\{4,2\}$ is not a minimal generating set for $\mu_{\frac{1}{2}}$. However, $\mu_{\frac{1}{2}}$ has a minimal generating set, namely $\{2\}$.

Proposition 4.14. Suppose μ is an L-ideal of R such that μ has the sup property. Let

$$S = \text{U}_{a \in [0,1]} S_a,$$

where $S_a \subseteq \{x_a \mid x \in R, \mu(x) = a\}$ if $a \in \text{Im}(\mu)$ and $S_a = \emptyset$ if $a \in L \setminus \text{Im}(\mu)$. If foot $(\text{U}_{b \geq a} S_b)$ is a minimal generating set for μ_a $\forall a \in \text{Im}(\mu)$; then S is a minimal generating set for μ.

Example 4.15. Let $L = [0,1]$. Consider the polynomial ring $Z[x]$ over the the ring of integers Z. Let $\mu = \langle 6_1, 2_{\frac{1}{2}}, x_{\frac{1}{2}} \rangle$. Let S denote the complement of $\langle 2 \rangle$ in $Z[x]$. Then $\text{Im}(\mu) = \{0, \frac{1}{2}, 1\}$, $\mu_1 = \langle 6 \rangle$, and $\mu_{\frac{1}{2}} = \langle 2, x \rangle$. Now $2 = 6.3^{-1} \in \langle 6 \rangle S^{-1}$ and so

$$\mu_1 S^{-1} = \langle 6 \rangle S^{-1} = \langle 2 \rangle S^{-1}$$

and

$$\mu_{\frac{1}{2}} S^{-1} = \langle 6, x \rangle S^{-1} = \langle 2, x \rangle S^{-1}$$

Thus μ has a fuzzy localized subring μ' in $Z[x] S^{-1}$. The level ideals of μ' are $\mu'_1 = \mu_1 S^{-1}$ and $\mu'_{\frac{1}{2}} = \mu_{\frac{1}{2}} S^{-1}$. Now $\{6_1, 2_{\frac{1}{2}}, x_{\frac{1}{2}}\}$ is a minimal generating set for μ, but $\{6_1, 2_{\frac{1}{2}}, x_{\frac{1}{2}}\}$ is not minimal generating sets for μ'. We note that $\langle 2 \rangle$ is the contracted ideal of μ'_1 and not $\mu_1 = \langle 6 \rangle$. In the proof of Theoren 4.10, we have $I_1 = \langle 2 \rangle$ and $I_{\frac{1}{2}} = \langle 2,x \rangle$.

Definition 4.16. Let S denote a set of L-singletons such that if x_a and $x_b \in S$, then $a = b > 0$. Let μ be an L-ideal of R is called an S-minimal generating set for μ if $\mu = \langle S \rangle \text{U} 0_{\mu(0)}$ and $\forall x \in$ foot (S), there does not exist $s \in S$ such that $sx \in$ (foot $(S) \setminus \{x\}$).

Proposition 4.17. Let S denote a set of L-singletons such that if x_a and $x_b \in S$, then $a = b > 0$. Let μ be an L-ideal of R such that μ has the sup property. If S is an S-minimal generating set for μ, then S is a minimal gnerating set for μ.

If x_a is an L-singleton of R, then $\phi\ (x)_a = \phi\ (x)_a$. Let μ and μ' be L-ideals of R and RS^{-1}, respectively, such that μ' is a localized L-subring of μ in RS^{-1}. If S is a set of L-singletons which generate μ, then $\{\phi\ (x)_a \mid x_a \in S\}$ generates μ' and we say that and μ and μ' have the same set of generators and we write $\phi\ (S)$ for $\{\phi\ (x)_a \mid x_a \in S\}$.

Theorem 4.18. Let λ, λ' be L-ideals of R, RS^{-1}, respectively, such that λ has the sup property. If λ has an S-minimal generating set and λ' is a locallized L-subring of λ in RS^{-1}, then λ and λ' have the same minimal generating sets and Im (λ) = Im (λ'). Conversely, if RS^{-1} is a localized ring at a prime ideal of R, λ and λ' have the same minimal generating sets and Im (λ) = Im (λ'), Then λ' is a localized L-subring of λ in RS^{-1}.

We now apply our results in the following example..

Example 4.19. Let $L = [0,1]$. Let R *denote the polynomial ring* R $[x,y,z]$ in the algebraically independent indeterminates x,y,z over the field of R of real numbers. Then the ideal $\langle x^2 - y, x^2 z\rangle$ represents the nonlinear system of equations

$$\begin{aligned} x^2 - y &= 0 \\ x^2\ y &= 0 \end{aligned}$$

and has the reduced primary representation

$$\langle x^2 - y,\ x^2 z\rangle = \langle x^2 - y, z\rangle \cap \langle x^2, y\rangle.$$

Hence

$$\sqrt{\langle x^2 - y, x^2 z\rangle} = \langle x^2 - y, z\rangle \cap \langle x, y\rangle.$$

and the prime ideals $\langle x^2 - y, z\rangle$ and $\langle x, y\rangle$ display the solution of the nonlinear system of equations via their corresponding irreducible affine avarieties.

Now consider the following nonlinear system of fuzzy intersection equation

$$\begin{aligned} (x_b)^2 - y_a &= 0_{\frac{1}{4}} \\ (x_b)^2 z_u &= 0_{\frac{1}{2}} \end{aligned}$$

Then this system is represented by the fuzzy ideal $\mu = \langle (x^2 - y)_{\frac{1}{4}},\ (x^2 y)_{\frac{1}{2}}\rangle$ and $S = \{(x^2 - y)_{\frac{1}{4}},\ (x^2 y)_{\frac{1}{2}}\}$ is a minimal generating set for μ. In order to examine the system locally we consider either of the prime ideals $\langle x^2 - y, z\rangle$ and $\langle xy\rangle$, say, $P = \langle x, y\rangle$, and we form the quotient ring R_p. Then in R_p, the extended ideal [16] of $\langle x^2 - y, z\rangle$ is

$$\langle x^2 - y,\ x^2 z\rangle^c = \langle x^2, y\rangle^c$$

Hence the the corresponding nonlinear system of fuzzy intersection equations is

$$y_a = 0_{\frac{1}{4}}$$

$$(x_b)^2 = 0_{\frac{1}{2}}$$

This system is represented by the fuzzy ideal $\gamma = \langle y_{\frac{1}{4}}, (x^2)_{\frac{1}{2}} \rangle$ in R. Now $S = \{y_{\frac{1}{4}}, (x^2)_{\frac{1}{2}}\}$ is a minimal generating set for γ. By Therorm 4.18, we have that S is a minimal generating set for the fuzzy localized subring γ' of γ in R_P. Hence γ' represents the same system of fuzzy intersection equations as γ does.

If we consider the prime ideal $N = \langle x^2 - y, z \rangle$, then R_N

$$\langle x^2 - y,\ x^2 z \rangle^c = \langle x^2 - y,\ z \rangle^c.$$

Hence the corresponding nonlinear system of fuzzy intersection equations is

$$(x_b)^2 - y_a = 0_{\frac{1}{4}}$$

$$z_u = 0_{\frac{1}{2}}$$

This system is represented by the fuzzy ideal $\lambda = \langle (x^2 - y)_{\frac{1}{4}}, z_{\frac{1}{2}} \rangle$ in R. We have that $\{(x^2 - y)_{\frac{1}{4}}, z_{\frac{1}{2}}\}$ is a minimal generating set for λ and also for the fuzzy localized subring λ' of λ in R_N.

5. LOCAL EXAMINATION OF FUZZY INTERSECTION EQUATIONS

In this section L denotes a dense chain and R a commutative ring with identity. The notion of algebraic L-varieties was introduced in order to use primary representation theory of L-ideals to examine the solution of fuzzy intersection equations. Local concepts of subrings were developed in order to lay the ground work for the examination of fuzzy intersection equations locally. In this section, we carry out a local examination of fuzzy interection equations. We show that a system of fuzzy intersection equations can be examined locally to obtain the general solution to the crisp part of the system. The details can be found in [2].

Let M be a multiplicative system in R [16, p. 46]. Let $N = \{x \in R \mid mx = 0$ for some $x \in R\}$. Then N is an ideal of R. If $N = \{0\}$, then M is said to be regular. Let h be the natural homomorphism of R onto $R/N \subseteq R_M$, the quotient ring of R with respect to M. If I is an ideal of R, then the ideal in R_M generated by $h(I)$ is called the extended ideal of I in R_M and is denoted by $h(I)^e$. If J is an ideal of R_M, then $h^{-1}(J)$ is called the contracted ideal of J in R.

Let μ be an L-ideal of R. Define the L-subset $h(\mu)^e$ of R_M by $\forall y \in R_M$, $h(\mu)^e(y) = \vee\{a \in L \mid y \in h(\mu)_{aM}\}$. Then $h(\mu)^e$ is an L-ideal of R_M. Let $a \in L$. Now $y \in h(\mu)_a \Leftrightarrow h(\mu)(y) \geq a \Leftrightarrow \vee\{\mu(x) \mid h(x) = y\} \geq a \Leftarrow \exists\, x \in \mu_a$ such that $h(x) \Leftrightarrow y \in h(\mu_a)$ where the "$\Leftarrow$" becomes "$\Leftrightarrow$" if μ has the sup property. Hence if μ has the sup property, then $h(\mu)_a = h(\mu_a)$ and so $(h(\mu))_{aM} = h(\mu_a)_M$. We use the notation m^e for $h(\mu)^e$ at times. If I is an ideal of R, *we sometimes use the notation* I^e for $h(I)^e$. If β is an L-ideal of R_M, then we use the notation β^c for $h^{-1}(\beta)$ at times. If J is an ideal of R_M, we sometimes use the notation J^c for $h^{-1}(J)$.

Suppose that μ has the sup property. Then $(\mu^e)(y) = a \Leftrightarrow h(\mu)^e(y) = a \Leftrightarrow \vee\{b/y \in h(\mu)_{bM} = h(\mu_b)_M\} = a \Leftrightarrow a$ is maximal in L such that $y \in h(\mu)_{aM} = h(\mu_a)_M = h(\mu_a)^e$ (since μ has the sup property) $= \mu_a^e$. Hence $(\mu^e)_a = \mu_a^e\ \forall a \in L$.

Theorem 5.1. Let β be a primary L-ideal of R_M Then $\sqrt{\beta^c} = \left(\sqrt{\beta}\right)^c$.

Theorem 5.2. Let μ be a primary L-ideal of R such that μ_* is disjoint from M.

(1) Then $\mu = \mu^{ec}$ and $\sqrt{\mu} = \left(\sqrt{\mu}\right)^{ec}$

(2) Then μ^e is is primary and $\sqrt{\mu^e} = \left(\sqrt{\mu}\right)^e$.

Lemma 5.3. Let μ and γ be L-ideals of R_M. Then $(\mu \cap \gamma)^c = \mu^c \cap \gamma^c$.

Theorem 5.4. Let μ be an ideal of R such that μ has a reduced primary representation $\mu = \cap_{i=1}^{n} \mu_i$. Suppose that for $1 \leq i \leq k$, $(\mu_i)_* \cap M = \varnothing$ and that for $k + 1 \leq i \leq n$, $(\mu_i)_* \cap M \neq \varnothing$. Then $\mu^e = \cap_{i=1}^{n} \mu_i^e$ is a reduced primary representation. Furthermore, $\mu^{ec} = \cap_{i=1}^{n} \mu_i$.

Example 5.5. Let R denote the polynomial ring $\mathbb{R}[x,y,z]$ in algebraically independent indeterminates x,y,z over the field $\mathbb{R}$ of real numbers. Then the ideal $\langle x^2 - y,\ x^2 z\rangle$ has the reduced primary representation

$$\langle x^2 - y,\ x^2 z\rangle = \langle x^2 - y,\ z\rangle \cap \langle x^2\ y\rangle.$$

We also have

$$\sqrt{\langle x^2 - y, x^2 z\rangle} = \langle x^2 - y,\ z\rangle \cap \langle x,y\rangle.$$

Now consider the nonlinear system of fuzzysingletons

$$\begin{aligned} (x_b)^2 - y_a &= 0_{\frac{1}{4}} \\ (x_b)^2\, z_a &= 0_{\frac{1}{2}} \end{aligned}$$

The solution to system (4) is

$$\{(0,\ 0,\ r) \mid r \in R\} \cup \{(s,\ s^2,\ 0) \mid s \in R\},$$

$a = \frac{1}{4}$, b Λ u = $\frac{1}{2}$. Let μ denote the fuzzy ideal $\langle x^2 - y)_{\frac{1}{4}}, (x^2z)_{\frac{1}{2}}\rangle \cup 0_1$. Then

$$\mu(r) = \begin{cases} 1 \text{ if } r = 0 \\ \frac{1}{2} \text{ if } r \in \langle x^2z\rangle \setminus \{0\} \\ \frac{1}{4} \text{ if } r \in \langle x^2 - y,\ x^2z\rangle \setminus \langle x^2z\rangle \\ 0 \quad \text{if } r \in R \setminus \langle x^2 - y,\ x^2z\rangle \end{cases}$$

Define the fuzzy subset μ_i of R, i = 1, ..., 5, as follow :

μ_1 (r) = 1 if r $\in \langle x^2,\ y\rangle$ and 0 otherwise ;

μ_2 (r) = 1 if r $\in \langle x^2 - y,\ z\rangle$ and 0 otherwise ;

μ_3 (r) = 1 if r $\in \langle x^2\rangle$ and $\frac{1}{4}$ otherwise ;

μ_4 (r) = 1 if r $\in \langle z\rangle$ and $\frac{1}{4}$ otherwise ;

μ_5 (r) = 1 if r $\in \langle 0\rangle$ and $\frac{1}{2}$ otherwise ;

Then μ_i is a primary fuzzy ideal of R, i = 1, ...,5 and $\mu = \cap_{i=1}^{5}\mu_i$ is a primary representationm of μ. Now

$\sqrt{\mu_1}$ (r) = 1 if r $\in \langle x,\ y\rangle$ and 0 otherwise ;

$\sqrt{\mu_2}$ (r) = 1 if r $\in \langle x^2 - y,\ z\rangle$ and 0 otherwise ;

$\sqrt{\mu_3}$ (r) = 1 if r $\in \langle x\rangle$ and $\frac{1}{4}$ otherwise ;

$\sqrt{\mu_4}$ (r) = 1 if r $\in \langle z\rangle$ and $\frac{1}{4}$ otherwise ;

$\sqrt{\mu_5}$ (r) = 1 if r $\in \langle 0\rangle$ and $\frac{1}{2}$ otherwise ;

Hence we see that $\mu = \cap_{i=1}^{5}\mu_i$ is a reduced primary representation of μ.

We see that the crisp part of the solution of system (4) is displayed by $\sqrt{\mu_1} \cap \sqrt{\mu_2}$ while the fuzzy part is displayed by $\sqrt{\mu_3} \cap \sqrt{\mu_4} \cap \sqrt{\mu_5}$. (In order to see this more clearly, one should consider the irreducible fuzzy algebraic varieties corresponding to the $\sqrt{\mu_i}$. Then one would be concerned with $c(\frac{1}{4}) = \frac{1}{2}$ rather than $\frac{1}{4}$ and $c(\frac{1}{2}) = \frac{1}{4}$ rather than $\frac{1}{2}$.)

Consider the quotient ring R_P where P is the prime ideal $\langle x,y\rangle$. Sicnce P $\cap$ cP $= \varnothing$, we have in R_P that $\langle x^2 - y,\ x^2z\rangle^e = \langle x^2,y\rangle^c =$ by [16, Theorem 17, p. 225]. Now

$$\mu_{1}. \cap cP = \langle x^2,y\rangle \cap cP = \varnothing$$

$$\mu_{3}. \cap cP = \langle x^2\rangle \cap cP = \varnothing,$$

$$\mu_{5}. \cap cP = \langle 0\rangle \cap cP = \varnothing,$$

while

$$\mu_{2}. \cap cP = \langle x^2 - y,z\rangle \cap cP \neq \varnothing,$$

$$\mu_{4}. \cap cP = \langle z\rangle \cap cP \neq \varnothing,$$

Thus by Theorem 5.4, we have in R_p that $\mu^e = \mu_1{}^e \cap \mu_3{}^e \cap \mu_5{}^e$ and so

$$\mu^e(r) = \begin{cases} 1 \text{ if } r \in R_p \backslash \langle x^2y\rangle^c \\ \frac{1}{4} \text{ if } r \in \langle x^2,\ y\rangle^e \backslash \langle x^2\rangle^c \\ \frac{1}{2} \text{ if } r \in \langle x^2\rangle^c \backslash \langle 0\rangle \\ 0 \text{ if } r \in \{0\} \end{cases}$$

Hence by Theorem 5.4,

$$\mu^{ec}(r) = \mu_1 \cap \mu_3 \cap \mu_5\,(r) = \begin{cases} 1 \text{ if } r \in R\backslash \langle x^2y\rangle \\ \frac{1}{4} \text{ if } r \in \langle x^2,\ y\rangle^e \backslash \langle x^2\rangle \\ \frac{1}{2} \text{ if } r \in \langle x^2\rangle \backslash \langle 0\rangle \\ 0 \text{ if } r \in \{0\} \end{cases}$$

Consider the nonlinear system of fuzzy singletons

$$y_a = 0_{\frac{1}{4}}$$

$$(x_b)^2 = 0_{\frac{1}{2}}$$

Then $\{(0,0,r) \mid r \in R\}$, $a = \frac{1}{4}$, $b = \frac{1}{2}$ is the solution to this system. It is represented by the fuzy ideal $\gamma = \langle (x_{\frac{1}{2}})^2, y_{\frac{1}{4}} \rangle \cup 0_1$. Now $\gamma = \mu_1 \cap \mu_3 \cap \mu_5$ is a reduced primary representation of γ. $\sqrt{\mu_1}$ displays the crisp part of the solution while $\sqrt{\mu_3} \cap \sqrt{\mu_5}$ displays the fuzzy part. Now consider the prime ideal $N = \langle x^2 - y, z \rangle$. Since $N \cap cN = \emptyset$, we have in R_N that $\langle x^2 - y, x^2 z \rangle^c = \langle x^2 - y, z \rangle^c$. Now

$$\mu_{2*} \cap cN = \langle x^2 - y, x^2 z \rangle \cap cN = \emptyset,$$
$$\mu_{4*} \cap cN = \langle z \rangle \cap cN = \emptyset,$$
$$\mu_{5*} \cap cN = \langle 0 \rangle \cap cN = \emptyset,$$

while

$$\mu_{1*} \cap cN = \langle x^2, y \rangle \cap cN \neq \emptyset,$$
$$\mu_{3*} \cap cN = \langle x^2 \rangle \cap cN \neq \emptyset,$$

Thus by Theorem 5.4, we have in R_p that $\mu^e = \mu_1^e \cap \mu_3^e \cap \mu_5^e$ and so

$$\mu(r) = \begin{cases} 0 \text{ if } r \in R_N \setminus \langle x^2 - y, z \rangle^c \\ \frac{1}{4} \text{ if } r \in \langle x^2 - y, z \rangle^c \setminus \langle z \rangle^c \\ \frac{1}{2} \text{ if } r \in \langle z \rangle^c \setminus \{0\} \\ 1 \text{ if } r \in \{0\} \end{cases}$$

Hence by Theorem 5.4,

$$\mu^{ec}(r) = \mu_2 \cap \mu_4 \cap \mu_5 (r) = \begin{cases} 0 \text{ if } r \in R \setminus \langle x^2 - y, z \rangle \\ \frac{1}{4} \text{ if } r \in \langle x^2 - y, z \rangle \setminus \langle z \rangle \\ \frac{1}{2} \text{ if } r \in \langle z \rangle \setminus \{0\} \\ 1 \text{ if } r \in \{0\} \end{cases}$$

Consider the nonlinear system of fuzzy singletons

$$(x_b)^2 - y_a = 0_{\frac{1}{4}}$$
$$z_u = 0_{\frac{1}{2}}$$

Then $\{(s, s^2, 0) \mid s \in R\}$, $b \wedge a = \frac{1}{4}$, and $u = \frac{1}{2}$ is the solution to this system. The system is represented by the fuzzy ideal $\lambda = \langle (x^2 - y)_{\frac{1}{4}}, z_{\frac{1}{2}} \rangle \cup 0_1$. Now $\lambda = \mu_2 \cap \mu_4 \cap \mu_5$ is a reduced primary representation of λ. displays the $\sqrt{\mu_2}$ displays the crisp part of the solutin while $\sqrt{\mu_4} \cap \sqrt{\mu_5}$ displays the fuzzy part.

We have examined the system (4) locally. From the two examinations, we obtain for the crisp part of the solution $\{(0, 0, r) \mid r \in R\}$ for (5) and $\{(s, s^2\ 0) \mid s \in R\}$ for (6). The union of these two qives us the crisp part of the solution to system (4). However the fuzzy solutions to (5) and (6) are $a = \frac{1}{4}$, $b = \frac{1}{2}$ and $b \wedge a = \frac{1}{4}$, $u = \frac{1}{2}$, respectively. The fuzzy part of the solution to (4) is $a = \frac{1}{4}$ and $b \wedge u = \frac{1}{2}$. The two "local" fuzzy solutions do not seem to give us the fuzzy part of the solution to (4), at least not immediately.

Consider all possible $\wedge$'s of the two fuzzy solutions above

$$\begin{aligned} a \wedge b \wedge a &= \frac{1}{4} \wedge \frac{1}{4} \\ a \wedge u &= \frac{1}{4} \wedge \frac{1}{2} \\ b \wedge u &= \frac{1}{2} \wedge \frac{1}{2} \end{aligned}$$

These equations reduce to

$$\begin{aligned} b \wedge a &= \frac{1}{4} \\ a \wedge u &= \frac{1}{4} \\ b \wedge a &= \frac{1}{4} \\ b \wedge u &= \frac{1}{2} \end{aligned}$$

Hence $a = \frac{1}{4}$ and $b \wedge u = \frac{1}{2}$ which is the solution to the original problem.

Problem : Determine a general procedure to find the solution to the fuzzy part of the original problem from the local solution.

An algorithm for solving fuzzy systems of intersection equations is given in [14] and an application to fuzzy graph theory is given in [15]. For a stury if *L*-intersection equations for *L* a complete distributive lattice, the reader is referred to [5].

REFERENCES

1. Alkhamees, Y. and Mordeson, J. N., Fuzzy localized subrings, Inform. Sci., 99(1997) 183-193.
2. Alkhamees, Y. and Mordeson, J. N., Local examination of fuzzy intersection equations, FSS, to appear.
3. Bourbaki, Elements of Mathematics, Commutative algebra, springer-Verlag, New York, 1989, Chaps. 1-7.
4. Eslami, E. and Mordeson, J. N., Completions and fuzzy power series subrings, FSS 82(1996) 97-102.
5. Liu Wangjin, On some systems of simultaneous equations in a completely distributive lattice, Inform. Sci. 50 (1990) 185-196.
6. Malik, D. S. and Modeson, J. N., Fuzzy primary representations of fuzzy ideals, Inform. Sci. 55 (1991) 151-165.
7. Malik, D. S. and Mordeson, J. N., Extensions of fuzzy subrings and fuzzy ideals, FSS 45 (1992) 245-251.
8. Malik, D. S. and Mordeson, J. N., *R*-primary *L*-representations of *L*-ideals, Inform. Sci. 88 (1996) 227-246.
9. McCoy, N. H., Rings and Ideals, The Carus Mathematical Monograms, Number 8, The Mathematical Associaltion of America, 1956.
10. Morderson, J. N., Fuzzy algebraic varieties, Rocky Mountain J. Math. 23 (1993) 1361-1377.
11. Mordeson, J. N., Fuzzy algebraic varieties II, Advances in Fuzzy Theory and Technology Vol. I (1993) (Edited by Paul Wang) 9-21.
12. Mordson, J. N., Fuzzy Coefficient fields of fuzzy subrings, FSS 58 (1993) 227-237.
13. Mordeson, J. N., Fuzzy intersection equations and primary representation, FSS, 83 (1996) 93-98.
14. Mordeson, J. N., and Peng, Chang-Shyh, Fuzzy intersection equations, FSS 60 (1993) 77-81.
15. Mordeson, J. N., and Peng, Change-Shyh, Operations on fuzzy graphs, Inform. Sci. (1994) 159-170.
16. Zariski, O. and Sammuel, P., Commutative Algebra vol. I, D. Van Nostrand Company, Inc., Princeton, New Jersey (1958).

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THE CONSTRUCTION OF A CLASS OF $\mathscr{P}$-REGULAR SEMIGROUPS *

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ABSTRACT : In this paper, we whall introduce a concept of regular transversal and characterize the structure of $\mathscr{P}$-regular semigroups with regular $*$-transversals by applying partial bands and regular $*$-semigroups.

1. INTRODUCTION

In [2-9], T. S. Blyth, R. McFadden, D. B. Mcalister and T. Saito introduced and studied multiplicative inverse transversals, inverse transversals, S-inverse transversal Q-inverse transversals and so on. Applying left regular bands, right regular bands and inverse semigroups, they characterized some special regular semigroups with inverse transversals. In [12], Tang proved that every inverse transversals of S is an S-inverse transversal. Especially, T. Saito described the structure of regular semigroups with S-inevrse transversal in [9]. In 1982, Yamada proved that a semigroup is a regular $*$-semigroup if and only if it has $\mathscr{P}$-system in [10]. Furthermore, Yamada and Sen introduced $\mathscr{P}$-regular semigroups in [10] which are generalized regular $*$-semigroups and orthodox semigroups.

First, we recall some concepts in order to introduce regular $*$-transversals. Let S be a regular semigroup. An inverse subsemigroup S^0 of S is an inverse transversal [4] if $| V(x) \cap S^0 | = 1$ for each $x \in S$, where $V(x)$ denotes the set of inverse of x. In this case, the unique element of $V(x) \cap S^0$ is denoted by x^0. Let $E(S^0) = E^0$ denote the semilattice of idempotents of S^0. The sets $\{e \in S \mid e = ee^0\}$ and $\{f \in S \mid f = f^0 f\}$ are denoted by I_S and Λ_S respectively, or simply I and Λ. Clearly, every element of these sets is idempotent. When both I and Λ are sub-bands of S, S^0 is called an S-inverse transversal [4]. An inverse transveral S^0 is multiplicative [4] if $x^0xyy^0 \in E^0$, and S^0 is weakly multiplicative [4] if $(x^0xyy^0)^0 \in E^0$ for every $x,y \in S$. A band B is left [resp.right] normal if $efg = egf$ [resp. $efg = feg$] for every $e, f, g \in B$. A subset Q of S is a quasi-ideal of S if $QSQ \subseteq Q$.

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Let S be a regular $*$-semigroup, E_s the set of idempotents of S, and P the set of projections of S (see[10]). Then (S, P) satisties the following.

(i) every L-class [R-class] of S contains a unique element of P;

(ii) For every element $a \in S$, $a^* \in V(a)$ such that $aP^1a^* \subseteq P$ and $a^*P^1a \subseteq P$;

(iii) $P^2 \subseteq E_s$.

On the other hand, let T be an orthodox semigroup and E_T the set of idempotents of T. Then (T, E_T) satisfies the following :

(i)' For every element $a \in T$, there exists $a' \in V(a)$ such tnat $aE_T^1a' \subseteq E_T$ and $a'E_T^1a \subseteq E_T$.

(ii)' $E_T^2a \subseteq E_T$.

Unifying these two concepts 'regular $*$-semigroups' and 'orthodox semigroups', Yamada and Sen introduced the concept '$\mathscr{P}$-regular semigroups'.

Let S be a regular semigroup, and E_s the set of idempotents of S. Let $P \subseteq E_s$. Then (S, P) is called a $\mathscr{P}$-regular semigroup [11] if it satisties the following :

(1) $P^2 \subseteq E_s$; (2) for $q \in P$, $qPq \subseteq P$ (3) for any $a \in S$, there exists $a^+ \in V(a)$ such that $aP^1a^+ \subseteq P$ and $a^+P^1a \subseteq P$.

Hereafter, (S,P) is denoted by $S(P)$, and a^+ is called a $\mathscr{P}$-inverse of a. $V_p(a)$ denotes the set of all $\mathscr{P}$-inverses of a. If (S, P) satisfies only (1), (2) and (3)' $P \cap L \neq \varnothing$ and $P \cap R \neq \varnothing$ for every L-class L and R-class R of S, (S, P) is called a weakly $\mathscr{P}$-regular semigroup, and denoted by $S[P]$. A subset P of E_s is called a characteristic set [weak characteristic set] (a C-set [weak C-set]) in S if it satisties (1)-(3) [(1)-(2), (3)']. This concept of a C-set is a generalization of both the concept of a $\mathscr{P}$-system [10] in a regular semigroup and the concept of the set of projections in a regular $*$-semigroup.

We know that

{regular $*$-semigroup} $\cup$ {orthodox semigroups} $\subset$ {$\mathscr{P}$-regular semigroups}.

In [14], we generalized $\mathscr{P}$-regular semigroups to abundant semigroups and established the structure theorem for the $\mathscr{P}$-regular semigroups and special weakly $\mathscr{P}$-regular semigroups.

In this paper, we investigate a special class of $\mathscr{P}$-regular semigroups which satisties that this class $\subset$ {$\mathscr{P}$-regular semigroups}.

First, we give a definition.

Definition 1.1. Let (S, P) be a $\mathscr{P}$-regular semigroup with C-set P and S^0 a subsemigroup of S. Then S^0 is called a regular $*$-transversal if it satisties that $| V_p(a) \cap S^0 | = 1$ for each $a \in S$.

Hereafter, this unique element of $V_p(a) \cap S^0$ is denoted by a^0.

Proposition 1.2. Let (S, P) be a $\mathscr{P}$-regular semigroup with C-set P and S^0 a regular $*$-transversal then (S^0, P^0) is a regular $*$-semigroup, where $P^0 = S^0 \cap P$ is the set of projections of S^0.

Further, it is clear that $e \in P^0$ if and only if $e = e^0$ for $e \in E_s$.

Proof. It is clear that (S^0, P^0) is $\mathscr{P}$-regular because it satisfies the following conditions.

(1) $(\forall p \in P^0)\ pP^0p \subseteq S^0 \cap pPp \subseteq S^0 \cap P = P^0$;

(2) $P^0P^0 \subseteq S^0S^0 \cap P^2 \subseteq S^0 \cap E_s = E_{s0}$; and

(3) $(\forall x \in S^0)\ xP^{01}x^0 \subseteq S^0 \cap xP^1x^0 \subseteq S^0 \cap P = P^0$ and $x^0P^{01}x \subseteq P^0$.

Applying Lemma 2.7 in [11] $P^0 \subseteq P$, for any $a \in S^0$, $\{\mathscr{P}$-inverse of a in S^0 $(P^0)\}$ (is denoted by $V_{p0}(a)) \subseteq \{\mathscr{P}$-inverse of a in $S(P)\}$. Furthermore, S^0 is a regular $*$-transversal of $S, P)$. So $V_{p0}(a)) \subseteq V_p(a) \cap S^0$. Therefore $| V_{p0}(a) | = 1$ since $| V_p(a) \cap S^0 | = 1$. This is that (S^0, P^0) is a regular $*$-semigroup and the other parts hold.

In the following discussions, we illustrate that $\{\mathscr{P}$-regular semigroup with a regular $*$-transversal $\} \subset \{\mathscr{P}$-regular semigroup$\}$ and a regular $*$-transversal may be not an inverse transversal.

Exmaple 1.1. Let $M(G ; 3, 4 ; Q)$ be a completely simple semigroup, where

$$Q = \begin{pmatrix} e & a & b \\ a^{-1} & e & c \\ b^{-1} & c^{-1} & e \\ b^{-1} & c^{-1} & e \end{pmatrix}$$

and a, b, c such that $ac \neq b$. Then $S = M(G ; 3, 4 ; Q)$ with C-set ([14]) $P = \{(1, e, 1), (2, e, 2), (3, e, 3), (3, e, 4))\}$. Let $S^0 = \{i, x, \lambda) \mid 1 \leq i, \lambda \leq 3, x \in G\}$ is a regular $*$-semigroup with $\mathscr{P}$-system

$$P^0 = S^0 \cap P = \{(i, e, i) \mid 1 \leq i \leq 3\}$$

Then, clearly $| V_p(x) \cap S^0 | = 1$ for each $x \in S$, and so S^0 is a regular $*$-transversal of S. But S^0 is not an orthodox semigroup because $(1, a, 2), (2, c, 3) \in E_S$ and

$(1, a, 2)(2, c, 3) = (1, ac, 3) \neq (1, b, 3)$. We know that every completely simple semigroup has an inverse transversal and there exists a completely simple semigroup T such that (T, P) is not $\mathscr{P}$-regular for any $P \subseteq E_T$.

Let $S(P)$ be a $\mathscr{P}$-regular semigroup with a regular $*$-transversal S^0. The sets $\{e \in S \mid ee^0 = e\}$ and $\{f \in S \mid f^0f = f\}$ are denoted by $I_{S(P)}$ and $\Lambda_{S(P)}$ are not sub-bands of $S(P)$. But $I_{S(P)}$, $\Lambda_{S(P)} \subseteq P$.

Proposition 1.3. Let $S(P)$ be a $\mathscr{P}$-regular semigroup with a regular $*$-transversal S^0. Then

(1) $(\forall a, b \in S(P))\ (ab)^0 = b^0a^0$;

(2) $P^0 = S^0 \cap P$ is a subsemilattice of $S(P)$ and $P^0 = E_{S^0}$ if $I_{S(P)}$ and $\Lambda_{S(P)}$ are two sub-bands of $S(P)$.

Proof. (1) since $V_P(a)\ V_P(b) \subseteq V_P(ba)$, $(ab)^0 = b^0a^0$. (2) clearly $P^0 = I_{S(P)} \cap \Lambda_{S(P)}$. Let $I_{S(P)}$ and $\Lambda_{S(P)}$. be two sub-bands of $S(P)$, and let $e, f \in P^0$. Then $ef \in I_{S(P}$ and $ef \in \Lambda_{S(P)}$ imply $ef \in I_{S(P)} \cap \Lambda_{S(P)} = P^0$ and so that $(ef)^0 = f^0e^0 = fe = ef$. Consequently, S^0 is an inverse subsemigroup and $P^0 = E_{S^0}$.

In general, $I_{S(P)}$ and $\Lambda_{S(P)}$ are not sub-bands of $S(P)$. By Example 1.1 and Proposition 1.3.

$$I_{S(P)} = \{(1, e, 1), (2, e, 2), (3, e, 3)\}$$

and

$$\Lambda_{S(P)} = (1, e, 1), (2, e, 2), (3, e, 3), 3, e, 4)\}$$

are not sub-bands of $S(P)$.

Let $S(P)$ be a $\mathscr{P}$-regular semigroup. Then $S(P)$ is called strongly $\mathscr{P}$-regular if $V_P(q) \subseteq P$ for every $q \in P$ ([15]).

Let $S(P)$ be a $\mathscr{P}$-regular semigroup. Then $S(P)$ is strongly $\mathscr{P}$-regular

$$\Leftrightarrow \text{“I } V_P(a) \cap V_P(b) \neq \varnothing \Rightarrow V_P(a) = V_P(b)\text{”}$$

$\Leftrightarrow (\forall p \in P)\ V_P(p) \subseteq P) \Leftrightarrow (\forall p,q \in P)$ “$pq \in P \Rightarrow qp \in P)$” (see [13], [14] or [15]).

Theorem 1.4. Let $S(P)$ be a strongly $\mathscr{P}$-regular semigroup with C-set P and let S^0 be a regular $*$-subsemigroup of $S(P)$ with $\mathscr{P}$-system $P^0 = S^0 \cap P$. Then S^0 is a regular $*$-transversal of $S(P)$ if and only if $V_P(x) \cap S^0 \neq \varnothing$ for each $x \in S(P)$.

Proof. The direct part is clear. Conversely, let $V_P(x) \cap S^0 \neq \varnothing$ for each $x \in S(P)$ and $S(P)$ be strongly $\mathscr{P}$-regular. If $x^0, x' \in V_P(x) \cap S^0$ then

$$x \in V_P(x^0) \cap V_P(x').$$

So $V_p(x^0) = V_p(x')$. (since $S(P)$ is strongly $\mathscr{P}$-regular). According to Lemma 2.7 in [11] and $P^0 = S^0 \cap P$, $V_{p^0}(a) \subseteq V_p(a)$ and $V_{p^0}(a) = V_p(a) \cap S^0$ ($V_{p^0}(a)$ dentotes the set of $\mathscr{P}$-inverse of a in S^0 (P^0)) for any $a \in S^0$. Let x^{00} be an $\mathscr{P}$-inverse of x^0 in S^0 (P^0). Applying $x^0, x' \in V_p(x) \cap S^0$ and $V_p(x^0) = V_p(x')$, $x^{00} \in V_p(x^0) = V_p(x')$. So x^0, x' are two $\mathscr{P}$-inverse of x^{00} in S^0 (P^0). But $P^0 = S^0 \cap P$ and S^0 is a regular $*$-subsemigroup of $S(P)$ with $\mathscr{P}$-system P^0. Therefore $x^0 = x'$.

In theorem 1.4, if $S(P)$ is only $\mathscr{P}$-regular, then $V_p(x) \cap S^0 \neq \varnothing$ for each $x \in S(P)$ does not imply that S^0 is a regular $*$-transversal of $S(P)$. For example, let $B = I \times \Lambda$ be a rectangular band, where $I = \{1, 2, 3, 4\}$. Take

$$P = \{(1, 1), (2, 2), (3, 3), (1, 4), (2, 4), (3, 4)\}$$

$B^0 = \{(i, j) | 1 \leq i, j \leq 3\}$, $P^0 = \{(i, i) | i = 1, 2, 3\}$. Then (B^0, P^0) is a regular $*$-semigroup with $\mathscr{P}$-system $P^0 = S^0 \cap P$ and (B, P) is $\mathscr{P}$-regular. Applying $(i, \lambda)\ P^1\ (i, \lambda) = \{(i, \lambda)\}$, we know that (1) $P^2 \subseteq B$; (2) $pPp = \{p\}$; and (3) if $i, \lambda \leq 3$ then $(i, \lambda)\ P^1\ (\lambda, i) = \{(i, i)\}, \subseteq P$ and $(\lambda, i)\ P^1\ (i, \lambda) = \{(\lambda, \lambda)\}, \subseteq P$; if $\lambda = 4$, then $(i, 4)\ P^1\ (i, 4) = \{(i, 4)\}$. Clearly (B^0, P^0) is a regular $*$-semigroup. For any $(i, 4)$, $(i, i) \in V_p((i, 4))$. So $V_p(a) \cap B^0 \neq \varnothing$ for any $a \in B$. But $| V_p(a) \cap B^0 | \neq 1$ for some $a \in B$ For example let $a = (3, 4)$. Then $(2, 3), (1, 3) \in V_p(a) \cap B^0$ and $(2, 3) \neq (1, 3)$.

Proposition 1.5. Let S^0 be a regular $*$-transversal of $S(P)$. Then $I_{s(P)} = \{aa^0 | a \in S(P)\}$ are of idempotents of P $\mathscr{L}$-related to an element of P^0, $\Lambda_{s(P)} = \{aa^0 | a \in S(P)\}$ are of idempotents of P $\mathscr{R}$-related to an element of P^0 and $P^0 = I_{s(P)} \cap \Lambda_{s(P)}$.

Proof. Let $\{aa^0 | a \in S(P)\}$ and $\Lambda' = \{a^0a | a \in S(P)\}$. Then, clearly $I_{s(P)} \subseteq I'$ and $\Lambda_{s(P)} \subseteq \Lambda'$. Conversely, $aa^0 (aa^0)^0 = aa^0 a^{00} a^0 = aa^0$ and $((a^0a))^0 a^0a = a^0 a^{00} a^0 a = a^0a$. Hance $aa^0 \in I_{s(P)}$ and $a^0a \in \Lambda_{s(P)}$. The remainder part is clear.

Theorem 1.6. Let S^0 be a regular $*$-transversal of $S(P)$. Then $I_{s(P)}$ and $\Lambda_{s(P)}$ are two sub-bands of $S(P)$ if and only if S^0 is an inverse transversal of S if and only if S^0 is an inverse subsemigroup of S.

Proof. Assume that $I_{s(P)}$ and $\Lambda_{s(P)}$ are sub-bands of S. By Proposition 1.3. we know that S^0 is an inverse semigroup. Let $x, y \in V(a) \cap S^0$. Then $x = xax$, $y = yay$ and $a = axa = aya$. So $x^0 = x^0a^0x^0$, $y^0 = y^0a^0y^0$ are $a^0 = a^0x^0a^0 = a^0y^0a^0$ imply $x^0, y^0 \in V(a^0) \cap S^0$, and so $x^0 = x^0$ and $x = y$, that is $| V(x) \cap S^0 | = 1$ for each $x \in S$.

Conversely, it is clear (reference to Tang [12]) and the other parts also hold.

In [2], D. B. McAlister and T. S. Blyth investigated a class of split orthodox semigroups. According to [4], we know that a band B has an inverse transversal if and only if B is split. But, for every hand B, let $P = B$. Then (B, P) is $\mathscr{P}$-regular and (B, P) has a regular $*$-transversal if and only if B has an inverse transversal if and only if B is split. So, for every band B, B may have not a regular $*$-transversal. According to [8], we easily obtain the following results.

Corollary 1.7. Let $S(P)$ be a $\mathscr{P}$-regular semigroup with an inverse transversal S^0. If $I_{s(P)}$ and $\Lambda_{s(P)}$ are two sub-bands of $S(P)$, then $I_{s(P)}$ $[\Lambda_{s(P)}]$ is a left [right] regular band.

2. ANOTHER DEFINITION OF REGULAR $*$-TRANSVERSAL

Throughout this section, let S *denote a regular semigroup and* E_s the set of all idempotents of S.

Theorem 2.1. Let S^0 be a regular $*$-transversal of $\mathscr{P}$-regular semigroup $S(P)$. Then there exists C-set P' in S such that

(1) $p' \in P'$ if and only if $p'^0 \in P^0$, and (2) (S^0, P^0) is a regular $*$-transversal of strongly $\mathscr{P}$-regular semigroup $S(P')$, where $P'^0 = P \cap S^0 = P'^0 \cap S^0$.

Proof. In the following discussions, x^0 always denotes the unique $\mathscr{P}$-inverse of x in S^0. Let $P' = \{x \in S|\ x^0 \in P^0\}$. Applying the $S(P)$ is $\mathscr{P}$-regular and Lemma 2.7 in [11], we know $P' \subseteq E_s$ (since $x \in E_S \Leftrightarrow V_P(x) \subseteq E_S$ in [11]. Clearly (S, P') Satisfies the following conditions :

(1) $(\forall p' \in P')\ p'P'p' \subseteq P'$ (since $p'^0 q^0 p'^0 \in p'^0 P^0 p'^0 \subseteq P^0$ for any $q \in P'$);

(2) $P'^2 \subseteq E_s$ (since $(xy)^0 = y^0x^0 \in P^0P^0 \subseteq PP \subseteq E_s$ for any $x, y \in P'$) ; and

(3) $(\forall a \in S)a'P'^1a^0 \subseteq P'$ and $a^0P'^1a \subseteq P'$ (since $(aa^0)^0 = a^{00}a^0$ and $(a^0a)^0 = a^0a^{00}$ belong to P^0).

Then (S, P') is a $\mathscr{P}$-regular semigroup and a^0 is a $\mathscr{P}$-inverse of a in S^0. Let $p,q \in P'$ and $pq \in P'$. Then $(pq)^0 = q^0p^0 \in P^0$ and $q^0, p^0 \in P^0$. According to Proposition 1.2, (S^0, P^0) is a regular $*$-semigroup. So it is strongly $\mathscr{P}$-regular. Therefore $(qp)^0 = p^0q^0 \in P^0$ and $qp \in P'$. Thus (S, P') is strongly $\mathscr{P}$-regular. Clearly $P^0 \subseteq P' \cap S^0$. Let $p \in P' \cap S^0$. Then $p^0 \in P^0$, Thus $p, p^0 \in V_P(p^0) \cap S^0$. But S^0 is a regular $*$-transversal of $S(P)$. Therefore $p = p^0 \in P^0$, that is $P' \cap S^0. \subseteq P^0$. By Theorem 1.4 and (3) above, we know that S^0 is a regular $*$-transversal of the $\mathscr{P}$-regular semigroup $S(P')$.

Now we may give another difinition of regular $*$-transversal.

Definition 2.2. Let S be a regular semigroup. Then S is called having a regular $*$-transversal map if there exists a map o from S into itself such that (1) $(\forall a \in S)a^0 \in V(a)$ and $a^0 \in E_s \Leftrightarrow a \in E_s$; (2) $(ab)^0 = b^0a^0$ for any $a, b \in S$; and (3) S^0 is a regular $*$-semigroup with $\mathscr{P}$-system $P^0 = \{p \in E_{s^0}\ |\ p^0 = p\}$ (S^0 is the image of map o and a^0 the image of a under map o). In this case, map o is called a regular $*$-transversal map.

Theorem 2.3. Let S be a regular semigroup. Then there exists $P \subseteq E_s$ such that (S, P) is $\mathscr{P}$-regular and it has a regular $\star$-transversal if and only if there exists a map o from S into S such that o is a regular $\star$-transversal map.

Proof. Let S have a regular $\star$-transversal S^0. Define a map o : $S \to S$, $a \to a^0$, where a^0 denotes the unique element of $V_p(a) \cap S^0$. Then the map o is a regular $\star$-transversal map. Conversely, let map o be a regular $\star$-transversal map and let S^0 denote the image of the map o. Let

$$P = \{x \in S \mid x^0 \in P^0\}$$

where $P^0 = \{p \in E_s \mid p = P^0\}$. Then (S, P) satisfies the following conditions :

(1) $P \subseteq E_s$; (2) $(\forall p \in P)\ pPp \subseteq P$ (since $(pqp)^0 = p^0q^0p^0 \in P^0$ for any $q \in P$); and (3) $(\forall a \in S)\ aP^1a^0 \subseteq P$ and $a^0P^1a \subseteq P$ (since $(apa^0)^0 = a^{00}p^0a^0 \in P^0$ and $(a^0pa)^0 = a^0p^0a^{00} \in P^0$). So (S, P) is $\mathscr{P}$-regular. Let $p, q \in P$ and $pq \in P$. Then $(pq)^0 = q^0p^0 \in P^0$ and $q^0, p^0 \in P^0$. Therefore $(qp)^0 = p^0q^0 \in P^0$ (since $S^0(P^0)$ is regular $\star$-semigroup and every regular $\star$-semigroup is strongly $\mathscr{P}$-regular) and $qp \in P$. Thus (S, P) is strongly $\mathscr{P}$-regular and a^0 is a $\mathscr{P}$-inverse of a in S^0. By Theorem 1.4, S^0 is a regular $\star$-transversal of $S(P)$.

Clearly, in Definition 2.2. (1)-(3) are equivalent to (1), (2) and (3)′ $(\forall a \in S)a^{000} = a^0$. Therefore we shall see that all $\mathscr{P}$-regular semigroups have regular $\star$-transversal is composed a interesting class which generalize split orthodox semigroups [2] from the following Theorem 2.4.

Theorem 2.4. Let $S(P)$ be a strongly $\mathscr{P}$-regular semigroup. Then there exists a regular $\star$-transversal in $S(P)$ if and only if there exists an homomorphism $\varphi : S(P) \mid \gamma_p \to S(P)$ such that $\varphi\natural$ is the identity map on $S(P) \mid \gamma_p$, where $\gamma_p = \{(a, b) \in S \times S \mid V_p(a) \cap V_p(b) \neq \varnothing\}$ and $\natural$ is the natural map induced by γ_p.

Proof. According to [13] or [14], we know that γp is the least strong $\mathscr{P}$-congruence [11] and $V_p(a) \cap V_p(b) \neq \varnothing\}$ if and only if $V_p(a) = V_p(b)$ (a $\mathscr{P}$-congruence is a congruence in a $\mathscr{P}$-regular semigroup. A $\mathscr{P}$-congruence σ is called strong if it satisfies that, for any $q \in P$ and $a \in S$, $q\sigma a$ implies $q\sigma a^*$ for all $\mathscr{P}$-inverses $a^* \in V_p(a)$). Let S^C be a regular $\star$-transversal of $S(P)$ and a map $\psi : S(P) \mid \gamma p \to S^0$, $a\gamma p \to a^{00}$. Then ψ is a homomorphism and it is surjective. Further, if $a^{00} = b^{00}$, then $a^0 = b^0$ and so $a\gamma_p = b\gamma_p$. Therefore ψ is an isomorphism from $S(P) \mid \gamma p$ into S^0 :

$$a\gamma_p\psi\natural = a^{00}\natural = a^{00}\gamma_p = a\gamma_p,$$

that is $\psi\natural = 1_{S(P)/\gamma P}$

Conversely, let ψ be a homomorphism from $S(P) \mid \gamma p$ into S such that $\psi\natural = 1_{S(P)/\gamma P}$. Let S^0 = image (ψ). Then S^0 is a regular $\star$-semigroup with $\mathscr{P}$-*system*

$\overline{P} = \{(p)\gamma_p\psi \mid p \in P\}$ because ψ is an isomorphism from $S(P) \mid \gamma p$ onto S^0. Furthermore, applying that $S(P)$ is strongly $\mathscr{P}$-regular, we have that $(p)\,\gamma p \subseteq P$ (applied $x \in (p)\,\gamma p$, $V_P\,(p) = V_P\,(x)$ and so $x \in V_P\,(p) \subseteq P$ for every $p \in P$. Let $p_1 = p\gamma_p\psi$ for any $p \in P$. Then $p_1\natural = p\gamma_p\psi\natural = p\gamma_p = p_1\gamma_p$. So $p_1 \in V_P\,(p) \subseteq P$, that is $\overline{P} \subseteq P$.

Let $p \in S^0 \cap P$. Then there exists $x \in S$ such that $p = (x\gamma_p)\psi$. So that $V_P\,(p) = V_P\,(x)$. Therefore $p = (p\gamma_p)\psi$, that is $\overline{P} \supseteq S^0 \cap P$. Put $a_1 = (a\gamma_p)\psi$. Then $(a\gamma_p)\psi\natural = a\gamma_p = (a_1)\natural = a_1\gamma_p$. So $V_P\,(a_1) = V_P\,(a)$. Let a_1^0 denote the unique $\star$-inverse if a_1 in S^0. Then $a_1^0 \in V_P\,(a) \cap S^0$, that is $V_P\,(a) \cap S^0 \neq \varnothing$ for any $a \in S$. By theorem 1.4, S^0 is a regular $\star$-transversal of $S(P)$.

3. SPLIT $\mathscr{P}$-REGULAR ORTHODOX SEMIGROUPS

In section 1, we know that $I_{s(p)}$ and $\Lambda_{s(p)}$ are not sub-bands in general. In this section, we shall investigate a special class of $\mathscr{P}$-regular semigroups, i.e. split $\mathscr{P}$-regular orthodox semigroups. In [2], D. B. McAlister and T. S. Blyth investigated split orthodox semigroups and special cases. An orthodox semigroup T is said to be split if the natural morphism $\natural : T \to T / \lambda$ splits, i.e if there is a (splitting) morphism : $\pi\ T / \lambda \to T$ such that $\pi\natural$ is the identity on T / λ. Here λ is the least inverse congruence in T. Therefore a $\mathscr{P}$-regular semigroup with a regular $\star$-transversal may be called a split $\mathscr{P}$-regular semigroup by Theorem 2.4.

Definition 3.1. Let S^0 be a regular $\star$-transversal of a $\mathscr{P}$-regular semigroup $S(P)$. Then S^0 is called a regular $\star$-inverse transversal if $I_{s(p)}$ and $\Lambda_{s(p)}$ are sub-bands of $S(P)$. In this case, $S(P)$ is also called split $\mathscr{P}$-regular orthodox semigroup. A regular $\star$-transversal S^0 is called $\mathscr{P}$-multiplicative if $x^0xyy^0 \in P^0 = S^0 \cap P$. In the following discussions, $E^0 = E_{s0}$ and $P^0 = P \cap S^0$.

Theorem 3.2. Let S^0 be a regular $\star$-transversal of a $\mathscr{P}$-regular semigroup $S(P)$. Then S^0 is $\mathscr{P}$-multiplicative if and only if S^0 is an inverse transversal of $S(P)$ and it is a quasi-ideal of $S(P)$.

Proof. let S^0 be $\mathscr{P}$-multiplicative. Then S^0 is a quasi-ideal (since $x^0ay^0 = x^0x^{00}x^0aa^0a^{00}a^0ay^0y^{00}y^0$). Let $p, q \in P^0$. Then $pq \in P^0$ and $(pq)^0 = q^0p^0 = qp = pq$. Therefore, S^0 is an inverse semigroup. By Theorem 1.6, we know that S^0 is an inverse transversal of $S(P)$. Conversely, it is clear.

Lemma 3.3. If S^0 is a regular $\star$-transversal of a $\mathscr{P}$-regular semigroup $S(P)$, then

(i) Each element of S can be writen in the form exf where $e = ee^0$, $f = f^0f$, $x \in S^0$, $x = e^0xf^0$;

(ii) Let e, f, x be as in (i) and let $g = gg^0$, $h = h^0h$, $y \in S^0$ with $y = g^0yh^0$. Then

$$exf = gyh \text{ if and only if } x = y,\ ex = gy,\ xf = yh.$$

Proof. (i) Let $a \in S$. Then

$$a = aa^0a^{00}a^0a.$$

By Proposition 1.5, $aa^0 \in I_{s(p)}$, $a^{00} \in S^0$. Let $e = aa^0$, $f = a^0a$, $x = a^{00}$. Then e, f, x satisfy the conditions in (i).

(ii) Note first that if e, x, f are as in (i), then since $x, e^0, f^0 \in S^0$, $x^0 = f^0 x^0 e^0 = (exf)^0$. It follows that $exf = gyh$ implies $x = y$, and on per-multiplying $exf = gyh$ by e^0, that is $xf = e^0ghy$. Therefore $x^0xf = x^0e^0gxh = x^0gxh = y^0g^0gyh = y^0g^0yh = y^0yh$, whence $xf = xh$; and similarly $ex = gx$. Conversely, $x = y$, $ex = gy$, $xf = yh$ imply $exf = gyh$.

In section 1, clearly Example 1.1 satisfies that S^0 is a quasi-ideal of $S(P)$, but

$$I_{S(P)} = \{(1, e, 1), (2, e, 2), (3, e, 3)\}$$

and

$$\Lambda_{S(P)} = \{(1, e, 1), (2, e, 2), (3, e, 3), (3, e, 4)\}$$

are not sub-bands.

Theorem 3.4. Let $S(P)$ be a $\mathscr{P}$-regular semigroup with regular $*$-transversal such that $I_{S(P)}$ and $\Lambda_{S(P)}$ are sub-bands. Then S is orthodox.

Proof. According to Theorem 1.6, S is orthodox. In fact, let $e, f \in E_s$. then e^0, $f^0 \in E_{S^0}$ and $(ef)^0 = f^0e^0 \in E_{S^0}$. Thus $ef \in E_{S^0}$.

But let $S(P)$ have a regular $*$-transversal S^0 and let S be orthodox. Then $I_{S(P)}$ and $\Lambda_{S(P)}$ may be not sub-bands.

Example 3.1. Let $S = \mathcal{M}(G\ ; I, \Lambda\ ; Q_1)$ be a completely simple semigroup, where $I = \Lambda = \{1, 2, 3\}$,

$$Q_1 = \begin{pmatrix} e & e & e \\ e & e & e \\ e & e & e \end{pmatrix},$$

$e \in G$ is identity.

Let $P = \{(1, e, 1), (2, e, 2), (3, e, 3)\}$. Then S is a regular $*$-semigroup with $\mathscr{P}$-system P. Let $S^0 = S$. Then S^0 is a regular $*$-transversal of $S(P)$. But $I_{S(P)} = P = \Lambda_{S(P)}$ is not sub-bands of S.

Proposition 3.5. Suppose that S contains a regular $*$-transversal S^0 that is a quasi-ideal of S, and let $e^0 \in P^0$, $g = gg^0 \in I$. Then $e^0g = (ge^0)^0 = e^0g^0 \in E^0 = E_{S^0}$.

Proof. Let $x = e^0g$. then $x = e^0gg^0 \in S^0$ beacuse S^0 is a quasi-ideal. So $x = e^0xg^0$ and $x^0 = g^0x^0e^0$. It follows that $gx^0xgx^0 = gx^0xg^0gx^0 = gx^0$ since $g^0g = g^0$; similarly $xgx^0x = x$, and so $x \in V(gx^0) \cap S^0$; further, $xgx^0 = xg^0gx^0 = xg^0x^0 = xx^0 = \in P$ and $gx^0x = gx^0xg^0 = gx^0(gx^0)^0 \in P$. So $x \in V_p(gx^0) \cap S^0$ that is $(gx^0) = x$. But also $x = (gx^0)^0 = (gg^0e^0)^0 = (ge^0)^0 = e^0g^0$, that is $e^0g = e^0g^0$.

Theorem 3.6. suppose that $S(P)$ contains a regular $*$-transversal S^0. Then the following statements are equivalent :

(1) S^0 is a quasi-ideal of $S(P)$; (2) $exf = ex^{00}f$ for any $x \in S(P)$ and $e, f \in P^0$; and (3) $x^0xyy^0 \in E^0$ for any $x, y \in S(P)$.

Proof. (1) ⇒ (2). Since $(exf)^0 = f^0x^0e^0 = fx^0e$ and $(exf)^{00} = ex^{00}f$, $exf = (exf)^{00} = ex^{00}f$.

(2) ⇒ (3). Let $e \in \Lambda_{s(p)}$, $f \in I_{s(p)}$. Then

$$ef = e^0eff^0 = e^0(ef)^{00}f^0 = e^0e^{00}f^{00}f^0 = e^0f^0 \in E^0.$$

(3) ⇒ (1). It is clear.

Proposition 3.7. Suppose that $S(P)$ contains a regular $*$-transversal S^0 then the sets

$$I_{s(p)} = \{e \in S \mid e = ee^0\},\ \Lambda_{s(p)} = \{f \in S \mid f = f^0f\}$$

are $\mathscr{R}$-unipotent and $\mathscr{L}$-unipoent partial sub-bands respectively of $S(P)$.

Proof. Let $e, f \in I_{s(p)}$. Then eRf implies e^0Lf^0. therefore $e = f$. Similarly $\Lambda_{s(p)}$ is $\mathscr{L}$-unipotent partial sub-band of $S(P)$.

Under the following discussions, we always suppose that a $\mathscr{P}$-regular semigroup $S(P)$ contains a regular $*$-inverse transversal. Then according to Theorem 1.6 and Definition 3.1, we know that S^0 is an inverse transversal of $S(P)$. Therefore, we have the following results similar to [5].

Proposition 3.8. Suppose that a $\mathscr{P}$-regular semigroup $S(P)$ contains a regular $*$-inverse transversal S^0 then

(1) $I_{S(P)}$ and $\Lambda_{S(P)}$ are $\mathscr{R}$-unipotent a $\mathscr{L}$-unipotent sub-bands respectively of $S(P)$; if S^0 is a quasi-ideal of $S(P)$, $I_{S(P)}$ and $\Lambda_{S(P)}$ are left normal and right normal respectively ;

(2) $I_{S(P)}$ and $\Lambda_{S(P)}$ have a common multiplicative inverse transversal P^0 ; further, if S^0 is quasi-ideal, then P^0 is a quasi-ideal respectively of $I_{S(P)}$, $\Lambda_{S(P)}$.

Proof. Notice Proposition 1.7 and Corollary 1.8 in [5].

Let $S(P_s)$ and $T(P_T)$ be two $\mathscr{P}$-regular semigroup and φ isomorphism from S into T. Then φ is called an $\mathscr{P}$-isomorphism if $P_T = (P_s)\varphi$. In this case, $S(P_s)$ and $T(P_T)$ are called $\mathscr{P}$-isomorphic and denoted by $S(P_S) \cong_p T(P_T)$. According to [5] and [13], we have the following results.

Theorem 3.9. Let S^0 be an inverse semigroup and let (B, P) be a $\mathscr{P}$-regular and with a regular $*$-inverse transversal E^0 which is a quasi-ideal such that E^0 is just the subsemilattice of S^0. Suppose there exists a $\Lambda \times I$ matrix Q over S^0 satisfying : $[f, e] = f^0e^0$ for all $e \in I$, $f \in \Lambda$, where $I = \{e \in B \mid e = ee^0\}$ and $\Lambda = \{f \in B \mid f = f^0f\}$ (in fact, applying Theorem 3.6, we have $f^0e^0 = fe$).

Let

$$T = \{e, x, f) \in I \times S^0 \times \Lambda \mid e\mathscr{L}xx^0,\ x^0x\mathscr{R}f\}$$

where x^0 is the inverse of x in S^0, and define multiplication in T by

$$(e, x, f)\ (g, y, h) = (eaa^0, a, a^0af),$$

where $a = x[f, g]y = xy$ (in fact $f^0 = x^0x$, $g^0 = yy^0$).

Let $P_1 = \{(pp^0, p^0, p^0p) \in I \times E^0 \times \Lambda \mid p \in P\}$. Then (T, P_1) is a $\mathscr{P}$-regular semigroup with a quas-ideal regular $*$-inverse transversal inomorphic to S^0.

Conversely, every $\mathscr{P}$-regular semigroup with a quasi-ideal regular inverse transversal can be constructed in this way.

Proof. Clearly $P \cap E^0 = E^0$ and $e^0 \in E^0$ and $e^0 \in V_p(e)$ for every $e \in E$. Let $T^0 = \{(yy^0, y, y^0y)\} \mid y \in S^0\}$. Then, $a[f, e]\,b = [af, eb]$ and $[f, e] = [f^0, e] = [f, e^0]$ for $a, b \in E^0$, $e \in I$ and $f \in \Lambda$. So applying Theorem 4.2 in [5] and Proposition 3 8, we have that T is a regular semigroup containing a quasi-ideal inverse transversal T^0 isomorphic to S^0. Under the following discussions, we prove the (T, P_1) is $\mathscr{P}$-regular with a regular $*$-inverse transversal T^0.

Clearly $(e, x, f) \in E_T \Leftrightarrow xf^0e^0x = x \Leftrightarrow x \in E^0$. Let

$$(p_1 p_1^0,\ p_1^0,\ p_1^0 p_1),\ (p_2 p_2^0,\ p_2^0,\ p_2^0 p_2) \in P_1$$

Then

(1) Clearly $P_1 \subseteq E_T$;

(2) $(p_1 p_1^0,\ p_1^0,\ p_1^0 p_1),\ (p_2 p_2^0,\ p_2^0,\ p_2^0 p_2)$

$= (p_1 p_1^0 p_1^0 p_1 p_2 p_2^0 p_1^0 p_2^0,\ p_1^0 p_1 p_2 p_2^0,\ p_1^0 p_2^0 p_1^0 p_1 p_2 p_2^0 p_2^0 p_2)$

$= (p_1 p_2 p_2^0 p_1^0,\ p_1^0 p_2^0,\ p_1^0 p_2^0 p_1 p_2) \in E_T$;

(3) $(p_1p_1^0, p_1^0, p_1^0p_1)(p_2p_2^0, p_2^0, p_2^0p_2)(p_1p_1^0, p_1^0, p_1^0p_1)$

$= (p_1p_2p_2^0p_1^0, p_1^0p_2^0, p_1^0p_2^0p_1p_2)(p_1p_1^0, p_1^0, p_1^0p_1)$

$= (p_1p_2p_2^0p_1^0p_1^0p_2^0, p_1^0p_2^0, p_1^0p_2^0p_1^0p_1)$

$= (p_1p_2p_2^0p_2^0p_1^0p_2^0, p_1^0p_2^0, p_1^0p_1^0p_2^0p_1^0p_1)$

$= (p_1p_2p_2^0p_2p_1p_1^0p_2^0, p_1^0p_2^0, p_1^0p_2^0p_1p_2p_1p_1^0p_1)$ (by Theorem 3.6)

$= (p_1p_2p_1(p_1p_2p_1)^0 p_1^0p_2^0, (p_1p_2p_1)^0(p_1p_2p_1)) \in p_1$. Similarly proof of Theorem 4.2 in [5], we have $(x^0x, x^0, xx^0) \in V(e, x, f)) \cap T^0$ (in fact, $(e, x, f)(x^0x, x^0, xx^0)(e, x, f)$

$= (exx^0(xx^0)^0, xx^0, (xx^0)^0xx^0xx^0)(e, x, f)$

$= (exx^0, xx^0, xx^0(e, x, f) = (exx^0, x, x^0xf) = (e, x, f)$

Similarly

$(x^0x, x^0, xx^0)(e, x, f)(x^0x, x^0, xx^0) = (x^0x, x^0, xx^0)$

Moreover,

$(e, x, f)(x^0x, x^0, xx^0) = (exx^0, xx^0, xx^0) = (e, xx^0, xx^0)(ee^0, e^0 e^0e) \in P_1$ and $(x^0x, x^0, xx^0)(e, x, f) = (x^0x, x^0x, x^0xf) = (x^0x, x^0x, f) = (ff^0, f^0 f^0f) \in P_1$ and

$(e, x, f)(p_1p_1^0, p_1^0, p_1^0p_1)(x^0x, x^0, xx^0)$

$= (exp_1^0p_1^0x^0, xp_1^0, p_1^0x^0xp_1^0p_1^0p_1 (x^0x, x^0, xx^0)$

$= (exp_1^0x^0, xp_1^0, p_1^0x^0xp_1^0p_1)(x^0x, x^0, xx^0)$

$= (exp_1^0x^0xp_1^0x^0xp_1^0x^0, xp_1^0x^0, xp_1^0x^0xp_1^0x^0xx^0)$

$= (exp_1^0x^0, xp_1^0x^0, xp_1^0x^0)$

$= (exp_1^0x^0(exp_1^0x^0)^0, (exp_1^0x^0)^0, (exp_1^0x^0)^0(exp_1^0x^0)) \in P_1 (exp_1^0x^0) \in I)$.

Similarly

$(x^0x, x^0, xx^0)(p_1p_1^0, p_1^0, p_1^0p_1)(e, x, f) \in P_1$.

So $(x^0x, x^0, x^0x) \in V_p((e, x, f))$. Therefore, (T, P_1) is $\mathscr{P}$-regular. Applying that T has a quasi-ideal inverse transversal T^0, we know that the $\mathscr{P}$-regular semigroup $T(P_1)$ has a regular $*$-transversal T^0.

Conversely, given a $\mathscr{P}$-regular semigroup $S(P)$ containing a regular $*$-quasi-ideal inverse transversal S^0 and applying Theorem 1.6 and Theorem 3.4., we know that the bands $B = E_{S(P)}$, $L_{S(P)}$, $\Lambda_{S(P)}$ and C-set P satisfy the hypotheses of the first part of the Theorem. Let $I = I_{S(P)}$ and $\Lambda_{S(P)}$. Define the $\Lambda \times I$ matrix Q by $Q[f, e] = [f, e] = fe$; since $f = f^0f$, $e = ee^0$, and S^0 is a quasi-ideal, Q is a matrix over S^0. So applying Theorem 3.6, we have $[f, e] = fe = f^0e^0$. Therefore all the conditions of the first part of the Theorem are satisfied and we can construct

$$T = \{(e, x, f) \in I \times S^0 \times \Lambda \mid e\mathscr{L}xx^0,\ f\mathscr{R}x^0x\}$$

under the multiplication specified in the first part. Let $P_1 = \{(pp^0, p^0\ p^0p) \in T \mid p \in P\}$ and $T^0 = \{(yy^0, y\ y^0y) \mid y \in S^0\}$. Then (T, P_1) is $\mathscr{P}$-regular with a regular $*$-quasi-ideal inverse transversal T^0. By Lemma 3.3, the mapping : $a \to (aa^0, a^{00}\ a^0a)$ is an $\mathscr{P}$-isomorphism from S onto T and S^0 is isomorphic to T^0.

In [15], T. Saito established a structure theorem for regular semigroups with inverse transversals. Similarly we have the following result for $\mathscr{P}$-regular semigroups with regular $*$-inverse transversals. $PT\ (A, B)$ denotes the set of all partial mappings from A into B.

Theorem 3.10. Let S^0 be an inverse semigroup with the semilattice E^0 of idempotents and (B, P) $\mathscr{P}$-regular band with a regular $*$-inverse transversal E^0, and let $I = \{aa^0 \mid a \in B\}$ and $\Lambda = \{a^0a \mid a \in B\}$. Then I and Λ are left regular band and right regular band respectively. Let $\Lambda \times I \to S^0$, $(f, e) \to f^0e^0 = f * e$. Suppose that, for each $(x, y) \in S^0 \times S^0$, there exist $\alpha_{(x,y)} \in PT\ (\Lambda \times I, I)$ and $\beta_{(x,y)} \in PT\ (\Lambda \times I, \Lambda)$ satisfying :

(a^0) $\mathrm{Dom}(\alpha_{(x,y)}) = \mathrm{Dom}(\beta_{(x,y)}) = R_{(x^0x)} \times L_{yy^0}$, $\mathrm{ran}(\alpha_{(x,y)}) \subseteq L_{xy\,(xy)^0}$ and $\mathrm{ran}\ (\beta_{(x,y)}) \subseteq R_{(xy)^0xy}$

(b^0) if $f \in R_{x^0x}$, $g \in L_{yy^0}$, $h \in R_{y^0y}$ and $k \in L_{zz^0}$, then

$$(f, g)\alpha_{(x,y)}((f, g)\beta_{(x,y)}h, k)\alpha_{(xy,z)} = (f, g(h, k)\alpha_{(y,z)})\alpha_{(x,yz)},$$

$$(f, g(h, k)\alpha_{(y,z)})\beta_{(xy,z)}(h, k)\beta_{(y,z)} = (f, g)\beta_{(x,y)}h, k)\beta_{(xy,z)}\ ;\ \text{and}$$

(c^0) $(x^0x, yy^0)\alpha_{(x,y)} = xy\,(xy)^0$ and $(x^0x, yy^0)\ \beta_{(x,y)} = (xy)^0\,(xy)$;

(d^0) $(f, e)\alpha_{(f^0, e^0)} = fef^0$ and $(f, e)\beta_{(f^0, e^0)} = e^0fe$

Define a multiplication on the set

$$W = \{(e, x, f) \in I \times S^0 \times \Lambda \mid e \in L_{xx^0},\ f \in R_{x^0x}\}$$

by $(e, x, f)\ (g, y, h) = (e(f, g)\alpha_{(x,y)}, xy, (f, g)\beta_{(x,y)}\ h)$.

Let

$$P_1 = \{(pp^0,\ p^0\ p^0p) \mid p \in P\}$$

be such that $(e,\ x,\ f)\ P_1(f^0,\ x^0,\ e^0) \subseteq P_1$ and $(f^0,\ x^0,\ e^0)\ P_1\ (e,\ x,\ f) \subseteq P_1$. Then $(W,\ P_1)$ is a $\mathscr{P}$-regular semigroup with a regular $*$-inverse transversal isomorphic to S^0, and $I_{w(p_1)} \simeq I$, $\Lambda_{w(p_1)} = \Lambda$ and $E_w(P_1) \simeq_p B(P)$. Conversely, every such semigroup can be so constructed.

Proof. According to theorem 3.6 in [9], Theorem 1.6 and Definition 3.1, we know that $(f^0,\ e)\alpha_{(f^0,e^0)} = f^0ef^0 = f^0e$ and $(f,\ e^0)\beta_{(f^0,e^0)} = e^0fe^0 = fe^0$ and w is orthodox with an inverse transversal $W^0 = \{(yy^0,\ y,\ y^0y) \mid y \in S^0\}$ isomorphic to S^0 because $(e,\ x,\ f)$ $(g,\ y,\ h) = (\ldots,\ x(f * g)\ y,\ \ldots)$. Therefore, we only need to prove that $(W,\ P_1)$ is $\mathscr{P}$-regular with a regular $*$-inverse transversal $W^0 =$ and $E_w(P_1) \simeq_p B(P)$.

Clearly, according to (a^0), we have $(e,\ x,\ f)(e,\ x,\ f) = (e,\ x,\ f)$ if and only if $x^2 = x$, and so $P_1 \subseteq E_w$.

Let $(p_1 p_1^0,\ p_1^0,\ p_1^0p_1),\ (p_2 p_2^0,\ p_2^0;\ p_2^0p_2) \in P_1$. Then

(1) Since

$$\operatorname{ran}\ (\alpha_{(p_1^0,p_2^0)}) \subseteq L_{p_1^0 p_2^0}$$

and

$$\operatorname{ran}\ (\beta_{(p_1^0,p_2^0)}) \subseteq R_{p_1^0 p_2^0}$$

$$\begin{aligned}
&(p_1 p_1^0,\ p_1^0,\ p_1^0p_1)\ (p_2 p_2^0,\ p_2^0,\ p_2^0p_2)\\
&= (p_1 p_1^0\ p_1^0p_1p_2\, p_2^0(p_1 p_1^0)^0,\ p_1^0\ p_2^0,\ (p_2 p_2^0)^0\ p_1^0p_1p_2\, p_2^0\ p_2^0p_2)\\
&= (p_1p_2\, p_2^0\, p_1^0,\ p_1^0\ p_2^0,\ p_2^0\, p_1^0p_1p_2) = (\ldots,\ p_1^0\ p_2^0,\ \ldots) \in E_w\ ;
\end{aligned}$$

further

(2) $$\begin{aligned}
&(p_1 p_1^0,\ p_1^0,\ p_1^0p_1)\ (p_2 p_2^0,\ p_2^0,\ p_2^0p_2)\ (p_1 p_1^0,\ p_1^0,\ p_1^0p_1)\\
&= (p_1p_2\, p_2^0\, p_1^0,\ p_1^0\ p_2^0,\ p_2^0\, p_1^0p_1p_2)\ (p_1 p_1^0,\ p_1^0,\ p_1^0p_1)\\
&= (p_1p_2\, p_2^0\, p_1^0\ p_2^0\, p_1^0p_1p_2p_1\, p_1^0(p_2^0\, p_1^0p_1p_2)^0,\ p_1^0\ p_2^0,\ (p_1 p_1^0)^0\, p_2^0\, p_1^0p_1p_2p_1\, p_1^0\ p_1^0p_1)\\
&= (p_1p_2\, p_2^0\, p_1^0\ p_2^0\, p_1^0p_1p_2p_1\, p_1^0\ p_2^0,\ p_1^0\ p_2^0,\ p_1^0\ p_2^0\, p_1^0p_1p_2p_1\, p_1^0\ p_1^0p_1)\\
&= (p_1p_2p_1\, p_1^0\ p_2^0,\ p_1^0\ p_2^0,\ p_1^0\ p_2^0p_1p_2p_1)\\
&= (p_1p_2p_1(p_1p_2p_1)^0,\ (p_1p_2p_1)^0,\ (p_1p_2p_1)^0p_1p_2p_1) \in P_1\ ;
\end{aligned}$$

(3) Since $(x^0x,\ x^0,\ xx^0) \in V((e,\ x,\ f))$,

$$(x^0x,\ x^0,\ xx^0)\ (e,\ x,\ f) = (f^0,\ x^0,\ e^0)\ (e,\ x,\ f)$$

$$= (f^0(e^0,\ e)\alpha_{(x^0,x)},\ x^0 \cdot x\ (e^0,\ e)\beta_{(x^0,x)}f)$$

$$= (f^0,\ f^0,\ f) = (ff^0,\ f^0,\ f^0f) \in P_1\ ;$$

Similarly $(e,\ x,\ f)\ (x^0x,\ x^0,\ xx^0) \in P_1$. $E_w(P_1) \simeq_p B(P)$.

Hence $(W,\ P_1)$ is $\mathscr{P}$-regular semigroup with a regular $\star$-inverse transversal W^0 isomorphic to S^0 by the definition of P_1.

Theorem 3.11. let $(S,\ P)$ be a strongly $\mathscr{P}$-regular semigroup. Then $S(P)$ has a regular *-inverse transversal if and only if $P = E_s$ and S is a split orthodox semigroup.

Proof. Let $e \in E_s$. Then $e^0 \in E^0 \subseteq P$ and we have the following diagram :

e	ee⁰
e⁰e	e⁰

Since ee^0, e^0 and $e^0e \in P$, $e \in P$, that is $P = E_s$. Applying Theorem 2.4, S is split orthodox semigroup. Conversely, it is obvious [2].

4. THE GENERAL CASE

In this section, we shall investigate general $\mathscr{P}$-regular semigroups with regular $\star$-transversal. In [13], Yamada and Nakagawa proved that, for every $\mathscr{P}$-regular semigroup $S(P)$, there exist a C-set P' such that $S(P')$ is a strongly $\mathscr{P}$-regular semigroup, in this case, P' is called a strong C-set. Therefore, in the following discussion, we always suppose that every $\mathscr{P}$-regular semigroup is strongly $\mathscr{P}$-regular. A partial groupoid E is called a regular partial band if there exists a regular semigroup S such that the partial groupoid E_s of idempotents of S is isomorphic to E [1].

Let P be a partial subgroupoid (that is a subset) of a regular partial band E. If P satisties the following conditions, then P is called a C-set in E :

(1) $e,\ f \in P$ implies that ef is defined in E ; (2) $P^2 = E$: and (3) for any $e,\ f \in P$, $(ef)e$ or $e(fe)$ is defined in E(hence both $(ef)e$ and $e(fe)$ are defined in E) ; further, $(ef)e = e(fe)$ is contained in P.

In this case, $(E,\ P)$ is called a regular $\mathscr{P}$-partial band and denoted by $E(P)$ ([14]). P is called a strongly C-set in E if, for any $p,\ q \in P$, $pq \in P$. In the following discussion, we construct $\mathscr{P}$-regular semigroups with regular $\star$-transversal.

Let $E(P)$ be a regular $\mathscr{P}$-partial band with C-set P and $V_p(e) = \{f \in E \mid e = efe, f = fef$ and $ef, fe \in P\}$. $f \in V_p(e)$ is called $\mathscr{P}$-inverse of e in $E(P)$. Then $E(P)$ is called a regular $\star$-partial band if $\mid V_p(e) \mid = 1$ for any $e \in E$. In this case, P is called $\mathscr{P}$-system of E. Let E be a regular partial band with C-set P. Then $E_1 \subseteq E$ is called a regular $\star$-transversal of $E(P)$ if it is a partial sub-band of E and, for any $e \in E$, $\mid V_p(e) \cap E_1 \mid = 1$. Let E_1 be a regular $\star$-partial sub-band of $E(P)$. If $E(P)$ is strong, then $\mid V_p(e) \cap E_1 \mid = 1 \Leftrightarrow V_p(e) \cap E_1 \neq \varnothing$ (its proof is similar to that Th.1.4).

Let $E(P)$ be a strongly regular $\mathscr{P}$-partial band with a regular $\star$-transversal E^0 For $e \in E$, $< e >$ denotes the set $\{x \in E \mid x \leq e\}$ ($x \leq e$ means $x = ex = xe$). Then $< e >$ is a partial sub-band of E. For $e, f \in E_s$, $< e >$ and $< f >$ are called isomorphic if there exists a bijective φ from $< e >$ onto $< f >$ such that, for any $x, y \in < e >$, xy is defined in E if and only if $(x) \varphi (y)\varphi$ is also and in this case, $(xy)\varphi = (x) \varphi (y)\varphi$. In this case $< e >$ and $< f >$ are denoted by $< e > \simeq < f >$. Further, φ is called strongly isomorphic if, for any $x \in < e >$, $(x) \varphi \in P$ if and only if $x \in P$. In this case, $< e > \simeq < f >$ is denoted by $< e > \simeq_s < f >$. Let $T_{e, f} = \{\alpha_{(e,f)} \mid \alpha_{(e,f)}$ is a strongly isomorphic from $< e >$ onto $< f >\}$ and $T_{(E,P)} = \cup_{e,f \in P} T_{(e,f)}$. Clearly, for $x \leq e$ ($x, e \in E$, $e \in P$), there exist $p,q \in P$ such that $p \leq e$, $q \leq e$ and $x = pq$. So that $\alpha_{(e,f)} = \beta_{(e,f)}$ if and only if $\alpha_{(e,f)} |_{< e > \cap P} = \beta_{(e,f)} |_{< e > \cap P}$. Therefore, in the following discussions, we always put that $< e > \subseteq P$ and $< f > \subseteq P$. Let $R_x^{(E,P)} = R_x \cap P$ and $L_x^{(E,P)} = L_x \cap P$. Sometimes, $R_x^{(E,P)}$ and $L_x^{(E,P)}$ are simply denoted by R_x and L_x respectively in no misunderstandings. Let $S(P)$ be a strongly $\mathscr{P}$-regular semigroup with a regular $\star$-transversal S^0. Then we have the following Lemma 4.1.

Lemma 4.1. For each $(x, y) \in S^0 \times S^0$, let α_x be mapping defined by $cx = x (c\alpha_x)$ (which is connecting mapping), where $c \in < xx^0 >$. Then $\alpha_x \in T_{(E_s,P)}$ and

(1) for each $(e, f) \in R_{x^0x}^{(E_s,P)} \times L_{yy^0}^{(E_s,P)}$, $(efe^0)\alpha_x^{-1} \in L_{xy(xy)^0}^{(E_s,P)}$ and $(f^0ef)\alpha_y \in R_{(xy)^0xy}^{(E_s,P)}$;

(2) if $f \in R_{x^0x}^{(E_s,P)}$, $g \in L_{yy^0}^{(E_s,P)}$, $h \in R_{y^0y}^{(E_s,P)}$ and $k \in L_{zz^0}^{(E_s,P)}$, then

$$(fgf^0)\alpha_x^{-1} \; [(gf^0g)\alpha_y hk((gf^0g)\alpha_y h)^0]\,\alpha_{xy}^{-1} = [f(g(hkh^0)\,\alpha_y^{-1}f^0]\,\alpha_x^{-1} \text{ and}$$

$$[k^0(g^0fg)\alpha_y hk]\alpha_z$$

$$= ([g(hkh^0)\,\alpha_y^{-1}]^0 \; f\,[g(hkh^0)\,\alpha_y^{-1}])\alpha_{yz} \, (k^0hk)\alpha_z \, ;$$

(3) $(x^0xyy^0x^0x) \; \alpha_x^{-1} = xy(xy)^0$ and

$(yy^0x^0xyy^0)\alpha_y = (xy)^0xy$ for any $x, y \in S^0$;

(4) if S^0 is an inverse transversal of S, then $(f^0ef)\alpha_{f^0}^{-1} = f^0e$ and $(efe^0)\alpha_{e^0} = fe^0$ for any $(f, e) \in \Lambda_{(E_s,P)} \times I_{(E_s,P)}$;

(5) S^0 is a regular $*$-subsemigroup such (4)′ $(f^0ef)\alpha_{f^0}^{-1} = f^0e$ and $(efe^0) = \alpha_{e^0} = fe^0$ for any $(f, e) \in \Lambda_{(E_s,P)} \times I_{(E_s,P)}$ if and only if S^0 is an inverse transversal.

Proof. Let $W = \{(e, x, f) \mid x \in S^0, e\mathscr{L}xx^0, x^0x\mathscr{R}f\}$. Defined multiplicative on W' by

$$(e, x, f)\,(g, y, h) = (e(fgf^0)\alpha_x^{-1}, xy, (g^0fg)\alpha_y h).$$

Defined a mapping φ from S into W :

$$a \to (aa^0, a^{00}, a^0a).$$

Since

$$(aa^0, a^{00}, a^0a)\,(bb^0, b^{00}, b^0b) = (aa^0(a^0abb^0a^0a^{00})\alpha_{a^{00}}^{-1}, a^{00}b^{00}, ((bb^0)^0a^0abb^0)\alpha_{b^{00}}b^0b)$$

$$= (aa^0, a^{00}(a^0abb^0a^0a^{00})a^0, a^{00}b^{00}, b^0(b^{00}b^0a^0abb^0)b^{00}b^0b) = (abb^0a^0, a^{00}b^{00}, b^0a^0ab), (ab)\varphi$$

$= a\varphi b\varphi$. Applying Lemma 3.3., φ is an isomorphism between S and W. So W is a semigroup on the multiplicative above.

(1) For each $(e,f) \in R_{x^0x}^{(E_s,P)} \times L_{yy^0}^{(E_s,P)}$, $(efe^0)\alpha_x^{-1} = x(efe^0)x^0$. Therefore $e^0 = x^0x$ and $f^0 = yy^0$ and $xefe^0x^0xy(xy)^0 = xefe^0x^0xyy^0x^0 = xefx^0xyy^0x^0xx^0 = xefyy^0x^0xyy^0x^0xx^0 = xefyy^0x^0xx^0 = xefe^0x^0$

and

$xy(xy)^0\,(xefe^0x^0) = xyy^0x^0xefe^0x^0 = xyy^0efe^0x^0 = xyy^0fefe^0x^0 = xyy^0fefx^0 = xyy^0fefex^0 = xyy^0fex^0$ $= xyy^0x^0$, so that $(efe^0)\,\alpha_x^{-1} \in L_{xy(xy)^0}^{(E_s,P)}$. Similarly

$$(f^0ef)\,\alpha_y \in R_{xy^0(xy)}^{(E_s,P)}$$

(2) and (3) are clear because $S \cong W$;

(4) Since $\Lambda_{(E_s,p)}$ and $I_{(E_s,p)}$ are a right regular band and a left regular band respectively, it is clear ;

(5) Suppose that S^0 is such that (4)′. Let $e, f \in I_{(E_s,p)}$. Then $ef(ef)^0 = eff^0e^0 = efe^0 = ee^0fe^0 = ee^0f = ef =$ So $ef \in I_{(E_s,p)}$, i.e., $I_{(E_s,p)}$ is a sub-band of S. Similarly, $\Lambda_{(E_s,p)}$ is also a sub-band of S. Applying Theorem 1.6, S^0 is an inverse transversal. Conversely, it is clear.

Theorem 4.2. Let S^0 be a regular $*$-semigroup with $\mathscr{P}$-system P^0, (E, P) a strongly regular $\mathscr{P}$-partial band with C-set P such that E_{S^0} is a regular $*$-transversal of $E(P)$ and $P^0 = P \cap E_{S^0}$ and P satisfies that $p \in P$ implies $p^0 \in P^0$ (p^0 is the unique element in $V_P(p) \cap E_{S^0}$).

Let $I_{(E,P)} = \cup \{L_a^{(E,P)} \mid a \in P\}$, $\Lambda_{(E,P)} = \cup \{R_a^{(E,P)} \mid a \in P\}$. Let $\Lambda \times I \to S^0$, $(f, e) \to f * e = f^0 e^0$ be a mapping. Let α be a mapping from S^0 into $T_{(E,P)}$ such that $(x)\alpha = \alpha_x \in T_{xx^0,\, x^0x}$ for any $x \in S^0$, and

(a*) for each $(e, f) \in R^{(E,P)}_{x^0x} \times L^{(E,P)}_{yy^0}$, $(efe^0)\alpha_x^{-1} \in L^{(E,P)}_{xy(xy)^0}$ and $(f^0ef)\alpha_y \in R^{(E,P)}_{(xy)^0xy}$; especially, if $y, x \in P^0$, then $(e)\alpha_{xy} = (xy)^0 exy$, where $e \in \langle xy(xy)^0 \rangle$;

(b*) if $f \in R^{(E,P)}_{x^0x}$, $h \in L^{(E,P)}_{yy^0}$, $h \in R^{(E,P)}_{y^0y}$ and $k \in R^{(E,P)}_{zz^0}$, then

$$(fgf^0)\alpha_x^{-1}\ [(g^0fg)\alpha_y hk\ ((g^0fg)\alpha_y h)^0]\alpha_{xy}^{-1} = [f(g(hkh^0)\alpha_y^{-1}\ f^0]\alpha_x^{-1}$$

and

$$[k^0(g^0fg)\alpha_y h)k]\alpha_z = ([(g(hkh^0)\alpha_y^{-1}]^0 f\, [g(hkh^0)\alpha_y^{-1}])\alpha_{yz}(k^0hk)\alpha_z\ ;$$

(c*) $(x^0xyy^0x^0x)\ \alpha_x^{-1} = (xy)\ (xy)^0$ and $(yy^0x^0xyy^0)\alpha_y = (xy)^0\, xy$.

Define a mulitification on the set

$$W = \{(e, x, f) \in I_{(E,P)} \times S^0 \times \Lambda(E,P) \mid e \in L^{(E,P)}_{xx^0},\ f \in R^{(E,P)}_{x^0x}\}$$

by

$$(e, x, f)\ (g, y, h) = (e\ (fgf^0)\alpha_x^{-1},\ xy,\ (g^0fg)\ (\alpha_y h).$$

Let $\overline{P} = \{(pp^0, p^0, p^0p) \mid p \in P\}$. Then $(W, \overline{P})$ is a strongly $\mathscr{P}$-regular semigroup with regular $*$-transversal $\mathscr{P}$-isomorphic to (S^0, P^0).

Conversely, every strongly $\mathscr{P}$-regular semigroup with a regular $*$-transversal can be constructed in this way.

Proof. We can easily show, by using (a*) and (b*), that W is a semigroup on the multiplication above (its proof is similar to TH. 2.5 in [9]). Let $W^0 = \{(e,x,f) \in W \mid e, f \in P^0\}$. Then $(e, x, f) \in W^0$ if and only if $e = xx^0$ and $f = x^0x$. By (c*), we obtain

$$(xx^0, x, x^0x)\ (yy^0, y, y^0y) = (xy\,(xy)^0, xy, (xy)^0\,xy),$$

which shows $(W^0, \overline{P^0}) \cong_p (S^0, P^0)$, where $\overline{P^0} = \{(p^0, p^0, p^0) \mid p^0 \in P^0\}$. So W^0 is a regular $\star$-subsemigroup of W.

Clearly, we have easily the following two results :

1. $(e, x, f) \in E_W$ if and only if $x \in E_{S^0}$;

2. $(e, x, f) \in \overline{P}$ if and only if $x \in P^0$ (in fact, (E,P) is strong and $x \in P^0$ implies $ef \in P$ and $x = fe$. So $(e, x, f) = (e, fe, f) = (ef\,(ef)^0, (ef)^0, (ef)^0\,ef)$).

For $(e, x, f) \in W$, we have $xx^0 = e^0$ and $(f, f^0) \in R^{(E,P)}_{x^0x} \times L^{(E,P)}_{x^0x}$. So $(f\,f^0f^0)\,\alpha_x^{-1} \in L^{(E,P)}_{xx^0=e^0}$ and $(f^0\,f\,f^0)\,\alpha_{x^0} \in R^{(E,P)}_{xx^0=e^0}$. Thus we have

$$(e, x, f)\ (f^0, x^0, e^0) = (e\ (f\,f^0f^0)\,\alpha_x^{-1},\ xx^0,\ (f^0\,ff^0)\,\alpha_{x^0}e^0),$$

$$= (e, e^0, e^0) = (ee^0, e^0, e^0e) \in \overline{P}$$

since $e \in P$. Moreover, $(e^0, e) \in R^{(E,P)}_{xx^0} \times L^{(E,P)}_{xx^0}$. Thus

$$(e^0\ e\ e^0)\ \alpha_{e^0}^{-1} \in L^{(E,P)}_{e^0x(e^0x)^0} = L^{(E,P)}_{e^0}$$

and

$$(e^0\,e^0\,e)\alpha_x \in R^{(E,P)}_{(e^0x)^0e^0x=x^0x=f^0}.$$

Thus we have

$$(e, x, f)\ (f^0, x^0, e^0)\ (e, x, f) = (e, e^0, e^0)\ (e, x, f)$$

$$= (e(e^0ee^0)\,\alpha_{e^0}^{-1},\ x,\ (e^0e^0e)\alpha_x f = (e, x, f)$$

Similarly, we have

$$(f^0, x^0, e^0)(e, x, f)(f^0, x^0, e^0) = (f^0, x^0, e^0).$$

Further, $(f^0, x^0, e^0)(e, x, f) = (f^0, f^0, f) \in \overline{P}$, and clearly $\overline{P} = \{(pp^0, p^0, p^0p) \mid p \in P\} \subseteq E_W$.

In the following discussions, we show that $(W, \overline{P})$ is strongly $\mathscr{P}$-regular with a regular $*$-transversal $(W^0, \overline{P^0})$.

(i) For any $p, q \in P$, we have

$$(pp^0, p^0, p^0p)(qq^0, q^0, q^0q) = (\ldots p^0q^0, \ldots) \in E_W.$$

since $p^0q^0 \in E$;

(ii) Since

$$(pp^0, p^0, p^0p)(qq^0, q^0, q^0q)(pp^0, p^0, p^0p) = (\ldots p^0q^0p^0, \ldots),$$

it is in $\overline{P}$ because $p^0q^0p^0 \in P$;

(iii) Since

$$(f^0, x^0, e^0)(pp^0, p^0, p^0p)(e, x, f) = (\ldots, x^0p^0x, \ldots),$$

it is in $\overline{P}$

Similarly $(e, x, f)(pp^0, p^0, p^0p)(f^0, x^0, e^0) \in \overline{P}$. Consequently, (f^0, x^0, e^0) is an $\mathscr{P}$-inverse of (e, x, f) in W^0. Therefore $(W, \overline{P})$ is $\mathscr{P}$-regular.

Let $(g, y, h) \in W^0$ be an $\mathscr{P}$-inverse of $(e, x, f) \in W$. Then

$$(e, x, f) = (e, x, f)(g, y, h)(e, x, f) = (\ldots, xyx, \ldots)$$

and

$$(g, y, h) = (g, y, h)(e, x, f)(g, y, h) = (\ldots, yxy, \ldots)$$

So $x = xyx$ and $y = yxy$. Clearly y is a $\mathscr{P}$-inverse of x in $W^0(\overline{P^0})$. Therefore, $y = x^0$, $g = x^0x$ and $y = xx^0$. Thus each element $(e, x, f) \in W$ has the unique $\mathscr{P}$-inverse (f^0, x^0, e^0) in W^0. So $(W, \overline{P})$ has a regular $*$-transversal $(W^0, \overline{P^0})$. Put (pp^0, p^0, p^0p),

$(qq^0, q^0, q^0q) \in \overline{P}$ for any $p, q \in P$. If $(pp^0, p^0, p^0p)\,(qq^0, q^0, q^0q) \in \overline{P}$, then

$$(pp^0, p^0, p^0p)\,(qq^0, q^0, q^0q) = (\ldots\, p^0q^0, \ldots) \in \overline{P}.$$

So $p^0q^0 \in P^0$. But (S^0, P^0) is a regular $*$-semigroup. Thus $p^0q^0 \in P^0$, and it implies

$$(qq^0, q^0, q^0q)\,(pp^0, p^0, p^0p) = (\ldots\, q^0p^0, \ldots) \in \overline{P}.$$

Therefore $(W, \overline{P})$ is strongly $\mathscr{P}$-regular with a regular $*$-transversal $(W^0, \overline{P^0})$. Conversely, it is clear by the first part of the proof of Lemma 4.1.

REFERENCES

1. G. B. baid, On semigroups and unigform partial bands, Semigroup Forum 4 (1972) : 185-188.
2. D. B. McAlister and T. S. Blyth, Split orthodox semigroups, J. Algebra 51 (1978) : 491-525.
3. T. S. Blyth and R. McFadden, Naturally ordered regular semigroups with a greatest idempotent, Proc. Royal Society of Edingburgh 91A (1981) : 107-122.
4. —, Regular semigroups with a multiplicative inverse transversals, Proc. Royal Society of Edingburg 92A (1982) : 253-270.
5. D. B. McAlister and R. B. NcFadden, Regular semigroups with inverse transversal, Quart. J. Math. Oxford (2) 34 (1983) : 459-474.
6. —, Maximum idempotents in naturally ordered regular semigroups, Proc. Edingburg Math. Society 26 (1983) : 213-220.
7. T. Saito, Structure of regular semigroups with a quasi-ideal inverse transversal, Semigroup Forum Vol. 31 (1985) : 305-309.
8. —, A note on regular semigroups with inverse transversal, Semigroup Forum Vol. 33 (1986) : 149-152.
9. —, Construction of regular semigroups with inversal transversal, Proc. Edingburg Math. Society 32 (1989) : 41-51.
10. M. Yamada, $\mathscr{P}$-systems in regular semigroups, Semigroup Forum 24 (1982) : 173-178.
11. —, and M. K. Sen, $\mathscr{P}$-regular semigroups, Semigroup Forum Vol. 39 (1989) : 157-178.

12. Xiling Tang, regular semigroups with inversetransversal, Semigroup Forum Vol. 55 (1997) : 24-32.
13. M. Yamada and H. Nakagawa, Some remarks on $\mathscr{P}$-regular semigroups, J. Pure Math. and Appl. Math., SerA, 2(1991) : 147-155.
14. Ronghua Zhang, The weakly $\mathscr{P}$-regular semigroups and its generalizations, D. Phil. Thesis, Univ. Sichuan 1997.
15. M. Yamada, $\mathscr{P}$-regularity in semigroups, and the structure of $\cap$ $\mathscr{P}$-regular semigroups, Proc. of the First International Symposium on Algebraic Structure and Number Theory, HongKong, (1988) : 331.
16. H. W. Zheng, Strong $\mathscr{P}$-congrucences on $\mathscr{P}$-regular semigroups, Semigroup Forum Vol. 51 (1995) : 217-223.

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FIXED POINT THEOREM ON P-METRIC SPACE AND RELATED PROBLEMS

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INTRODUCTION

A complete pseudometric space was introduced by G. Kureppa [5]. This pseudometric space was completely different from the usual one. L Collatz [2] obtained a fixed point theorem for an operator on this pseudometric space. But the fixed point theorem of Collatz suffered from the limitation that it could not be applied easily to given problems.

In this paper, we obtain a common fixed point theorem for the same iteration of two operators on a pseudometric space. The corresponding fixed point theorem for a single operator on the pseudometric space is given in the form of a corollary. As an application of this corollary, it is shown that if two operators on two pseudometric spaces satisfy conditions which ensure their fixed points, then it is possible to construct an operator on the tensor product of the given pseudometric spaces such that it possesses a fixed point on the tensor product space.

To avoid confusion, we replace the word pseudometric by P-metric and limit our discussions in four sections. Section 1 contains some definitions as given by L. Collatz [2]. Section 2 considers the tensor product of two P-metric spaces. Section 3 contains our main common fixed point theorem and its corollary. Section 4 shows an application of this corollary.

1. SOME KNOWN DEFINITIONS

Definition 1.1. A sequence $\{\rho_n\}$ of a partially ordered linear space H over F is said to converge to a limit element ρ in H (written as $\rho_n \to \rho$), if $\{\rho_n\}$ satisfies the following properties :

(i) $\{\rho_n\} = \{\sigma\}$ (σ being a fixed element of H) $\Rightarrow \rho_n \to \sigma$.

(ii) $\rho_n \to \rho \Rightarrow \rho_{n_m} \to \rho$ ($\{\rho_{n_m}\}$ being a subsequence of $\{\rho_n\}$)

(iii) $\rho_n \to \rho, c_n \to c$ ($\{c_n\} \subset F$, $c \in F$) $\Rightarrow c_n\rho_n \to c\,\rho$

(iv) $\rho_n \to \rho, \sigma_n \to \sigma \Rightarrow \rho_n + \sigma_n \to \rho + \sigma$

(v) $\rho_n \to \rho, \rho_n \geq \theta_H \Rightarrow \rho \geq \theta_H$, where θ_H is the additive identity of H

(vi) $\theta_H \leq \rho_n \leq \sigma_n$ and $\sigma_n \to \theta_H \Rightarrow \rho_n \to \theta_H$, where $\{\sigma_n\} \subset$ H.

Definition 1.2. A space X is called a P-metric space if any two elements f, g of X are associated with a map ρ such that ρ (f, g) is an element of a partially ordered linear space H over the field F and ρ (f, g) satisfies the following properties :

(i) ρ (f, g) $\geq \theta_H$

(ii) ρ (f, g) = ρ (g, f)

(iii) ρ (f, h) $\leq \rho$ (f, g) + ρ (g, h) ; where h $\in$ X

Definition 1.3. A sequecnce $\{f_n\} \subset (X, \rho, H)$ is said to converge to a limit element f in X (written as $f_n \to f$) if

(i) $\rho(f_n, f) \to \theta_H$.

If X is also a linear space over a field F, then $\{f_n\}$ is said to converge to f, if in addition to (i), the following conditions hold :

(ii) $f_n \to f, g_n \to g$, ($\{f_n\} \subset X$, $\{g_n\} \subset X$, f, g $\subset$ X) $\Rightarrow f_n + g_n \to f + g$

(iii) $f_n \to f, c_n \to c$, ($\{c_n\} \subset F$, $c \in F$) $\Rightarrow c_n f_n \to c f$

Definition 1.4. A sequence $\{f_n\} \subset (X, \rho, H)$ is said to be a Cauchy sequence, if $\lim\limits_{m, n\to\infty} \rho(f_m, f_n) \to \theta_H$.

Definition 1.5. A proper (improper) subset N of a P-metric space (X, ρ, H) is said to be complete, if for every Cauchy sequence $\{f_n\} \subset$ N, there exists some f $\in$ N such that $\rho(f_n, f) \to \theta_H$.

Definition 1.6. Let (X, ρ, H) and (Y, σ, L) be two P-metric spaces. An operator T : D $\subset$ X $\to$ Y is said to be continuous if for every $\{f_n\} \subset$ D such that $f_n \to f \in$ D, $Tf_n \to Tf$ in Y.

Definition 1.7. Let (X, ρ, H) and (Y, σ, L) be two partially ordered linear F-metric spaces over F, then $T : D \subset X \to Y$ is said to be positive if for every $f \in X$ such that $f \geq \theta_x$, $Tf \geq \theta_y$, where θ_x and θ_y are the additive identities of X and Y respectively.

Definition 1.8. Let (X, ρ, H) and (Y, σ, L) be two P-metric spaces. An operator $T : D \subset X \to Y$ is said to be bounded if there exists a positive linear continuous operator $Q : H \to L$ such that

$\sigma(Tf_1, Tf_2) \leq Q\rho(f_1, f_2)$, $\forall f_1, f_2 \in D$. Obviously, if T is bounded, then T is continuous.

Definition 1.9. Let (X, ρ, E) be a P-metric space where E is a partially ordered linear space over the field F. Then a map $f : X \to E$ is said to be bounded if for any x_1, x_2, of X, there exists a positive, continuous, linear operator Q on E such that $f(x_1) - f(x_2) \leq Q\,\rho\,(x_1, x_2)$.

2. TENSOR PRODUCT OF TWO P-METRIC SPACES

2.1. DEFINITION OF TENSOR PRODUCT

Let (X, ρ, E) and (Y, σ, E) be two linear P-metric spaces over the field F where E is a partially ordered normed algebra over F. Let X^* consist of all bounded linear maps $f : X \to E$. Then X^* is a linear space over F under pointwise operations when $\| f \|$ is defined as $\| f \| = \inf \| Q \|$, where $f(x_1) - f(x_2) \leq Q\rho\,(x_1, x_2)$, $\forall\, x_1, x_2 \in X$ and for some positive continuous linear operator Q on E and $\| Q \| = \inf \alpha$, $\alpha > 0$ such that $\| Q\,(x) \cdot \|_E \leq |\, \alpha \,| \leq \|\, x \,\|_E$, $\forall\, x \in E$. Similarly, Y^* may be defined and shown to be a normed linear space over F. Now, let $BL\,(X^*, Y^*\,;\,E) = \{\phi \,/\, \phi : X^* \times Y^* \to E$, where ϕ is bilinear and $\|\, \phi\,(f, g)\, \| \leq \beta \,\|\, f \,\|\, \|\, g \,\|$, for some $\beta > 0$, $\forall\, f \in X^*$, $\forall\, g \in Y^*\}$. Then $BL(X^*, Y^*; E)$ is a normed linear space over F under pointwise operations and $\| \phi \|$ defined as $\| \varphi \| = \inf \beta$. We now make the following definition:

Definition 2.1.1. Let $x \in X$, $y \in Y$; let $x \otimes y$ be an element of $BL(X^*, Y^*\,;\,E)$ such that $(x \otimes y)\,(f, g) = f(x)\,g\,(y)$, $\forall f \in X^*$, $\forall\, g \in Y^*$. Then the algebraic tensor product $X \otimes Y$ of X and Y is defined to be the linear span of $\{\, x \otimes y : x \in X, y \in Y \}$ in $BL\,(X^*, Y^*\,;\,E)$.

2.2. SOME RESULTS ON TENSOR PRODUCT

Proposition 2.2.1. Let (X, ρ, E) and (Y, σ, E) be two linear P-metric spaces over F where E is a partially ordered normed algebra over F. Then $(X \otimes Y, d, E)$ is a complete linear P-metric space over F, where for any two elements u, v of $X \otimes Y$,

$$u = \sum_{i=1}^{n} x_i \otimes y_i,\ v = \sum_{j=1}^{n} (x'_j \otimes y'_j),\ d\,(u, v) = \sum_{r=1}^{n} \rho\,(x_r, x'_r) + \sum_{r=1}^{n} \sigma\,(y_r, y'_r)$$

Proof. Clearly, $d(u, v) \geq \theta_E$, $d(u, v) = \theta_E$ iff $u = v$, and $d(u, v) = d(v, u)$.

Let $w = \sum_{j=1}^{n} x''_j \otimes y''_j$ be any other element in $X \otimes Y$.

Then

$$d\,(u, w) = \sum_{r=1}^{n} \rho\,(x_r, x''_r) + \sum_{r=1}^{n} \sigma\,(y_r, y''_r) \leq \sum_{r=1}^{n} \rho\,(x_r, x'_r) + \sum_{r=1}^{n} \rho\,(x'_r, x''_r)$$

$$+ \sum_{r=1}^{n} \sigma\,(y_r, y'_r) + \sum_{r=1}^{n} \sigma\,(y'_r, y''_r)$$

$$= \{\sum_{r=1}^{n} \rho\,(x_r, x'_r) + \sum_{r=1}^{n} \sigma\,(y_r, y'_r)\} + \{\sum_{r=1}^{n} \rho\,(x'_r, x''_r) + \sum_{r=1}^{n} \rho\,(y'_r, y''_r)\}$$

$$= d(u, v) + d(v, w)$$

Hence $d : (X \otimes Y) \times (X \otimes Y) \to E$ is a P-metric on $X \otimes Y$. Now let $\{u_m\}$ be an arbitrary Cauchy sequence in $X \otimes Y$. Then $d(u_m, u_n) \to \theta_E$ as $m, n \to \infty$ where

$$u_m = \sum_{i=1}^{\alpha} x_i^{(m)} \otimes y_i^{(m)} \text{ and } u_n = \sum_{i=1}^{\alpha} x_i^{(n)} \otimes y_i^{(n)}.$$

Now $d(u_m, u_n) = d(\sum_{i=1}^{\alpha} x_i^{(m)} \otimes y_i^{(m)}, \sum_{i=1}^{\alpha} x_i^{(n)} \otimes y_i^{(n)})$

$$= \sum_{i=1}^{\alpha} \rho(x_r^{(m)}, x_r^{(n)}) + \sum_{i=1}^{\alpha} \sigma(y_r^{(m)}, y_r^{(n)}).$$

So there exists a positive integer N such that $d(u_m, u_n) \to \theta_E$, $\forall$ m, n $\geq$ N. Hence $\rho(x_r^{(m)}, x_r^{(n)}) \to \theta_E$, and $\sigma(y_r^{(m)}, y_r^{(n)}) \to \theta_E$ as m, n→∞. Hence for each r, $\{x_r^{(m)}\}$ and $\{y_r^{(m)}\}$ are Cauchy sequences in X and Y respectively. But X and Y are complete. So, $x_r^{(m)} \to x_r$ (say). $y_r^{(m)} \to y_r$ (say) as m→∞ where $x_r \in X$ and $y_r \in Y$ for each r = 1, 2,..., α.

Let $u = \sum_{r=1}^{\alpha} x_r \otimes y_r$. Then $u \in X \otimes Y$. We now show that $u_m \to u$ as m→∞. In fact, $d(u_m, u) = d(\sum_{i=1}^{\alpha} x_i^{(m)} \otimes y_i^{(m)}, \sum_{r=1}^{\alpha} x_r \otimes y_r)$

$= \sum_{j=1}^{\alpha} \rho\ (x_j^{(m)}, x_j) + \sum_{j=1}^{\alpha} \sigma\ (y_j^{(m)}, y_j) \to \theta_E$ as m→∞, (Since for each r = 1, 2, ..., α, $x_r^{(m)} \to x_r$ and $y_r^{(m)} \to y_r$ as m→∞).

So, $u_m \to u$ as m→∞. Thus $(X \otimes Y, d\ ; E)$ is a complete P-metric space.

3. COMMON FIXED POINT THEOREM IN A P-METRIC SPACE

Theorem 3.1. Let (X, ρ, E) be a complete P-metric space where E is a normed linear space over F. Let $x_0 \in X$ be given and let $S = \{u \in X / \rho\ (u, x_0) \leq e$, for some $e \geq \theta_E\}$. Let I be the identity operator on E. Let T_1^m and T_2^m (m ≥ 1) : $D \subset X \to X$ be such that

(i) $\rho(T_1^m x, T_2^m y) \leq Q\rho\ (x, y)$ ...... (3.1)

for all $x, y \in S$ and for some positive continuous linear operator Q on E such that $Q < I$, $I-Q$ is injective and $\{Q^n\}$ is convergent in BL(E, E).

(ii) $\rho(x_0, T_1^m x_0) < (I-Q)\ e$ ...... (3.2)

Then T_1^m and T_2^m have a unique common fixed point in S.

Proof. Let $x_n = T_1^m(x_{n-1})$ when n is an odd integer and $x_n = T_2^m(x_{n-1})$ when n is an even integer. First we verify that $x_n \in S$ if n is even. As $e \geq \theta_E$, so $Q(e) \geq Q(\theta_E) = \theta_E$.

Thus $-Q(e) \leq \theta_E$. Hence, (I-Q) e ≤ e. Now by (3.2),

$$\rho(x_0, x_1) < (I\text{-}Q)\, e \qquad \text{......} \; (3.3)$$

So $\rho(x_0, x_1) < (I\text{-}Q)\,e$. Hence $x_1 \in S$. Let us asume that $x_1, x_2, \ldots, x_{n-1} \in S$. We now show that $x_n \in S$. Using (3.1) and (3.2), we have

$$\rho(x_1, x_2) = \rho(T_1^m(x_0), T_2^m(x_1)) \leq Q\rho(x_0, x_1) < Q\,(I\text{-}Q)\,e$$

$$\rho(x_2, x_3) = \rho(T_2^m(x_1), T_1^m(x_2)) \leq Q\rho(x_1, x_2) < Q^2\,(I\text{-}Q)\,e$$

$$\rho(x_{n-1}, x_n) = \rho(T_1^m(x_{n-2}), T_2^m(x_{n-1})) \leq Q\rho(x_{n-2}, x_{n-1}) < Q^{n-1}\,(I\text{-}Q)\,e$$

So, $\rho(x_0, x_n) \leq \rho(x_0, x_1) + \rho(x_1, x_2) + \ldots\ldots + \rho(x_{n-1}, x_n)$

$< (I\text{-}Q)\,e + Q\,(I\text{-}Q)\,e + Q^2\,(I\text{-}Q)\,e + \ldots\ldots + Q^{n-1}\,(I\text{-}Q)\,e = (I\text{-}Q^n)\,e,$

Again $e \geq \theta_E \Rightarrow Q^n(e) \geq \theta_E$, so $(I\text{-}Q^n)\,e \leq e$.

Thus $\rho(x_0, x_n) < e$. Hence, $x_n \in S$. It may be similarly shown that $x_n \in S$ when n is odd. So $\{x_n\} \subseteq S$, $\forall\, n \in N$. We now show that $\{x_n\}$ is a Cauchy sequence in S. We have,

$\rho(x_1, x_2) \leq Q\rho(x_0, x_1)$; $\rho(x_2, x_3)$; $\leq Q^2\rho(x_0, x_1), \ldots, \rho(x_n, x_{n-1}) \leq Q^n\rho(x_0, x_1)$.

So, $\rho(x_n, x_{n+p}) \leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \ldots + \rho(x_{n+p-1}, x_{n+p})$

$\leq (Q^n + Q^{n+1} + \ldots\ldots + Q^{n+p-1})\,\rho(x_0, x_n)$

Hence

$$[(Q^n + Q^{n+1} + \ldots\ldots + Q^{n+p-1})\,\rho(x_0, x_1) - \rho(x_n, x_{n+p})] \geq \theta_E \qquad \text{......} \; (3.4)$$

Again $I > Q \Rightarrow Ix > Qx$, $\forall\, x > \theta_E$ i.e. $(I\text{-}Q)\,x > \theta_E$, $\forall\, x > \theta_E$. So (I-Q) is a positive operator. Hence from (3.4), it follows that

$$(I\text{-}Q)\,[(Q^n + \ldots\ldots + Q^{n+p-1})\,\rho(x_0, x_1) - \rho(x_n, x_{n+p})] \geq \theta_E \qquad \text{......} \; (3.5)$$

So $(Q^n - Q^{n+p})\,\rho(x_0, x_1) - (I\text{-}Q)\,\rho(x_n, x_{n+p}) \geq \theta_E$

i.e. $(I\text{-}Q)\,\rho(x_n, x_{n+p}) \leq (Q^n - Q^{n+p})\,\rho(x_0, x_1)$.

Now as n → ∞, R. H. S. tends to θ_E, for p = 1, 2, 3 ... ($\{Q^n\}$ being convergent in BL(E,E)).

So (I-Q) $\rho(x_n, x_{n+p}) \to \theta_E$, as n → ∞, p = 1, 2, 3 ... . Since I ≠ Q and (I-Q) is injective, so $\rho(x_n, x_{n+p}) \to \theta_E$, as n → ∞, p = 1, 2, 3 ... . Hence $\{x_n\}$ is a Cauchy sequence in S. As S is complete, So $x_n \to x \in S$. We now show that x is a common fixed point of T_1^m and T_2^m. In fact,

$$\rho(x, T_1^m x) \leq \rho(x, x_n) + \rho(x_n, T_1^m x)$$

$$= \rho(x, x_n) + \rho(T_2^m x_{n-1}, T_1^m x) \text{ (taking n even)}$$

$$\leq \rho(x, x_n) + Q\rho(x_{n-1}, x).$$

Now $\rho(x, x_n) \to \theta_E$, and $\rho(x_{n-1}, x) \to \theta_E$, as n → ∞.

Again $\rho(x_{n-1}, x) \to \theta_E \Rightarrow Q\rho(x_{n-1}, x) \to \theta_E$, (as Q is continuous on E).

So $\rho(x, T_1^m x) = \theta_E$ and hence $T_1^m x = x$. Similarly, taking n as an odd positive integer, it may be shown that $T_2^m x = x$. Thus x is a common fixed point of T_1^m and T_2^m. Now we prove that x is unique. If possible, let $T_1^m y = y = T_2^m y$, for some $y \in S$ where $x \neq y$. Then $\rho(x, y) = \rho(T_1^m x, T_2^m y) \leq Q\rho(x, y)$. So (I-Q) $\rho(x, y) \leq \theta_E$. But (I-Q) is a positive operator and $\rho(x, y) \geq \theta_E$. Hence (I-Q) $\rho(x, y) \geq \theta_E$. Therefore, (I-Q) $\rho(x, y) = \theta_E$. As I ≠ Q and I-Q is injective, so $\rho(x, y) = \theta_E$ and hence $x = y$ This completes the proof.

Corollary 3.2. Let (x, ρ, E) be a complete P-metric space where E is a normed linear space over F. Let $x_0 \in X$ be given and let $S = \{u \in X / \rho(u, x_0) \leq e$, for some $e \geq \theta_E\}$. Let I be the identity operator on E and T : D ⊂ X → X be such that:

(i) $\rho(Tx, Ty) \leq Q\rho(x, y)$,

for all $x, y \in S$ and some positive continuous linear operator Q on E such that Q < I, I-Q is injective and $\{Q^n\}$ is convergent in BL (E, E).

(ii) $\rho(x_0, Tx) < (I-Q)e$.

Then T has a unique fixed point is S.

Remark 3.3. The conditions of Theorem 3.1 are sufficient but not necessary.

Example 3.4. Let R^2 be the 2-dimensional real vector space. Let $x, y \in R^2$ where $x = (x_1, x_2)$, $y = (y_1, y_2)$. Let $x \leq y$ if $x_i \leq y_i$, i = 1 2. Let $\rho(x, y) = (|x_1 - y_1|, |x_2 - y_2|)$.

Let two operators T_1 and T_2 on R^2 be defined as $T_1(x) = 2x$, $T_2(y) = \frac{y}{4}$, $x, y \in R^2$.

Obviously, R^2 is a P-metric space with associated partially ordered linear space R^2 itself. Also origin is a common fixed point of T_1^m and T_2^m, where m is a finite positive integer.

But T_1^m, T_2^m do not satisfy the conditions of Theorem 3.1, which may be shown as follows.

If possible, let the conditions of Theorem 3.1 be satisfied by T_1^m and T_2^m, for some positive integer m. Then for any two points $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$

$$\begin{aligned} \rho(T_1^m z_1, T_2^m z_2) &= \left(\left| 2^m x_1 - \frac{1}{4^m} x_2 \right|, \left| 2^m y_1 - \frac{1}{4^m} y_2 \right|\right) \\ &< (\alpha I)\, \rho(z_1, z_2), \text{ for some } \alpha,\ 0 < \alpha < 1 \\ &= \alpha(|x_1 - x_2|, |y_1 - y_2|) \end{aligned}$$

In particular, let us choose $z_1 = (2, 2)$, $z_2 = (1, 1)$.

Then we have $\left| 2^{m+1} - \frac{1}{4^m} \right| < \alpha$, $0 < \alpha < 1$, and m is a positive integer. This is a contradiction. Therefore, T_1^m and T_2^m do not satisfy the conditions of Theorem 3.1, although they have a common fixed point. This justifies Remark 1.

Remark 3.5. If the conditions of Theorem 3.1 are satisfied only when $m \neq 1$, then also T_1^m and T_2^m have a unique fixed point.

Proof. In this case, there exists a unique $x_0 \in S$ such that $T_1^m x_0 = x_0 = T_2^m x_0$, $m \neq 1$.

Now $T_1 x_0 = T_1(T_1^m x_0) = T_1^{m+1}(x_0) = T_1^m (T_1 x_0)$. Thus $T_1 x_0$ is a fixed point of T_1^m. But x_0 is the unique fixed point of T_1^m. Hence $T_1 x_0 = x_0$. So x_0 is a fixed point of T_1.

Similarly, it may be shown that x_0 is a fixed point of T_2. Thus x_0 is a common fixed point of T_1 and T_2. To show that x_0 is unique, let y_0 be another common fixed point of T_1 and T_2 where $y_0 \neq x_0$. Obviously, x_0 and y_0 are also common fixed points of T_1^m and T_2^m where $m \neq 1$. As by given conditions, T_1^m and T_2^m have unique common fixed point when $m \neq 1$, so $x_0 = y_0$.

Remark 3.6. If in a given problem, the conditions for the existence of common fixed point of two operators T_1 and T_2 do not hold, then the question of existence of common fixed points remains indecisive, as the conditions are sufficient but not necessary. However, in this case, similar conditions for higher order iterations of T_1 and T_2 may be tried. If such conditions are found to hold for some iterations of T_1 and T_2, it may be concluded immediately from Remark 3.5, that T_1 and T_2 have a common fixed point. To illustrate this point, let us consider the following example.

Example 3.7. Let R^n be the n-dimensional real vector space. Let $x, y \in R^n$ where $x = (x_1, x_2, \dots x_n)$, $y = (y_1, y_2, \dots y_n)$. Let $x \le y$ if $x_i \le y_i$; i = 1, 2, ... n.

Let ρ be defined by

$\rho(x, y) = (| x_1 - y_1 |, | x_2 - y_2 |, \dots\dots, | x_n - y_n |)$. Let $E_0 = (e_1, e_2, \dots\dots, e_n)$ be the standard basis of R^n ; let T_1 and T_2 be two operators on R^n defined as follows :

$$T_1 (e_1) = \frac{1}{2} e_2,\ T_1 (e_2) = \frac{1}{2} e_1,\ T_1 (h) = \frac{1}{2} h, \text{ for all } h \in E_0,\ h \ne e_1, e_2$$

$$\text{and } T_2 (e_3) = \frac{1}{2} e_4,\ T_2 (e_4) = \frac{1}{2} e_3,\ T_1 (h') = \frac{1}{2} h', \text{ for all } h' \in E_0,\ h' \ne e_3, e_4.$$

For $x = \sum_{i=1}^{n} x_i e_i$, let $T_1 (x) = \sum_{i=1}^{n} x_i T_1 (e_i)$, $T_2 (x) = \sum_{i=1}^{n} x_i T_2 (e_i)$. It may be verified easily that R^n is a complete P-metric space with associated partially ordered linear space R^n itself. Now for any $x \in X$, $T_1 (x) = \frac{1}{2} x''$, where $x'' = (x_2, x_1, x_3, x_4 \dots\dots, x_n)$ and $T_1^2(x) = \frac{1}{4} (x_1, x_2, \dots\dots, x_n) = \frac{1}{4} x$.

Similarly, for any $y \in X$, $T_2 (y) = \frac{1}{2} y''$, where $y'' = (y_1, y_2, y_4, y_3 \dots\dots, y_n)$ and $T_2^2(y) = \frac{1}{4} y$.

Obviously T_1, T_2 do not satisfy the conditions ensuring their fixed points.

Now,

$$\rho(T_1^2 x, T_2^2 y) = \rho(\frac{1}{4} x, \frac{1}{4} y) = \frac{1}{4} (| x_1 - y_1 |, | x_2 - y_2 |, \ldots\ldots, | x_n - y_n |)$$

$< Q\rho(x, y)$, where $Q = \frac{1}{2} I < I$. Hence, by Theorem 3.1,

T_1^2, T_2^2 have a unique common fixed point. So by Remark 3.2, T_1 and T_2 will also have the same unique common fixed point.

Remark 3.8. The example considered in Remark 3.6, also reveals that conditions of Theorem 3.1 (when m = 1) are also sufficient, but not necessary.

4. APPLICATION OF FIXED POINT THEOREM

Theorem 4.1. Let (x, ρ, E) and $(Y\ \sigma, E)$ be two complete linear P-metric spaces over F where E is a partially ordered normed algebra over F. Let $T_1 : X \to X$, $T_2 : Y \to Y$, be two linear operators on X and Y respectively such that $\forall\ x_1, x_2 \in X$, $\forall\ y_1, y_2 \in Y$,

$\rho(T_1 x_1, T_1 x_2) \leq Q_1\rho(x_1, x_2)$ and $\sigma(T_2 y_1, T_2 y_2) \leq Q_2\rho(y_1, y_2)$ for some operators $Q_1, Q_2 : E \to E$, where both of Q_1 and Q_2 are positive, linear and continuous and either $Q_1 < Q_2 < I$ or $Q_2 < Q_1 < I$; moreover both of $I\text{-}Q_1$ and $I\text{-}Q_2$ are injective and $\{Q_1^n\}$, $\{Q_2^n\}$ are convergent in BL(E, E).

Let $T : X \otimes Y \to X \otimes Y$ be such that for

$$u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y,\ T(u) = T(\sum_{i=1}^{n} x_i \otimes y_i)$$

$$= \sum_{i=1}^{n} \{ T_1(x_i) \otimes T_2(y_i)\}.$$ Let $v = \sum_{j=1}^{n} (x'_j \otimes y'_j) \in X \otimes Y$ and let

$$d(u,v) = d(\sum_{i=1}^{n} (x_i \otimes y_i), \sum_{j=1}^{n} (x'_j \otimes y'_j) = \sum_{r=1}^{n} \rho\ (x_r \otimes x'_r) + \sigma\ (y_r \otimes y'_r).$$

Then T has a unique fixed point in $(X \otimes Y, d)$.

Proof. $d(Tu, Tv) = d[T(\sum_{i=1}^{n} x_i \otimes y_i), T(\sum_{j=1}^{n} x'_j \otimes y'_j)]$

$$= d[\sum_{i=1}^{n} \{ T_1(x_i) \otimes T_2(y_i)\}, \sum_{j=1}^{n} \{ T_1(x'_j) \otimes T_2(y'_j)\}]$$

$$= \sum_{r=1}^{n} \rho \{ T_1(x_r), T_1(x'_r)\} + \sum_{r=1}^{n} \sigma \{ T_2(y_r), T_2(y'_r)\}$$

$$\leq \sum_{r=1}^{n} Q_1\rho (x'_r \otimes x'_r) + \sum_{r=1}^{n} Q_2\sigma (y_r \otimes y'_r)$$

$$< Q_1[\sum_{r=1}^{n} \rho (x_r \otimes x'_r) + \sum_{r=1}^{n} \sigma (y_r \otimes y'_r)] \text{ (assuming } Q_2<Q_1).$$

$$= Q_1 \, d(u, v).$$

Similarly, if $Q_1 < Q_2$, then $d(Tu, Tv) < Q_2 d(u, v) \ \forall \ u, v \in X \otimes Y$. Thus $(X \otimes Y, d)$ is a complete P-metric space with $T : X \otimes Y \to X \otimes Y$ satisfying $d(Tu, Tv) < Q_1 d(u, v)$, or $d(Tu, Tv) < Q_2 d(u, v)$. Hence by Corollary of the Theorem 3.1, T has a unique fixed point in $X \otimes Y$.

REFERENCES

1. F. F. Bonsall and J. Duncan, Complete normed Algebras, Springer Verlag, Berlin, Heidelburg, New York, 1973.
2. L. Collatz, Functional Analysis and Numerical Mathematics, Academic Press, New York and London, 1966.
3. B. Choudhury and S. Nanda, Functional Analysis with applications, Wiley Eastern Ltd., New Delhi, Bombay, Calcutta, 1991.
4. R. Kannan, Some Results on fixed points, Bull. Cal. Math. Soc. 60 (Nos. 1 and 2) (1968) (71-76)
5. G. Kureppa, Tableaux ramifies d'ensembles, espaces pseudodostancies, C. R. 198 (1934), 1563-1565.
6. S. P. Singh, Some Results on fixed points. Yokohama Math. Jour., XVII (2) 1969, 61-64.
7. A. E. Taylor, Introduction to functional Analysis, John Willey and Sons, Inc. New York, London ; 1958.

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GROWTH PROPERTIES OF HOLOMORPHIC FUNCTIONS IN SEVERAL COMPLEX VARIABLES

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ABSTRACT : Let H_D be the family of all holomorphic functions in a polydomain $D \subset C^n$, the complex n-space. For $f, g \in H_D$, the Hadamard Product $f * g$ is defined. A few growth properties of $f * g$ and their partial derivatives are obtained. A few properties of the entire functions have also been obtained.

1. Notations. We denote the complex and real n-spaces by C^n and R^n respectively. I^n will denote the cartesian product of n-copies of I, the set of non-negative integers. We indicate the points $(z_1, \dots z_n)$, $(m_1, \dots m_n)$ etc. of C^n and R^n of their corresponding unsuffixed symbols z, m etc. For $z, w \in C^n$ and $\alpha \in C$ we define $z = w$ iff $z_i = w_i$, $i = 1, \dots n$, $\alpha z = (\alpha z_1, \dots, \alpha z_n)$ $z + w = (z_1 + w_1, \dots z_n + w_n)$ and $| z | = \{ | z_1 |^2 + \dots + | z_n |^2\}^{1/2}$. The positive hyperoctant R_+^n is the set $R_+^n = \{ X : X \in R^n, x_i \geq 0, i = 1, \dots, n\}$. For $k \in R_+^n$ we set $\| k \| = k_1 + \dots + k_n$ and for an $m \in I^n$, $m! = m_1! \dots m_n!$. For any $P \in I$, $\bar{p}$ will stand for the n-tuple $(p, \dots, p)$. Also for $z \in C^n$ and $k \in R_+^n$ we shall write $z^k = z_1^{k_1} \dots z_n^{k_n}$. For $x, y \in R^n$ we say that

(i) $x \leq y$ iff $x_i \leq y_i$, $i - 1, \dots, n$.

(ii) $x < y$ iff $x \leq y$ but $x \neq y$;

(iii) $x \ll y$ iff $x_i < y_i$, $i = 1, \dots, n$. For a holomorphic function f, $f^{(k)}$ will denote the function $\dfrac{\partial^{\|k\|} f}{\partial z_1^{k_1} \dots \partial z_n^{k_n}}$, $k \in I^n$.

2. Let H_D be the family of all holomorphic functions represented by multiple power series of the form

(2.1) $f(z) = \sum_{\|m\|=0}^{\infty} a_m z^m \quad a_m \in C,$

each being holomorphic in a closed polydisk $D = \{z : | z_i | \le R_i,\ i = 1, \ldots, n\}$. For $f, g \in H_D$ we define Hadamard product $f * g$ by

(2.2) $(f * g)(z) = \sum_{\|m\|=0}^{\infty} a_m b_m z^m$, where

$g(z) = \sum_{\|m\|=0}^{\infty} b_m z^m$. Evidently, $f * g \in H_D$. Corresponding to $f \in H_D$ we define the maximum modulus function $M_f(r)$, maximum term $\mu_f(r)$, and central indices $\nu_j(r)$ of $\mu_f(r)$, $j = 1, \ldots, n$ on R_+^n by $M_f(r) = \max_{|z_j| = r_j} | f(z) |$, $\mu_f(r) = \max_{m \in I^n}\{| a_m | r^m\}$, $\nu_j(r) = \max\{m_j : | a_m | r^m = \mu_f(r),\ j = 1, \ldots n\}$. We call $\nu = (\nu_1, \ldots, \nu_n)$ as the rank of the maximum term $\mu_f(r)$. Throughout this paper $M_{(f*g)^{(k)}}(r)$ will be denoted by $M(r, k)$, $k \in I^n$.

Theorem 1. Let $f, g \in H_D$ and $A(r) \ge 0$ denote the maximum of Re $(f * g)$ on $| z_j | = r_j$, $j = 1, \ldots, n$. Then for $\bar{0} \ll r \ll R$,

$$M(r, k) \le \frac{k!\,2^{\|k+1\|}\prod_{j=1}^{n} R_j}{(R-r)^{k+1}} \left(4\,A(R) + 3\,| f(\bar{0}) * g(\bar{0}) |\right)$$

Proof. We take any point z such that $| z_j | = r_j$. By Cauchy's integral formula,

$(f(z) * g(z))^{(k)} = \dfrac{k!}{(2\pi i)^n} \displaystyle\int_C \frac{f(w) * g(w)}{(w_1 - z_1)^{k_1+1} \ldots (w_n - z_n)^{k_n+1}}\, dw_1 \ldots dw_n$, where $C = C_1 \times \ldots \times C_n$, $C_j : | w_j - z_j | = d_j$ and $d_j = \dfrac{1}{2}(R_j - r_j)$.

Now on each C_j, $w_j = z_j + d_j e^{i\theta_j}$, $0 \le \theta_j \le 2\pi$, and so $| W | \le r_j + d_j < R_j$. Hence,

(2.3) $$| (f(z) * g(z))^{(k)} | \le \frac{k!}{(2\pi)^n \prod_{j=1}^{n} d_j^{k_j+1}} \int_C | f(w) * g(w) |\, dw_1 \ldots dw_n$$

Also we have [2]

(2.4) $$| f(w) * g(w) | \le \prod_{j=1}^{n} \frac{R_j}{(R_j - r_j - d_j)} \left(4\,A(R) + 3\,| f(\bar{0}) * g(\bar{0}) |\right)$$

From (2.3) and (2.4) we get;

$$M(r,\ k) \le \frac{k!2^{\|k+1\|}\Pi_{j=1}^{n}R_j}{(R-r)^{k+1}}\ (4\ A(R) + 3\ |f\ (\bar{0}) * g\ (\bar{0})|)$$

Theorem 2. Let f, $g \in H_D$ and A_1 (r) be the maximum of $|\ Re(f * g)\ (z)|$ on $|\ z_j\ | = r_j$, $j = 1, \ldots, n$. Then for $r \le R$, $|(f\ (\bar{0}) * g\ (\bar{0}))^k| \le \frac{k!2A_1(r)}{r^k}$.

Proof. It is known [2] that $\frac{1}{2}(2\pi)^n a_k r^k = \int_0^{2\pi} \ldots \int_0^{2\pi} e^{-\|k\theta\|}\ Re\ f\ (z)\ d\theta_1 \ldots d\theta_n$, where $k\theta = (k_1\theta_1, \ldots, k_n\theta_n)$. Let $f(z) * g(z) = u + iv$. Then,

$\frac{1}{2}(2\pi)^n |\ a_k b_k |r^k \le \int_0^{2\pi} \ldots \int_0^{2\pi}\ |\ u\ |\ d\theta_1 \ldots d\theta_n$. Also it is known ([1], P31) that

$$k!a_k = \frac{\partial^{\|k\|}\ f(\bar{0})}{\partial_{z_1}^{k_1} \ldots \partial_{z_n}^{k_n}}.\ \text{Hence,}\ \left|\frac{\partial^{\|k\|}\ f(0)*(0)}{\partial_{z_1}^{k_1} \ldots \partial_{z_n}^{k_n}}\right| \le \frac{k!}{(2\pi)^n}\ 2 \int_0^{2\pi} \ldots \int_0^{2\pi}\ |\ u\ |\ d\theta_1 \ldots d\theta_n$$

$$= \frac{k!2A_1(r)}{r^k}.$$

Theorem 3. let $f \in H_D$. Then for $\bar{0} \ll r \ll R$,

$$M_f\ (r) - |f\ (\bar{0})| \le \textstyle\sum_{i=1}^{n} r_i M_f\ (r,\ i) \le M_f(R)\ \sum_{i=1}^{n} \frac{r_i}{R_i - r_i}$$

where $M_f(r,\ i) = max_{|z_j| = r_j} \left| \frac{\partial f}{\partial z_i} \right|$, $i = 1, \ldots, n$.

Proof. For the sake of simplicity we prove the theorem for $n = 2$. The proof is exactly similar for the case of more variables. We know ([3], P38) that

$$f\ (z) = f\ (0,\ 0) + \int_{(0,0)_{(L)}}^{(z_1,z_2)} df = f(0,\ 0) + \int_{(0,0)_{(L)}}^{(z_1,z_2)} \left(\frac{\partial f}{\partial z_1} dz_1 + \frac{\partial f}{\partial z_2} dz_2\right)$$

where L is a piece-wise smooth curve in D, joining the points (0, 0) and $(z_1,\ z_2)$. Since the value of the integral is independent of the path L([3], P38) we take the path L as $L = L_1 \times L_2$ where L_1, L_2 are two straight paths in D such that $L_1 \subset \{z : |z_1| \le r_1, z_2 = \text{constant}\}$, $L_2 \subset \{z : |z_2| \le r_2, z_1 = \text{constant}\}$. Let L_1 be the "line segment" joining the points (0, 0) and $(z_1, 0)$ and L_2 be the "line segment" joining the points $(z_1, 0)$ and (z_1, z_2). Then,

$$f(z_1, z_2) = f(0, 0) + \int_{(0,0)(L_1)}^{(z_1,0)} \frac{\partial f}{\partial z_1} dz_1 + \int_{(z_1,0)(L_2)}^{(z_1,z_2)} \frac{\partial f}{\partial z_2} dz_2. \text{ Hence,}$$

$|f(z_1, z_2)| \le |f(0, 0)| + r_1 M_f(r, 1) + r_2 M_f(r, 2)$, $\forall z \in D$. That is,

(2.5) $M_f(r) - |f(0, 0)| \le \sum_{i=1}^{2} r_i M_f(r, i)$. Again, for any $z \in$ Int D, by Cauchy's integral formula,

$$f(z_1, z_2) = -\frac{1}{4\pi^2} \int_{C_1} \times \int_{C_2} \frac{f(t_1, t_2)}{(t_1 - z_1)(t_2 - z_2)} dt_1 dt_2,$$

where $C_k : |t_k - z_k| = R_k - r_k$, $k = 1, 2$. Hence,

$$\left| \frac{\partial f}{\partial z_1} \right| = \frac{1}{4\pi^2} \left| \int_{C_1} \times \int_{C_2} \frac{f(t_1, t_2)}{(t_1 - z_1)^2 (t_2 - z_2)} dt_1 dt_2 \right| \le \frac{M_f(R)}{R_1 - r_1}.$$

That is, $M_f(r, 1) \le \frac{M_f(R)}{R_1 - r_1}$. Similarly, $M_f(r, 2) \le \frac{M_f(R)}{R_2 - r_2}$. Hence

(2.6) $\sum_{i=1}^{2} \le r_i M_f(r, i) \le M_f(R) \sum_{i=1}^{2} \frac{r_i}{R_i - r_i}$

Combining (2.5) and (2.6) we get the result.

3. Lemma. Let $f(z) = \sum_{\|m\|=0}^{\infty} a_m z^m$ be a function holomorphic in the closed polydisk $D = \{z : |z_i| \le R_i, i = 1, \dots, n\}$, $Re\ f(z) = u(r, \theta)$ and $\phi(r) = max_{0 \le \theta_j \le 2\pi} u(r, \theta)$ for $z_j = r_j e^{i\theta_j}$, $j = 1, \dots, n$. Then $|\alpha_m| \le \frac{2(\phi(r) - \alpha_0^-)}{r^m}$ and $|\beta_m| \le \frac{2(\phi(r) - \alpha_0^-)}{r^m}$, where $a_m = \alpha_m + i\beta_m$, $m \in I^n$

Proof. Let $f(z) = u(r, \theta) + iv(r, \theta)$. Then ([2], 1963),

(3.1) $u(r, \theta) = \sum_{\|m\|=0}^{\infty} r^m (\alpha_m cos \| m\theta \| - \beta_m sin \| m\theta \|)$, where $m\theta = (m_1\theta_1, \ldots, m_n\theta_n)$. Since the series on the r.h.s. of (3.1) is uniformly convergent, we can multiply both sides of (3.1) by *cos* $\| m\theta \|$ or by *sin* $\| m\theta \|$ and integrate term by term between 0 to 2π and we get,

$$(2\pi)^n \alpha_{\bar{0}} = \int_0^{2\pi} \ldots \int_0^{2\pi} u(r, \theta)\, d\theta_1 \ldots d\theta_n,$$

$$\frac{1}{2}\alpha_m r^m (2\pi)^n = \int_0^{2\pi} \ldots \int_0^{2\pi} u(r, \theta) \cos \| m\theta \|\, d\theta_1 \ldots d\theta_n \text{ and}$$

$$\frac{1}{2}\beta_m r^m (2\pi)^n = -\int_0^{2\pi} \ldots \int_0^{2\pi} u(r, \theta) \sin \| m\theta \|\, d\theta_1 \ldots d\theta_n, \; m \neq \bar{0}.$$

Therefore,

$$(3.2) \qquad \frac{1}{2}(2\alpha_{\bar{0}} \pm \alpha_m r^m)(2\pi)^n = \int_0^{2\pi} \ldots \int_0^{2\pi} (1 \pm \cos \| m\theta \|)\, u(r, \theta)\, d\theta_1 \ldots d\theta_n \text{ and}$$

$$(3.3) \qquad \frac{1}{2}(2\alpha_{\bar{0}} \pm \beta_m r^m)(2\pi)^n = \int_0^{2\pi} \ldots \int_0^{2\pi} (1 \mp \sin \| m\theta \|)\, u(r, \theta)\, d\theta_1 \ldots d\theta_n.$$

From (3.2) and (3.3) we have,

$$(3.4) \qquad \frac{1}{2}(2\alpha_{\bar{0}} \pm \alpha_m r^m)(2\pi)^n \leq \phi(r) \int_0^{2\pi} \ldots \int_0^{2\pi} (1 \pm \cos \| m\theta \|)\, d\theta_1 \ldots d\theta_n =$$

$(2\pi)^n \phi(r)$ and

$$(3.5) \qquad \frac{1}{2}(2\alpha_{\bar{0}} \pm \beta_m r^m)(2\pi)^n \leq (2\pi)^n \phi(r).$$

The lemma now follows from (3.4) and (3.5).

Theorem 4. If the real part $u(r, \theta)$ of an entire function f satisfies the condition: $u(r, \theta) \leq \phi(r) \leq r^k$ for some $k \gg \bar{0}$ and for suffciently large $\| r \|$, $\phi(r)$ being defined as in the lemma, then f is a polynomial of degree not greater than $\| k \|$.

Proof. By the previous lemma we have,

$$| \alpha_m | \leq \frac{2(\phi(r) - \alpha_{\bar{0}})}{r^m} \text{ and } | \beta_m | \leq \frac{2(\phi(r) - \alpha_{\bar{0}})}{r^m}. \text{ Hence,}$$

$$| \alpha_m | \leq \frac{2(r^k - \alpha_{\bar{0}})}{r^m} \text{ and } | \beta_m | \leq \frac{2(r^k - \alpha_{\bar{0}})}{r^m} \; \forall \text{ large } \| r \|.$$

If $m \in I^n$ be such that $\| m \| > \| k \|$, then $\alpha_m \to 0$, $\beta_m \to 0$ as $\| r \| \to \infty$. This proves the theorem.

4. Let $D \subset C^n$ be an arbitrary bounded polydisk with centre at the origin of coordinates. Then for an entire function f we define

$M_{f,D}(R_1) = \operatorname{Sup}_{z \in D_{R_1}} |f(z)|$, $R_1 > 0$, where the point $z \in D_{R_1}$ iff the

point $\frac{z}{R_1} = \left(\frac{z_1}{R_1}, \ldots, \frac{z_n}{R_1}\right) \in D$.

Theorem 5. If f is a transcendental entire function and P is a polynomial, then

$$\lim_{R_1 \to \infty} \frac{M_{f,D}(R_1)}{M_{P,D}(R_1)} = \infty,$$ where D is an arbitrary bounded polydisk.

Proof. Let $P(z) = \sum_{\|t\|=0}^{\|m\|} b_t z^t$ be a polynomial in $z_1, \ldots, z_n$ of degree $\| m \|$, where $b_m \neq 0$ and D is given by $D = \{z : | z_i | < r_i,\ i = 1, \ldots, n\}$. Then

(4.1) $M_{P,D}(R_1) \le | b_m | r^m R_1^{\|m\|} (1 + \varepsilon)$ for large R_1 and $\varepsilon > 0$. *Let* $f(z) = \sum_{\|k\|=0}^{\infty} a_k z^k$. We fix a term of f in its Taylor series expansion whose degree is $\| p \|$ such that $p \gg m$ and $a_p \neq 0$. By Cauchy's inequality, $| a_p | \le \frac{M_{f,D}(R_1)}{r^p R_1^{\|p\|}}$ and so

(4.2) $M_{f,D}(R_1) \ge | a_p | r^p R_1^{\|p\|}$

for sufficiently large R_1. Hence from (4.1) and (4.2) we have for large R_1,

$$\frac{M_{f,D}(R_1)}{M_{P,D}(R_1)} \ge \frac{|a_p| r^p R_1^{\|p\|}}{|b_m| r^m R_1^{\|m\|}(1+\epsilon)} \to \infty$$ as $R_1 \to \infty$. This proves the theorem.

5. Let $D \subset C^n$ be an n-circular domain with centre at z^0. We denote by $| D |$ the image in R_+^n of the domain D under the mapping $r_j = | z_j - z_j^{(0)} |$, $j = 1, \ldots, n$ i.e.

$| D | = \{r \in R_+^n : | z_j - z_j^{(0)} | = r_j,\ j = 1, \ldots, n\}$

Let f_1, f_2 be two entire functions and $\mu_1(r), \mu_2(r), \mu(r)$ be respectively the maximum terms of f_1, f_2 and $f_1 * f_2$. Also let $v_1(r)$, $v_2(r)$ and $v(r)$ be the ranks of $\mu_1(r)$, $\mu_2(r)$ and $\mu(r)$ respectively. Then we have the following theorem.

Theorem 6. $\mu(r) \le \frac{\mu_1(r)\mu_2(r)}{r^{v(r)}}$ for $r \in |C^n| - D_1$, where D_1 is the set of all discontinuities of $v_1(r)$, $v_2(r)$ and $v(r)$ in $|C^n|$.

Proof. Let $f_1(z) = \sum_{\|m\|=0}^{\infty} a_m z^m$; $f_2(z) = \sum_{\|m\|=0}^{\infty} b_m z^m$. Then

$(f_1 * f_2)(z) = \sum_{\|m\|=0}^{\infty} a_m b_m z^m$, $\mu_1(r) = |a_{v_1(r)}|\, r^{v_1(r)}$, $\mu_2(r) = |b_{v_2(r)}|\, r^{v_2(r)}$.

Now

$$\begin{aligned} \mu(r) = |a_{v(r)}\, b_{v(r)}|\, r^{v(r)} &= \frac{|a_{v(r)}|r^{v(r)}|b_{v(r)}|r^{v(r)}}{r^{v(r)}} \\ &\le \frac{|a_{v_1(r)}|r^{v_1(r)}|b_{v_2(r)}|r^{v_2(r)}}{r^{v(r)}} \\ &= \frac{\mu_1(r)\mu_2(r)}{r^{v(r)}} \end{aligned}$$

Theorem 7. Let $\sum_{n=0}^{\infty} f_n(z)$ be a series of entire functions which converges absolutely and uniformly to $f(z)$ on any complete n-circular domain in C^n. Then, $\mu_f(r) \le \sum_{n=0}^{\infty} \mu_{f_n}(r)$, where $\mu_f(r)$ and $\mu_{f_n}(r)$ denote respectively the maximum terms of f and f_n for $r \in |C^n| - D' \cup D$, $D = \bigcup_{n=1}^{\infty} D_n$, D_n is the set of all discontinuities of the rank $v_n(r)$ of $\mu_{f_n}(r)$ in $|C^n|$ and D' is the set of all discontinuities of $v(r)$, the rank of $\mu_f(r)$.

Proof. Clearly, $f(z)\ \sum_{n=0}^{\infty} f_n(z)$ is an entire function.

Let $f_n(z) = \sum_{\|m\|=0}^{\infty} a_m^{(n)} z^m$. Since the series is uniformly convergent,

$$f(z) = \sum_{n=0}^{\infty} f_n(z) = \sum_{n=0}^{\infty} \sum_{\|m\|=0}^{\infty} a_m^{(n)} z^m = \sum_{\|m\|=0}^{\infty} \sum_{n=0}^{\infty} a_m^{(n)} z^m = \sum_{\|m\|=0}^{\infty} a_m z^m,$$

where $a_m = \sum_{n=0}^{\infty} a_m^{(n)}$. Now $\mu_f(r) = |a_v|\, r^v = |\sum_{n=0}^{\infty} a_v^{(n)}|\, r^v \le \sum_{n=0}^{\infty} |a_v^{(n)}|\, r^v = \sum_{n=0}^{\infty} \mu_{f_n}(r)$.

REFERENCES

1. Bochner, S and Martin, W.T., Several Complex variables, Princeton University Press, 1948.
2. Bose, S. K. and Sharma, D, Integral functions of two complex variables, Composito Math 15 (1963), 210-226.
3. Fuks, B. A., Introduction to the theory of analytic functions of several complex variables, Amer. Math. Soc. Providence, R.I. (1963).

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ON THE CONVERGENCE OF THE EIGENFUNCTION EXPANSION CORRESPONDING TO A MIXED STURM-LIOUVILLE PROBLEM (II)

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ABSTRACT : The convergence of the eigenfunction expansion corresponding to a Sturm-Liouville Problem (SLP) on a finite interval [a,b] is well-known (vide Titchmarsh [3]), when each of the boundary conditions (BCs) involves only one end point, either a or b. Such BCs are known as separated BCs. If, instead, the BCs involve both the end points a,b simultaneously, they are referred to as mixed BCs. The convergence of the eigenfunction expansion corresponding to a SLP, having mixed BCs, has been established here under usual restrictions.

Key words : Mixed Sturm-Liouville problem, Eigenfunction expansion.

American Mathematical Society Classification : 34B

1. INTRODUCTION

A second-order STURM-LIOUVILLE equation is a differential equation (DE) of the form

$$L[y] = -[p(x)\, y^{(1)}(x)]^{(1)} + q(x)\, y(x) = \lambda\, y(x), \qquad \dots (1.1)$$

where (i) $p : [a, b] \to R$ (the set of real numbers) is absolutely continuous,

(ii) $q : [a, b] \to R$ is continuous,

(iii) $\lambda \in C$ (the set of complex members) is a parameter,

(iv) $-\infty < a < b < \infty$.

If the Sturm-Liouville equation (1.1) is associated with boundary conditions, the corresponding boundary value problem is known as a Sturm-Liouville Problem (SLP).

E.C. Titchmarsh [3] has discussed thoroughly those SLPs where the BCs are of the form

$$\alpha_1\, y(a) + \alpha_2\, y^{(1)}(a) = 0, \qquad \dots (1.2a)$$

$$\beta_1\, y(b) + \beta_2\, y^{(1)}(b) = 0, \qquad \dots (1.2b)$$

$\alpha_1, \alpha_2, \beta_1, \beta_1$ being presscribed real numbers. Such SLPs are called **separated** SLPs (SSLPs), since here the BCs are separated, as they involve one end point at a time. Titchmarsh has proved that there exist a countable number of real numbers, μ_n, (n = 0,1,2,...), such that the DE (1.1), with $\lambda = \mu_n$, has a non-trivial solution φ_n, that satisfies the BCs (1.2a) - (1.2b). Since the DE (1.1) and the BCs (1.2) are all homogeneous, it can be so arranged that

$$\int_a^b \varphi_n^2(x) = 1. \qquad \text{... (1.3)}$$

μ_n is called an **eigenvalue** and φ_n is called a **normalised eigenfunction** corresponding to the eigenvalue μ_n of the above-mentioned SSLP, henceforth to be referred to as SSLP (I). For suitable f : [a,b] → R, we define

$$C_n = \int_a^b f(x)\, \varphi_n(x)\, dx \quad (n = 0, 1, 2, \ldots), \qquad \text{... (1.4)}$$

The infinite series $\sum_{n=0}^{\infty} C_n \varphi_n(x), \quad x \in [a, b]$ ... (1.5)

is known as the eigenfunction expansion of f(x), corresponding to the SSLP (I). Titchmarsh's [3] investigation centred round the convergence of the infinite series (1.5). His result can be stated as follows :

The eigenfunction expansion (1.5) of f(x) converges, whenever, the ordinary Fourier series of f(x) converges. Further, the two series converge to the same sum.

If the BCs (1.2a) – (1.2b) are replaced by the BCs

$$\cup_1[y] \equiv a_1 y(a) + a_2 y^{(1)}(a) + a_3 y(b) + a_4 y^{(1)}(b) = 0, \qquad \text{... (1.6a)}$$

$$\cup_2[y] \equiv b_1 y(a) + b_2 y^{(1)}(a) + b_3 y(b) + b_4 y^{(1)}(b) = 0, \qquad \text{... (1.6b)}$$

where a_i, b_i (i = 1,2,3,4) are prescribed real numbers, then the SLP, consisting of (1.1) - (1.6a) - (1.6b), is called a **mixed** SLP (henceforth referred to as the MSLP (II)), since here the BCs involve both the end points a, b simultaneously. In order to avoid the case where the BCs (1.6a) - (1.6b) are dependent, we assume that the rank of the matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix} \text{ is two.}$$

It has been shown by M.S.P. Eastham [2] that, under suitable restriction on the a_i's and b_i's, the MSLP (II) also possesses a countable number of **real** eigenvalues

λ_n (n = 0, 1, 2...). That each λ_n is simple will be proved in the sequel. Let ψ_n denote the normalised eigenfunction corresponding to the eigenvalue λ_n, so that

$$\int_a^b \psi_n^2 (x)\, dx = 1. \qquad \dots (1.7)$$

$$\text{If } d_n = \int_a^b f(x)\, \psi_n (x)\, dx, \ (n = 0, 1, 2, \dots), \qquad \dots (1.8)$$

$$\text{the infinite series } \sum_{n=0}^{\infty} d_n\, \psi_n (x) \qquad \dots (1.9)$$

is known as the eigenfunction expansion of f(x) corresponding to the MSLP (II). Our aim here is to establish the convergence of the infinite series (1.9) under suitable restrictions. Fortunately, the result obtained for the MSLP (II) is exactly the same as that obtained by Titchmarsh [3] for the SSLP (I). It may be stated as follows :

Theorem. Let

(1) $-\infty < a < b < \infty$,

(2) p : [a, b] → R be absolutely continuous,

(3) q : [a, b] → R be continuous,

(4) $\lambda \in C$ be a parameter,

(5) $[p(a)]^{-1} (a_1 b_2 - a_2 b_1) = [p(b)]^{-1} (a_3 b_4 - a_4 b_3)$,

(6) f : [a, b] → $\Re$ be such that $\int_a^b |f(x)|^2\, dx < \infty$,

(7) λ_n (n = 0, 1, 2...) denote the eigenvalues of the MSLP (II),

(8) ψ_n denote the normalised eigenfunction corresponding to the eigenvalue λ_n,

(9) $d_n = \int_a^b f(x)\, \psi_n (x)\, dx$, (n = 0, 1, 2,...),

(10) $x \in [a, b]$ be prescribed.

Then the infinite series $\sum_{n=0}^{\infty} d_n\, \psi_n (x)$ behaves, as regards convergence, in exactly the same way as the ordinary Fourier series of f(x) does.

e.g., if f is of bounded variation in a neighbourhood of x,

$$\sum_{n=0}^{\infty} d_n\, \psi_n (x) \text{ will converge to } \frac{1}{2}\{f(x + 0) + f(x - 0)\}.$$

Such conditions on f that ensure the convergence of the ordinary Fourier series of f are referred to as "Fourier conditions", and the convergence of $\sum\limits_{n} c_n \psi_n(x)$ or $\sum\limits_{n} d_n \psi_n(x)$ under such conditions is referred to as "convergence under Fourier conditions". Accordingly the above theorem merely states that the infinite series $\sum\limits_{n} d_n \psi_n(x)$ **converges under Fourier conditions provided (1) – (10) above hold good.**

In this paper an honest attempt has been made to imitate the way of attack of Titchmarsh of SSLP (I) in tackling the MSLP (II). This has led us to surface the differences between SSLP (I) and MSLP (II) which are worth-noting.

2. SOME PRELIMINARIES

2.1 The bilinear concomitant. The bilinear concomitant associated with the differential expression L [.], for f, g : [a, b] → C, which are differentiable, is denoted by $[f, g]_x$ and is defined by

$$[f, g]_x = p(x)\,\{f(x)\,\bar{g}^{(1)}(x) - f^{(1)}(x)\,\bar{g}(x)\},\ \forall\ x \in [a, b]. \qquad \ldots (2.1)$$

The following properties of $[f, g]_x$ are worth-noting :

(a) $[f, g]_x = -\overline{[f,g]}_x\ \forall\ x \in [a, b]$. ... (2.2)

(b) Green's identity : $\forall\ x_1, x_2, \in [a, b]$,

$$\int_{x_1}^{x_2} \{\bar{g}\ L[f] - f\ \overline{L[g]}\}\,dx = [f, g]_{x_2} - [f, g]_{x_1}. \qquad \ldots (2.3)$$

(c) If φ, ψ satisfy the DE (1.1) for the same λ, then

$[\varphi, \bar{\psi}]_x$ is idependent of x. ... (2.4)

(d) If f, g, u, v : [a, b] → C are differentiable, then $\forall\ x \in [a, b]$,

$$[f, g]_x\,[u, v]_x = [f, \bar{u}]_x\,[\bar{g}, v]_x + [f, v]_x\,[u, g]_x. \qquad \ldots (2.5)$$

2.2 Some special solutions of (1.1).

(1) Let η_1, η_2 denote the solutions of (1.1) that satisfy

$$\eta_1(a, \lambda) = 1, \qquad p(a)\eta_1^{(1)}(a, \lambda) = 0, \qquad \text{... (2.6)}$$

$$\eta_2(a, \lambda) = 0, \qquad p(a)\eta_2^{(1)}(a, \lambda) = 1, \qquad \text{... (2.7)}$$

Clearly $[\eta_1, \overline{\eta_2}]_x = 1.$ ... (2.8)

(2) If a solution θ of (1.1) be such that, for some $c \in [a, b]$, both $\theta(c, \lambda)$ and $\theta^{(1)}(c, \lambda)$ are real, then

$$\overline{\theta}(x, \lambda) = \theta(x, \bar{\lambda}) \qquad \forall\, x \in [a, b]. \qquad \text{... (2.9)}$$

(3) Let φ_i, χ_i (i = 1,2) be the solutions of (1.1) that satisfy

$$p(a)\varphi_1(a, \lambda) = a_2, \qquad p(a)\varphi_1^{(1)}(a, \lambda) = -a_1, \qquad \text{... (2.10)}$$

$$p(a)\varphi_2(a, \lambda) = b_2, \qquad p(a)\varphi_2^{(1)}(a, \lambda) = -b_1, \qquad \text{... (2.11)}$$

$$p(b)\chi_1(b, \lambda) = a_4, \qquad p(b)\chi_1^{(1)}(b, \lambda) = -a_3, \qquad \text{... (2.12)}$$

$$p(b)\chi_2(b, \lambda) = b_4, \qquad p(b)\chi_2^{(1)}(b, \lambda) = -b_3, \qquad \text{... (2.13)}$$

Then $\overline{\varphi_i}(x, \lambda) = \varphi_i(x, \bar{\lambda}), \quad \overline{\chi_i}(x, \lambda) = \chi_i(x, \bar{\lambda}), \ (i = 1,2),$ ... (2.14)

The BCs (1.6a) and (1.6b) respectively can now be expressed as

$$[y, \overline{\varphi_1}]_a + [y, \overline{\chi_1}]_b = 0 \qquad \text{... (2.15)}$$

and $[y, \overline{\varphi_2}]_a + [y, \overline{\chi_2}]_b = 0$ ... (2.16)

2.3 The condition of selfadjointness.

Let $\mathscr{D} = \{f \in L^2[a, b] : L[f] \in L^2[a, b],\ U_1[f] = 0 = U_2[f]\}$ ... (2.17)

An operator $\mathscr{L} : \mathscr{D} \to L^2\,[a, b]$ is defined by stipulating

$$\mathscr{L}f = L[f] \qquad \forall\, f \in \mathscr{D}. \qquad \text{... (2.18)}$$

An operator $\mathscr{L}$ is called self-adjoint if (1) $\mathscr{D}$ is dense in L^2 [a, b] and

$$\langle \mathscr{L}f, g\rangle = \langle f, \mathscr{L}g\rangle \quad , \forall\ f, g \in \mathscr{D},$$

i.e. $$\int_a^b \bar{g}\ L\,(f]\ dx = \int_a^b f\ \overline{L[\,g\,]}\ dx, \quad \forall f, g \in \mathscr{D}. \qquad \text{... (2.19)}$$

It is cusomary to call the MSLP (II) self-adjoint if $\mathscr{L}$ is self-adjoint.

While the SSLP (I) is selfadjoint for all α_1, α_2, β_1, $\beta_2 \in R$, the MSLP (II) is self-adjoint if and only if the a_i's and b_i's satisfy

$$[p(a)]^{-1}(a_1b_2 - a_2b_1) = [p(b)]^{-1}(a_3b_4 - a_4b_3) \qquad \text{... (2.20)}$$

The condition (2.20) of selfadjointness can be conveniently expressed as

$$[\varphi_1, \varphi_2]_a = [\chi_1, \chi_2]_b. \qquad \text{... (2.21)}$$

The operator $\mathscr{L}$ being self-adjoint, all its eigenvalues are real. It is to be noted that the eigenvalues of $\mathscr{L}$ are none but the eigenvalues of the MSLP (II).

2.4. The eigenvalues of the MSLP (II)

The solutions η_1, η_2 of (1.1) being linearly independent, any other solution y of (1.1) can be expressed as

$$y(x) = y(x, \lambda) = A_1\ \eta_1(x, \lambda) + A_2\ \eta_2(x, \lambda), \qquad x \in [a, b], \qquad \text{... (2.22)}$$

where A_1, A_2 are arbitrary complex numbers, possibly dependent on λ. If y satisfies the boundary conditions (1.6a) - (1.6b), then

$$A_1\ \alpha_1\ (\lambda) + A_2\ \alpha_2\ (\lambda) = 0 \qquad \text{... (2.23)}$$

$$A_1\ \beta_1\ (\lambda) + A_2\ \beta_2\ (\lambda) = 0 \qquad \text{... (2.24)}$$

where

$$\alpha_1(\lambda) = a_1 + a_3\,\eta_1(b, \lambda) + a_4\ \eta_1^{(1)}(b, \lambda), \qquad \text{... (2.25)}$$

$$\alpha_2(\lambda) = \frac{a_2}{p(a)} + a_3\,\eta_2(b, \lambda) + a_4\ \eta_2^{(1)}(b, \lambda), \qquad \text{... (2.26)}$$

$$\beta_1(\lambda) = b_1 + b_3\,\eta_1(b, \lambda) + b_4\ \eta_1^{(1)}(b, \lambda), \qquad \text{... (2.27)}$$

$$\beta_2(\lambda) = \frac{b_2}{p(a)} + b_3\,\eta_2(b, \lambda) + b_4\ \eta_2^{(1)}(b, \lambda). \qquad \text{... (2.28)}$$

If y is an eigenfunction, we must have $(A_1, A_2) \neq (0, 0)$. The values of λ for which (2. 23) - (2. 24) yield $(A_1, A_2) \neq (0, 0)$ are therefore the eigenvalues of the MSLP (II). Hence λ is an eigenvalue of the MSLP (II) if and only if

$$\Delta(\lambda) = \begin{vmatrix} \alpha_1(\lambda) & \alpha_2(\lambda) \\ \beta_1(\lambda) & \beta_2(\lambda) \end{vmatrix} = 0. \quad \ldots (2.29)$$

2.5. The boundary condition functions.

Let φ, χ be the solutions of (1.1), defined for all $x \in [a, b]$, by

$$\varphi(x) = \varphi(x, \lambda) = \alpha_2(\lambda)\eta_1(x, \lambda) - \alpha_1(\lambda)\eta_2(x, \lambda), \quad \ldots (2.30)$$

$$\chi(x) = \chi(x, \lambda) = \beta_2(\lambda)\eta_1(x, \lambda) - \beta_1(\lambda)\eta_2(x, \lambda). \quad \ldots (2.31)$$

It can be easily verified that

$$U_1[\varphi] = 0 \quad \text{and} \quad U_2[\chi] = 0. \quad \ldots (2.32)$$

Hence φ and χ are known as boundary condition functions. The importance of the boundary condition functions φ, χ is revealed from the following relation :

$$\Delta(\lambda) = [\varphi, \bar{\chi}](\lambda) = p(x)W_\lambda(\varphi, \chi, x), \quad \ldots (2,33)$$

where $W_\lambda(\varphi, \chi; x)$ represents the Wronskian of φ, χ ; the suffix λ indicates the dependence of the Wronskian on λ. We know that $[\varphi, \bar{\chi}]_x$ is independent of x, and so we have omitted the suffix x, instead its dependence on λ has been indicated.

If $\lambda = \lambda_n$ is an eigenvalue of the MSLP (II), it follows from (2. 33) that there exists a real number K_n ($\neq 0$ or ∞) such that

$$\chi(x, \lambda_n) = K_n \varphi_1(x, \lambda_n). \quad \ldots (2.34)$$

We also note that

$$\varphi(x, \lambda) = \varphi_1(x, \lambda) + \chi_1(x, \lambda), \quad \ldots (2.35)$$

$$\chi(x, \lambda) = \varphi_2(x, \lambda) + \chi_2(x, \lambda). \quad \ldots (2.36)$$

2.6. Lemma : Each eigenvalue of the MSLP (II), under the restrictions (1) - (5) of # 1, is real and is a simple zero of $\Delta(\lambda)$.

It should be noted that we need not refer to the operator $\mathscr{L}$ in order to establish that the eigenvalues of the MSLP (II) are all real. This can be achieved directly from the Green's identity (b) of # 2.1, using other results of # 2.1 and the selfadjointness condition (2.21).

To prove that each eigenvalue λ_n is a simple zero of $\Delta(\lambda)$ we note that, λ_n being real,

$$(\lambda - \lambda_n) \int_a^b \varphi(x, \lambda)\, \varphi(x, \lambda_n)\, dx$$

$$= [\varphi(x, \lambda), \varphi(x, \lambda_n)]_b - [\varphi(x, \lambda), \varphi(x, \lambda_n)]_a . \qquad \dots (2.37)$$

Hence two cases are to be considered :

(1) $[\varphi_1, \varphi_2]_a = [\chi_1, \chi_2]_b \neq 0,$

(2) $[\varphi_1, \varphi_2]_a = [\chi_1, \chi_2]_b = 0.$

In case (2), the boundary conditions $U_i[y] =$ (i= 1, 2) become separated and so this case does not come under our present purview.

In case (1), we note that, using (2. 5), (2. 2), (2. 4) and that χ_1, $\chi_i^{(1)}$ (i = 1, 2) take values at b that are independent of λ and φ_1, $\varphi_i^{(1)}$ (i = 1,2) take values at a that are independent of λ, it can be proved that

$$[\varphi(x, \lambda), \varphi(x, \lambda_n)]_b\, [\chi_1(x, \lambda_n), \chi_2(x, \lambda_n)]_b$$

$$= [\varphi(x, \lambda), \varphi(x, \lambda_n)]_a\, [\varphi_1(x, \lambda_n), \varphi_2(x, \lambda_n)]_a - [\varphi_1(x, \lambda_n), \chi_1(x, \lambda_n)]_a\, \Delta(\lambda) \quad \dots (2.38)$$

Further, if $[\varphi_1(x, \lambda_n), \chi_1(x, \lambda_n)]_a = 0$, using $\Delta(\lambda_n) = 0$ we can prove that $\varphi(x, \lambda_n) = 0$, which contradicts the fact that η_1, η_2 are two linearly independent solutions of (1.1).

From (2.37) and (2.38) we then have

$$\frac{\Delta(\lambda)}{\lambda - \lambda_n} = - \frac{[\varphi_1(x, \lambda_n), \varphi_2(x, \lambda_n)]a}{[\varphi_1(x, \lambda_n), \chi_1(x, \lambda_n)]a} \int_a^b \varphi(x, \lambda), \varphi(x, \lambda_n)]\, dx. \qquad \dots (2.39)$$

So, $\lim\limits_{\lambda \to \lambda_n} \Delta(\lambda) / (\lambda - \lambda_n) \neq 0$, which indicates that $\lambda = \lambda_n$ is a simple zero of $\Delta(\lambda)$.

3. THE FUNCTION Ψ

Let $\lambda \in C$ be such that $\Delta(\lambda) \neq 0$, i.e., λ is not an eigenvalue of the MSLP (II). For each $f \in L^2[a, b]$ and each $x \in [a, b]$, let

$$\Phi(x, \lambda) = \frac{\chi(x, \lambda)}{\Delta(\lambda)} \int_a^x \varphi(t, \lambda)\, f(t)\, dt + \frac{\varphi(x, \lambda)}{\Delta(\lambda)} \int_x^b \chi(t, \lambda)\, f(t)\, dt. \qquad \dots (3.1)$$

It can be easily verified that for each such λ, the function Φ satisfies the DE

$$L[y] = \lambda y - f. \qquad \ldots (3.2)$$

One is then automatically tempted to expect Φ to play the role of the inverse of the operator $\lambda - \mathscr{L}$, $\mathscr{L}$ being defined as in # 2. This requires Φ to satisfy the BCs (1.6a) – (1.6b), which it obviously does not. Nevertheless, we can choose complex numbers C and D, possibly dependent on λ, such that if

$$\Psi(x, \lambda) = \Phi(x, \lambda) + C(\lambda)\, \varphi(x, \lambda) + D(\lambda)\, \chi(x, \lambda), \qquad \ldots (3.3)$$

$\forall\, x \in [a, b]$ and $\forall\, \lambda \in C$ with $\Delta(\lambda) \neq 0$, then Ψ will satisfy the DE (3.2) and the BCs (1.6a) – (1.6b). Clearly Ψ will serve as the inverse of the operator $\lambda - \mathscr{L}$.

We further note that both C and D are integral functions of λ.

3.1. Lemma :

$$\Psi(x, \lambda) = \sum_{n=0}^{\infty} \frac{d_n \psi_n(x)}{\lambda - \lambda_n}, \qquad \forall\, x \in [a, b], \qquad \forall\, \lambda \neq \text{any } \lambda_n. \qquad \ldots (3.4)$$

where $$\Psi_n(x) = \left\{\frac{k_n}{\Delta'(\lambda_n)}\right\}^{1/2} \varphi(x, \lambda_n), \qquad \Delta'(\lambda_n) = \left| \frac{d}{d\lambda}\Delta(\lambda) \right|_{\lambda=\lambda_n}. \qquad \ldots (3.5)$$

and $$d_n = \int_a^b \psi_n(t)\, f(t)\, dt. \qquad \ldots (3.6)$$

Since, for each $x \in [a, b]$, $\Psi(x, \lambda)$ is analytic except for simple poles at λ_n ($n = 0, 1, 2, \ldots\ldots$), it is enough to show that the residue of $\Psi(x, \lambda)$ at λ_n is $d_n \Psi_n(x)$. From (3.3), the residue of $\Psi(x, \lambda)$ at λ_n

$$= \lim_{\lambda \to \lambda_n} (\lambda - \lambda_n)\, \Psi(x, \lambda) = \lim_{\lambda \to \lambda_n} (\lambda - \lambda_n)\, \Phi(x, \lambda)$$

$$= \frac{\chi(x, \lambda_n)}{\Delta'(\lambda_n)} \int_a^x \varphi(t, \lambda_n)\, f(t)\, dt + \frac{\varphi(x, \lambda_n)}{\Delta'(\lambda_n)} \int_x^b \chi(t, \lambda_n)\, f(t)\, dt$$

where ' denotes differentiation with respect to λ

$$= \frac{K_n}{\Delta'(\lambda_n)}\, \varphi(x, \lambda_n) \int_a^b \varphi(t, \lambda_n)\, f(t)\, dt = d_n\, \psi_n(x).$$

4. THE PROOF OF THE THEOREM

If Γ is a closed contour in the λ-plane that intersects the real λ-axis at points which are not eigenvalues of the MSLP (II), then, from (3. 3), we have

$$\int_{\Gamma} \Psi(x, \lambda)\, d\lambda = \int_{\Gamma} \Phi(x, \lambda)\, d\lambda. \qquad \ldots (4.1)$$

The method of procedure of Titchmarsh is to prove that

$$\int_{\Gamma} \Psi(x, \lambda)\, d\lambda \to \text{a finite limit as } \Gamma \text{ recedes to infinity.} \qquad \ldots (4.2)$$

Unfortunately, this is not straight forward; it requires, among other requisites, suitable orders for λ_n and Ψ_n as $n \to \infty$.

Upto this point the SSLP (I) and the MSLP (II) can be described to have run parallel to each other. The proof of (4. 2) can not be achieved following Titchmarsh. So we are forced to look for a detour. Let Γ_N be a closed contour in the complex λ-plane such that the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_N$ of the MSLP (II) lie within Γ_N, while all the other eigenvalues of MSLP (II) lie outside Γ_N. We are to show that

$$I_N = \int_{\Gamma_N} \Psi(x, \lambda)\, d\lambda \to \text{a finite limit as } N \to \infty. \qquad \ldots (4.3)$$

Since the corresponding investigation in the case of SSLP has been dealt with successfully, it is natural to try to relate the integral I_N of (4. 3) with the corresponding integral J_N associated with a suitable SSLP (I_1). Our aim then will be to show that $I_N - J_N \to 0$ as $N \to \infty$, so that the eigenfunction expansion corresponding to the mixed SLP (II) will converge whenever the eigenfunction expansion corresponding to the separated SLP (I_1) converges.

Let us consider the SSLP (I_1) consisting of the DE (1.1) and the BCs

$$a_1 y(a) + a_2 y^{(1)}(a) = 0 = a_3 y(b) + a_4 y^{(1)}(b). \qquad \ldots (4.4)$$

Let

$$\Delta_1(\lambda) = p(x)\, [\varphi_1(x, \lambda)\, \chi_1^{(1)}(x, \lambda) - \varphi_1^{(1)}(x, \lambda)\, \chi_1(x, \lambda)]. \qquad \ldots (4.5)$$

Clearly the zeros of $\Delta_1(\lambda)$ are precisely the eigenvalues $\{\mu_n\}_{n=0}^{\infty}$ of the SSLP (I). For each $x \in [a, b]$, and for each $\lambda \in C$ with $\Delta_1(\lambda) \neq 0$, let

$$\Phi_1(x, \lambda) = \frac{\chi_1(x, \lambda)}{\Delta_1(\lambda)} \int_a^x \varphi_1(t, \lambda)\, f(t)\, dt + \frac{\varphi_1(x, \lambda)}{\Delta_1(\lambda)} \int_x^b \chi_1(t, \lambda)\, f(t)\, dt. \qquad \ldots (4.6)$$

Let the contour Γ_N be so shosen that Γ_N contains within it the eigenvalues μ_0, $\mu_1, \ldots, \mu_{N'}$, while all the other eigenvalues lie outside Γ_N and let

$$J_N = \int_{\Gamma_N} \Phi_1(x, \lambda)\, d\lambda. \qquad \ldots (4.7)$$

We note that $N' \to \infty$ as $N \to \infty$

The fact that both Ψ and Φ_1 satisfy the nonhomogeneous DE (3.2), implies that $\Psi - \Phi_1$ satisfies the homogeneous DE (1.1). φ_1 and χ_1 being two linearly independent solutions of the DE (1.1), there exist A_1, B_1, $\in C$ such that, for all $x \in [a, b]$,

$$\Psi(x, \lambda) - \Phi_1(x, \lambda) = A_1\, \varphi_1(x, \lambda) + B_1\, \chi_1(x, \lambda). \qquad \ldots (4.8)$$

To determine A_1, B_1, we use the fact that Ψ satisfies the BCs (1.6a) – (a.6b). For brevity we use the notation

$$\int_a^b f(t)\, \overline{g(t)}\, dt = \langle f, g \rangle \qquad \ldots (4.9)$$

From (2.20) we have

$$(a_1 b_2 - a_2 b_1) / p(a) = (a_3 b_4 - a_4 b_3) / p(b) = K \text{ (say)}. \qquad \ldots (4.10)$$

In # 2.6 we have seen that $K = 0$ leads to separated BCs. So, for our present MSLP we may assume that $K \neq 0$.

It can now be verified that

$$I_N - J_N = \int_{\Gamma_N} \frac{K\{\langle \varphi_1 + \chi_1, f \rangle\}\, \{\varphi_1(x, \lambda) + \chi_1(x, \lambda)\}}{\Delta_1(\lambda)\, \{[\varphi_2, \overline{\chi}_1] - [\varphi_1, \overline{\chi}_2] - 2K\}}\, d\lambda \qquad \ldots (4.11)$$

We are to show that $I_N - J_N \to 0$ as $N \to \infty$.

Using the estimates of φ_1, χ_1, φ_2, χ_2, as given in (1.7.3), (1.7.6) and (1.7.8.), it can be shown that the integrand in (4.11) is ($\lambda = s^2$)

$$0\left(\int_a^b \frac{1}{|s|^2}\, e^{|t|\,(y-b)}\, |f(y)|\, dy\right) + 0\left(\int_a^b \frac{1}{|s|^2}\, e^{|t|\,(a-y)}\, |f(y)|\, dy\right) \text{ as } |s| \to \infty.$$

Proceeding as in # 1.9 of [3] it can then be proved that

$$I_N - J_N = \underline{0}\,(1) \text{ as } N \to \infty.$$

This completes the proof of the theorem.

5. REMARKS

Titchmarsh [3] remarked on page 22 that separated boundary conditions can be replaced by more general ones, viz., by mixed boundary conditions. This may lead one to think that such a replacement is just a routine work. That this is not so has been revealed here. Although the desired convergence of the eigenfunction expansion corresponding to a MSLP has already been obtained [1], the importance of the present endeavour is to clearly expose the similarities and the dissimilarities that lie between a SSLP and a MSLP. The Φ – function for the MSLP (II), constructed following Titchmarsh, does not serve as the resolvent operator for the MSLP (II). This calls for a deviation from the method of Titchmarsh [3] and it has been exhibited here. As in [1], a comparison with a suitable SSLP has become necessary, however the calculations seem to be simpler here than in [1].

REFERENCES

1. Jyoti Das and Arnab Kumar Chakravarty, Convergence of the eigenfunction expansion corresponding to a mixed Sturm-Liouville problem, Jour. Pure Math., 14 (1997), 47-55.
2. M.S.P. Eastham, Theory of Ordinary Differential Equations, Van Nostrand Reinhold company (1970) London, New York, Cincinnati, Toronto, Melbourne.
3. E.C. Titchmarsh, Eigenfunction Expansions Associated with Second-order Differential Equations, Part I. Oxford at the Clarendon Press (1962).

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ON PSEUDO CONFORMALLY SYMMETRIC KÄHLER SPACES

U. C. DE, B. BARUA AND B. K. MAZUMDER

1. INTRODUCTION

In a recent paper [2] the first author and H. A. Biswas introduced and studied a type of non-flat Riemannian space (M^n,g) $(n > 3)$ whose conformal curvature tensor C^h_{ijk} defined by

$$C^h_{ijk} = R^h_{ijk} - \frac{1}{n-2}\left(g_{ij} R^h_k - g_{ik} R^h_j + \delta^h_k R_{ij} - \delta^h_j R_{ik}\right) +$$

$$\frac{R}{(n-1)(n-2)}\left(g_{ij}\,\delta^h_k - g_{ik}\,\delta^h_j\right) \qquad \text{... (1.1)}$$

satisfies the condition

$$\nabla_l C^h_{ijk} = 2\lambda_l C^h_{ijk} + \lambda^h C_{lijk} + \lambda_i C^h_{ljk} + \lambda_j C^h_{ilk} + \lambda_k C^h_{ijl} \qquad \text{... (1.2)}$$

where R^h_{ijk} is the curvature tensor of type (1,3), R^h_k is the (1,1) Ricci tensor, R is the scalar curvature, λ_l is a non-zero covariant vector and '∇_l' denotes covariant differentiation with respect to the metric tensor g_{ij}. Such space is called a pseudo conformally symmetric space and λ_l is called its associated vector. An n-dimensional space of this kind has been denoted by $(PCS)_n$. If λ_l becomes zero in (1.2), then the space reduces to a conformally symmertic space [1].

It is to be noted that dimension of M^n has been taken > 3, because it is known that $C^h_{ijk} = 0$, when n = 3.

In an earlier paper [3] we have proved the existence of a $(PCS)_n$ $(n > 3)$ by an example. The present paper deals with a type of n-dimensional $(n = 2m, m \neq 1,2)$

Kähler space which satisfies (1.2). In general the scalar curvatrue of $(PSC)_n$ is not necessarily a constant. It is shown that a pseudo conformally symmetric Kähler space is Ricci symmetric.

2. PRELIMINARIES

Let F_i^h be the structure tensor and g_{ji} be the positive definite Riemannian metric of a real n-dimensional (n = 2m, m ≠ 1, 2) Kähler space. Then

$$\left.\begin{aligned} & F_j^r F_r^i = -\delta_j^i,\ g_{rs} F_j^r F_i^s = g_{ij} \\ & F_{ji} = g_{ri} F_j^r = -F_{ij},\ F^{ji} = g^{ir} F_r^i = -F^{ij} \\ & \nabla_k F_j^i = 0,\ \nabla_k g_{ji} = 0,\ g_{jm} F^{ji} = F_m^i. \end{aligned}\right] \quad \dots (2.1)$$

Let R_{ijk}^h be the Riemann curvature tensor and $R_{ij} = R_{ijh}^h$ be the Ricci tensor. Also let

$$R_{hijk} = R_{ijk}^r g_{rh},\ R = R_{ji} g^{ji} \text{ and } H_{ij} = \frac{1}{2} R_{ijkl} F^{kl}.$$

Then the following relations hold [6] ;

$$H_{ij} = -H_{ji}, \quad \dots (2.2)$$

$$R_{ks} F_j^s = H_{kj}, \quad \dots (2.3)$$

$$H_{ks} F_j^s = -R_{kj}, \quad \dots (2.4)$$

$$H_{kj} F^{kj} = -R, \quad \dots (2.5)$$

$$\nabla_l H_{kj} + \nabla_k H_{jl} + \nabla_j H_{lk} = 0, \quad \dots (2.6)$$

$$R_k^s F_s^h - R_s^h F_k^s = 0. \quad \dots (2.7)$$

3. PSEUDO CONFORMALLY SYMMETRIC KÄHLER SPACE

In a recent paper [4] De and Barua proved that in a $(PCS)_n$ (n > 3) the following relation holds :

$$\nabla_h C_{ijk}^h = 0 \quad (3.1)$$

Multiplying both sides of (1.1) by F^{jk} and taking covariant derivative of the result thus obtained we get by virtue of $\nabla_h F^{jk} = 0$

$$\nabla_m C^h_{ijk} F^{jk} = \nabla_m R^h_{ijk} F^{jk} - \frac{1}{n-2} [g_{ij} \nabla_m R^h_k F^{jk} - g_{ik} \nabla_m R^h_j R^{jk} + \delta^h_k \nabla_m R_{ij} F^{jk} -$$

$$\delta^h_j \nabla_m R_{ik} F^{jk}] + \frac{\nabla_m R}{(n-1)(n-2)} [\delta^h_k g_{ij} F^{jk} - \delta^h_j g_{ik} F^{jk}]$$

Contracting h and m we have

$$\nabla_h C^h_{ijk} F^{jk} = \nabla_h R^h_{ijk} F^{jk} - \frac{1}{n-2} [\nabla_h R^h_k g_{ik} F^{jk} - g_{ik} \nabla_h R^h_j R^{jk} + \delta^h_k \nabla_h R_{ij} F^{jk} -$$

$$\delta^h_j \nabla_h R_{ik} F^{jk}] + \frac{\nabla_h R}{(n-1)(n-2)} [\delta^h_k g_{ij} F^{jk} - \delta^h_j g_{ik} F^{jk}]$$

By virtue of (3.1) and using the property $[\nabla_h R^h_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik}]$, we can write the above equation

$$0 = (\nabla_k R_{ij} - \nabla_j R_{ik}) F^{jk} - \frac{1}{n-2} [\nabla_h R^h_k g_{ij} F^{jk} - \nabla_h R^h_j g_{ik} F^{jk} + \nabla_k R_{ij} F^{jk} - \nabla_j R_{ik} F^{jk}]$$

$$+ \sigma \nabla_k R g_{ij} F^{jk} - \sigma \nabla_j R g_{ik} F^{jk}$$

where $\sigma = \dfrac{1}{(n-1)(n-2)}$

Using (2.1) we get

$$(\nabla_k R^r_i F^k_r + \nabla_j R^r_i F^j_r) - \frac{1}{n-2} [\nabla_h R^h_k F^k_i + \nabla_h R^h_j F^j_i + \nabla_k R^r_i F^k_r +$$

$$\nabla_j R^r_i F^j_r] + \sigma \nabla_k R F^k_i + \sigma \nabla_j R F^j_i = 0.$$

Or, $$\frac{n-3}{n-2} (\nabla_k R^r_i F^k_r + \nabla_j R^r_i F^j_r) - \frac{1}{n-2} [\nabla_h R^h_k F^k_i + \nabla_h R^h_j F^j_i] +$$

$$2\sigma \nabla_j R F^j_i = 0.$$

By virtue of (2.7) and using the property $[\nabla_h R^h_j = \frac{1}{2}\nabla_j R]$, we obtain from the above

$$\frac{n-3}{n-2}\left(\frac{1}{2}\nabla_r R\, F^r_j + \frac{1}{2}\nabla_r R\, F^r_i\right) - \frac{1}{n-2}\left[\frac{1}{2}\nabla_k R\, F^k_i + \frac{1}{2}\nabla_j R\, F^j_i\right] + 2\sigma\, \nabla_j R F^j_i = 0.$$

or, $\frac{n-3}{n-1}\nabla_j R F^j_i = 0$

Hence $\nabla_j R F^j_i = 0$ ... (3.2)

From (3.2) it follows that $\nabla_j R = 0$

Therefore R is a constant. ... (3.3)

Thus we can state the following theorem :

Theorem 1. In a pseudo conformally symmetric Kähler space the scalar curvature is constant.

Now by virtue of (3.1) we have [5]

$$(\nabla_k R_{lj} - \nabla_j R_{lk}) - \frac{1}{2(n-1)}(\nabla_k R\, g_{lj} - \nabla_j R\, g_{lk}) = 0 \qquad \text{... (3.4)}$$

By virtue of (3.3), the equation (3.4) reduces to

$$\nabla_k R_{lj} - \nabla_j R_{lk} = 0 \qquad \text{... (3.5)}$$

Again from (2.6) we get

$$(\nabla_k H_{lj} + \nabla_l H_{jk} + \nabla_j H_{kl} = 0$$

Transvecting with F^j_l we get

$$F^j_l(\nabla_l H_{jk}) = \nabla_j R_{kl} - \nabla_k R_{jl}$$

By virtue of (3.5) this reduces to

$$F^j_l(\nabla_l H_{jk}) = 0$$

Transvecting this with $F_t^k \; F_h^l$ we get

$$F_t^k \; F_l^j \; F_h^l (\nabla_i \, H_{jk}) = 0$$

or, $- \; F_t^k \; (\nabla_h \, H_{jk}) = 0$

$$\text{Hence } \nabla_h \, H_{jt} = 0$$

Thus the space is Ricci symmetric.

Hence we can state the following theorem :

Theorem 2. A pseudo conformally symmetric Kähler space is Ricci symmetric.

REFERENCES

1. M. C. Chaki and B. Gupta, On conformally symmetric spaces, Indian J. Math. 5, (1963), 113-122.
2. U. C. De and H. A. Biswas On pseudo conformally symmetric manifolds, Bull. Cal. Math. Soc. 85 (1993), 479-486.
3. U. C. De and B. K. Mazumder Conformal transformation of a pseudo conformally symmetric spaces (to appear).
4. U. C. De and B. Barua, A note on pseudo conformally symmetric spaces, Jour. Pure. Math., 9 (1992), 1-4.
5. L. P. Eisenhart, Riemannian Geometry, Princeton University Press (1954), 92.
6. K. Yano Differential geometry on complex and almost complex spaces, Pergamon Press (1965), 70-72.

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THE EFFECT OF CERTAIN TRANSFORMATIONS ON LEBESGUE MEASURABLE SETS AND SETS HAVING THE BAIRE-PROPERTY IN R^n

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ABSTRACT : In [5], Pal has extended a theorem of Kurepa [4] and there is a result in [1] from which Pal's theorem follows immediately. Here we investigate the effects of certain transformations on the class of Lebesgue measurable sets and sets having the Baire-property [6] in R^n the aim of which is to generalize the results in [1].

Key words and phrases : Baire-property, metric density, $\mathscr{I}$-density, difference set.

Kurepa [4] proved the following theorem :

If $A \subseteq R^n$ (n-dimensional euclidean space) is Lebesgue measurable having a positive measure and $\alpha_1, \alpha_2, \ldots \alpha_p$ ($\alpha_k \neq 0$) are p real numbers, then there exists an open ball K_r of radius r with centre at the origin such that for each $x \in K_r$, there exist vectors $a_0(x)$, $a_1(x), \ldots a_p(x)$ in A stisfying the relation : —

$$x = \frac{a_1(x) - a_0(x)}{\alpha_1} = \frac{a_2(x) - a_0(x)}{\alpha_2} = \ldots = \frac{a_p(x) - a_0(x)}{\alpha_p}$$

The following extension of Kurepa's theorem is due to Pal [5].

Let A and B be Lebesgue measurable sets of positive measures in R^n. Also let $\alpha_1, \alpha_2, \ldots \alpha_p$ ($\alpha_k \neq 0$) be any system of p real numbers and let $\left\{\beta_k\right\}_{k=1}^{\infty}$ be any sequence of non-zero real numbers containing 1 such that $\lim\limits_{k\to\infty} \beta_k = 1$. Then there exist a positive integer N_0 and balls K (with centre at the origin) and K_1 such that for any vector x in K, for a system of p vectors $z_1, z_2, \ldots, z_p$ in K_1 and for a system of p real numbers

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$\beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_p}$ of the sequence $\left\{\beta_k\right\}_{k=1}^{\infty}$ with $k_i \geq N_0$, there exist vectors

$$a(x\ ;\ z_1, z_2, \ldots, z_p, \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_p}) \in A \text{ and}$$

$$b_i(x\ ;\ z_1, z_2, \ldots, z_p, \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_p}) \in B \quad (i = 1,2,\ldots,p)$$

such that

$$x = \frac{\frac{1}{\beta_{k_i}} b_i(x;\ z_1, z_2, \ldots z_p, \beta_{k_1}, \beta_{k_2}, \quad , \beta_{k_p})}{\alpha_i}$$

$$- \frac{a(x;\ z_1, z_2, \ z_p, \beta_{k_1}, \beta_{k_2}, \quad , \beta_{k_p}) + \frac{z_i}{\beta_{k_i}}}{\alpha_i} \quad (i = 1,2,\ldots,p).$$

The open balls K and K_1 (in Pal's theorem) have been constructed using metric density points of A and B and the choice of the sequence $\left\{\beta_k\right\}_{k=1}^{\infty}$. Also the positive integer N_0 depends on $\left\{\beta_k\right\}_{k=1}^{\infty}$. Since the formulation of pal's theorem rests heavily on some measure-theoritic properties of measurable sets of positive measures (the most important of these being the existence of metric density points) they cannot be avoided in the constructions of K and K_1. But the other dependences and the artificial restriction (the condition that $1 \in \left\{\beta_k\right\}_{k=1}^{\infty}$) could be waived. The following theorem [1] which may be viewed as a modification of Pal's theorem does this to some extent.

If A and B are Lebesgue measurable sets of positive measures in R^n, $\left\{\alpha_k\right\}_{k=1}^{\infty}$ be any convergent sequence of non-zero reals having a finite limit ; then given any positive integer p, we can determine an interval I in R (with positive abscissae and 1 as an interior point), balls K_0 (with centre at the origin) and K_1 in R^n such that for every vector x from K_0, sequence $\left\{z_k\right\}_{k=1}^{\infty}$ of vectors from K_1 (with a limit point at the centre of K_1) and a sequence $\left\{\beta_k\right\}_{k=1}^{\infty}$ of real numbers from I (having 1 as a limit point), there exist subsequences $\left\{z_{k_i}\right\}_{i=1}^{\infty}$ of $\left\{z_k\right\}_{i=1}^{\infty}$ whose first p terms are z_1, $z_2, \ldots, z_p$; $\left\{\beta_{k_i}\right\}_{i=1}^{\infty}$ of $\left\{\beta_k\right\}_{k=1}^{\infty}$ whose first p terms are $\beta_1, \beta_2, \ldots, \beta_p$ satisfying the following property :—

There exist vectors $a(x) \in A$, $b_i(x) \in B$ $(i = 1,2,3\ldots)$ such that :—

$$x = \frac{\frac{b_i(x)}{\beta_{k_i}} - \left(a(x) + \frac{z_{k_i}}{\beta_{k_i}}\right)}{\alpha_{k_i}} \quad (i = 1,2,3\ldots), \text{ where } \left\{\alpha_{k_i}\right\}_{i=1}^{\infty} \text{ is a subsequence}$$

of $\left\{\alpha_k\right\}_{k=1}^{\infty}$ with its first p terms as $\alpha_1, \alpha_2, \ldots, \alpha_p$.

The word 'limit point' of a sequence (in the above theorem) has been used and shall henceforth be used in the sense of subsequential limit.

One observes in the above theorem that the balls K and K_1 do not depend on the choice of any sequence $\left\{\beta_k\right\}_{k=1}^{\infty}$. Also the positive integer N_0 may be taken as 1 for all sequences $\left\{\beta_k\right\}_{k=1}^{\infty}$ that lie in the interval I and are convergent at 1. Indeed the result is much stronger compared to Pal's theorem for the later follows almost immediately from the former (this can be easily cheked). In spite of these, there is another notable distinction. Here, unlike as in Pals' theorem, each point x in K has been expressed in an infinitely many ways. Also there runs in [1] the exact Baire-category analogue of the said result that extends the categorical version of Kurepa's theorem.

Although in presentation, the result in [1] possesses greater uniformity compared to Pal's theorem, yet its formulation depends on the choice of the sequence $\left\{\alpha_k\right\}_{k=1}^{\infty}$ (more to say on its first p members $\alpha_1, \alpha_2, \ldots, \alpha_p$). In the present paper, our main aim is to show that even this dependence could be ruled out and this is possible in some more general settings. Here we begin with a general class of transformations which in particular include those giving rise to Pal's theorem and theorem 1 of [1], prove a general theorem from which all the results hitherto mentioned follows as particular cases and illustrate that in one special situation it yields a strong modification of theorem 1 of [1] in complete tune with the situation we are desiring. This would be followed by somewhat an analogous result in the realm of Baire-category where it is found that even stronger theorem holds. We end up this article with a counterexample showing that no such extension is however possible in the measure-theoritic case.

A close scrutiny into the nature of the proofs reveals that the statements of Pal's theorem and theorem 1 of [1] ought to have more refined expressions than in the way they are expressed. For example, on reflection on the later one finds that corresponding to each point a (say) of density of A and each point b(say) of density of B, there exists (by construction) a triplet $\{K, K_1, I\}$ where K is an poen ball with origin as centre, K_1 with centre at the point $c = b - a$ and an interval I in the real line having positive abscissae and 1 as interior point for which the said formulation holds. This is the main reason why Kurepa's theorem may be considered as one particular case of the above result (for therein, since $A = B$ we may choose $a = b$ and select the sequences according to our needs). In all the results that are to follow from now onwards, we will lay stress upon this factor in order to induce greater strength into our results.

Throughout the sequel, unless otherwise specified, we are to consider only subsets of R^n. Measure is used in the sense of Lebesgue and the term Baire-category is used with reference to the well-accepted notion called 'Baire-property' of sets. We give below a list of symbols which are to be used frequenty in what follows.

(i) N and R (resp, R^+) are the set of natural and the real numbers (resp, positive real numbers).

(ii) m stands for the Lebesgue measure and $\mathscr{L}$ for the class of Lebesgue measurable sets.

(iii) $A + x$, $(\lambda_1, \lambda_2, \ldots, \lambda_n)A$ $((\lambda_1, \lambda_2, \ldots, \lambda_n) \in R^n)$, $\overline{A}$, $A \setminus B$, $A \Delta B$, A' and $[-1, 1]^n$ for the x-trans late of A, the set $\{(\lambda_1.x_1, \lambda_2.x_2, \ldots, \lambda_n.x_n) ; (x_1, x_2, \ldots, x_n) \in A\}$, the closure of A, difference and symmetric difference of two sets A and B, the complement of A and the unit interval in R^n.

(iv) χ_A and $\mathscr{D}(A)$ are the characteristic function and the difference set of A (that is the set $\{x - y ; x, y \in A\}$). For any sequence $\left\{A_n\right\}_{n=1}^{\infty}$ of sets, $\limsup\limits_{n \to \infty} A_n$ and $\liminf\limits_{n \to \infty} A_n$ designates its upper and lower limits.

(v) $\|.\|$, dist (A,B) and $\delta(A)$ stand for the usual norm, the distance between two sets A and B and the diameter of A in R^n. In particular, $|.|$ denotes the norm in R.

(vi) $\mathscr{T}$ and $\mathscr{B}$ represent the σ-ideal of first category sets and the σ-algebra of sets with the Baire-property.

To begin with, let Ω be any arbitrary metric space and $f : R^n \times R^n \times \Omega \to R^n$ be a transformation satisfying the following auxilliary conditions :—

(i) $f(A,y,\omega) \in \mathscr{L}$ for every $A \in \mathscr{L}$, $y \in R^n$ and $w \in \Omega$.

(ii) $\lim_{k\to\infty} m(f(A, y_k, \omega_k)\, \Delta f(A, y, \omega)) = 0$ for every $A \in \mathscr{L}$ and convergent sequences $\{y_k\}_{k=1}^{\infty} \subseteq R^n$ and $\{\omega_k\}_{k=1}^{\infty} \subseteq \Omega$ with $\lim_{k\to\infty} y_k = y$ and $\lim_{k\to\infty} \omega_k = \omega$. In particular, there exists a fixed element $\omega_0 \in \Omega$ such that $\lim_{k\to\infty} m(f(A, y_k, \omega_k)) = m(A)$ whenever $\lim_{k\to\infty} \omega_k = \omega_0$.

(iii) If $\lim_{k\to\infty} y_k = y$, $\lim_{k\to\infty} \omega_k = \omega_0$, then $\lim_{k\to\infty} f(x, y_k, \omega_k) = x - y$.

(iv) To each $a \in R^n$, there is associated an open ball K_a (with centre at the point a) such that for convergent sequences $\{\omega_k\}_{k=1}^{\infty} \subseteq \Omega$ and $\{y_k\}_{k=1}^{\infty} \subseteq \Omega$ with $\lim_{k\to\infty} \omega_k = \omega_0$, there exist a number $M(y_k ; \omega_k) > 0$ and $N_a(y_k ; \omega_k) \in N$ such that $\| f(x, y_k, \omega_k) - f(x', y_k, \omega_k) \| \le M_a(y_k ; \omega_k) \| x - x' \|$ for all x, x' ($\in K_a$) and $k \ge N_a(y_k ; \omega_k)$.

The following are some non-trivial examples of the above classes of transformations :—

Example 1. It we choose $\Omega = R\backslash\{0\}$ (with respect to the metric induced by the usual metric on R), $\omega_0 = 1$ and define $f ; R^n \times R^n \times \Omega \to R^n$ by the formula $f(x, y, \omega) = \dfrac{x-y}{\omega}$ ($x, y \in R^n$, $\omega \in \Omega$), then f satisfies all the conditions (i) – (iv) stated above.

Example 2. Let Ω denotes the collection of all nxn matrices (a_{ij}) and we define the distance between two elements (a_{ij}) and (b_{ij}) in Ω by the formula $d((a_{ij}), (b_{ij})) = \sum_{i=1}^{n} \sum_{j=1}^{n} | a_{ij} - b_{ij} |$; ω_0 being the identity matrix and we define $f ; R^n \times R^n \times \Omega \to R^n$ by the formula $f(x,y,\omega) = (z_1, z_2,\ldots,z_n)$ where $z_1 = \sum_{j=1}^{n} a_{ij} (x_j - y_j)$ $(1 = 1, 2,\ldots,n)$, $x = (x_1, x_2\ldots x_n)$, $y = (y_1, y_2, \ldots, y_n)$ ($\in R^n$) and $\omega = (a_{ij}) \in \Omega$. Then f satisfies all the conditions (i) – (iv) stated above.

Definition 1. [7] A point $a \in R^n$ is called a 'point of density' of a set $A \in \mathscr{L}$ if $\lim\limits_{k\to\infty} \dfrac{m(A \cap C_k)}{m(C_k)} = 1$ for all sequences $\{C_k\}_{k=1}^{\infty}$ of cubes satisfying $a \in C_k$ for all $k \in N$ and $\lim\limits_{k\to\infty} \delta(C_k) = 0$.

For any $A \in \mathscr{L}$, we write

$\Phi(A) = \{x \in R^n ;\ x \text{ is a point of density of } A\}$.

We state below two standard results on density points which we would be using in the sequel. The proofs are easy and follow from the definition given above and some simple measure-theoritic computations.

Proposition 1. Let A, B ($\in \mathscr{L}$). Then $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$.

Proposition 2. Let A, B ($\in \mathscr{L}$) ; $a \in \Phi(A)$, $b \in \Phi(B)$, and $c = b - a$. Then $a \in \Phi(A \cap (B - C))$.

Proposition 2 follows from propositon 1 chiefly by the use of translation invariance of the lebesgue measure.

Finally, our results are —

Theorem 1. Let A,B($\in \mathscr{L}$) with m(A), (>0), $\Lambda \subseteq R$ be any bounded open set and $f : R^n \times R^n \times \Omega \to R^n$ be any transformation satisfying condition (i) – (iv) stated above. Then given any $p \in N$, for every $a \in \Phi(A)$, $b \in \Phi(B)$, there exist an open ball W in Ω (with its centre at ω_0), open balls K_0 (with centre at the origin) and K_c (with centre at the point $c = b - a$) such that for every $x \in K_0$, sequences $\{\alpha_k\}_{k=1}^{\infty} \subseteq \Lambda \setminus \{0\}$, $\{z_k\}_{k=1}^{\infty} \subseteq K_c$ and $\{\omega_k\}_{k=1}^{\infty} \subseteq W$, there exist subsequences $\{\alpha_{k_i}\}_{i=1}^{\infty}$ whose first p terms are $\alpha_1, \alpha_2, \ldots, \alpha_p$; $\{z_{k_i}\}_{i=1}^{\infty}$ whose first p terms are $z_1, z_2, \ldots, z_p$ and $\{\omega_{k_i}\}_{i=1}^{\infty}$ whose first p terms are $\omega_1, \omega_2, \ldots, \omega_p$ respectively satisfying the following property : — there exist vectors $a(x) \in A$, $b_i(x) \in B$ ($i = 1, 2, 3, \ldots$) such that : —

$$x = \frac{f(b_i(x), z_{k_i}, \omega_{k_i}) - a(x)}{\alpha_{k_i}} \quad (i = 1, 2, 3, \ldots).$$

A proof of the theorem depends on the following lemmas

Lemma 1. Let $A \in \mathscr{L}$, $x \in R^n$. Then $\lim\limits_{x\to 0} m(A\Delta(A + x)) = 0$

We note that, $m(A\Delta(A + x)) \le 2 \int\limits_{R^n} \| \chi_A (z) - \chi_A (z - x)\| \, dz \to 0$

(as $x \to 0$) [8]

Hence the lemma.

Lemma 2. Let A, B and C ($\in \mathscr{L}$). Then $\| m(A \cap B) - m(A \cap C) \| \le m (B \Delta C)$.

The lemma is a simple consequence of the subadditivity of m.

Lemma 3. Let B be a closed set with $m(B) > 0$; $q \in N$ and $c \in R^n$. Let K_b be an open ball with centre at b for which condition (iv) stated above holds. Then

$$\lim_{\substack{x\to 0\\ z^{(k)}\to 0\\ \omega^{(k)}\to 0}} m((B \cap K_b - c) \,\Delta \bigcap_{k=1}^{q} \{f(B \cap K_b, z^{(k)}, \omega) - x\})) = 0$$

where x, $z^{(k)}$ and $\omega^{(k)}$ ($k = 1, 2, \ldots, q$) are independent continuous variables in R^n and Ω tending in the limit to 0,c and ω_0 independently of each other and simultaneously.

Proof. It suffices to prove that

$$\lim_{\substack{x\to 0\\ z\to c\\ \omega\to\omega_0}} m((B \cap K_b - c) \,\Delta\, \{f(B \cap K_b, z^{(k)}, \omega) - x\}) = 0,$$

for the remaining is a consequence of using some simple measuretheoritic properties of m. Let $\varepsilon > 0$ be given and we choose an open set $G \supseteq B \cap K_b - c$ such that $m (G \setminus (B \cap K_b - c)) < \frac{\varepsilon}{3}$.

We set $\hat{B} = B \cap K_b - c$ and let $\rho = \text{dist}\,(\hat{B}, G')$. Let $\{\omega_k\}_{k=1}^{\infty} \subseteq \Omega$ and $\{x_k\}_{k=1}^{\infty}$, $\{z_k\}_{k=1}^{\infty}$ ($\subseteq R^n$) be sequences such that $\lim\limits_{k\to\infty} \omega_k = \omega_0$, $\lim\limits_{k\to\infty} x_k = 0$ and $\lim\limits_{k\to\infty} z_k = c$. Then on account of condition (iv), there exist a number $M_b(z_k ; \omega_k) > 0$ and $N_b(z_k ; \omega_k) \in N$ such that $\| f(x, z_k, \omega_k) - f(y, z_k, \omega_k) \| \le M_b(z_k ; \omega_k). \| x - y \|$ for all $k \ge N_b(z_k ; \omega_k)$ and x, y ($\in K_b$).

We now set $\eta = \dfrac{\rho}{M_b(z_k : \omega_k)+3}$ and consider the covering $\mathscr{F} = \{S(x, \eta) ; x \in B \cap K_b\}$ of $B \cap K_b$ by open balls $s(x, \eta)$. The set $B \cap K_b$ being compact, there exists a finite collection $\{y_1, y_2, \ldots, y_m\}$ of points in $B \cap K_b$ for which $B \cap K_b \subseteq \bigcup\limits_{i-1}^{m} s(y_i, \eta)$. We set $F_i = (B \cap K_b) \cap S(y_i, \eta)$ (i = 1, 2,...,m). We claim that for each i, there exists $N_i \in N$ such that $f(F_i, z_k, \omega_k) - x_k \subseteq G$ for all $k \ge N_i$ (i = 1, 2, ..., m). For if $x \in F_i$, then by the triangle inequality, $\| f(x_i, z_k, \omega_k) - x_k - (x - c) \le \| f(x, z_k, \omega_k) - f(y_i, z_k, \omega_k) \| + \| f(y_i, z_k, \omega_k) - (y_i - c) \| + \| y_i - x \| + \| x_k \|$ for all $k \in N$ and i = 1, 2,..., m.

Now since $x \in F_i$, $y_i \in K_b$, be condition (iv), $\| f(x, z_k, \omega_k) - f(y_i, z_k, \omega_k) \| \le M_b(z_k ; \omega_k) \| x - y_i \| < M_b(z_k ; \omega_k)\eta$ for all $k \ge N_b(z_k ; \omega_k)$.

Again by condition (iii), there exists $\hat{N}_i \in N$ such that $\| f(y_i, z_k, \omega_k) - (y_i - c) \| < \eta$ for all $k \ge \hat{N}_i$.

Finally, let $N_0 \in N$ such that $\| x_k \| < \eta$ for all $k \ge N_0$.

If now we set $N_i = \text{Max}\{\hat{N}_i, N_0, N_b(z_k ; \omega_k)\}$, then $\| f(x, z_k, \omega_k) - x_k - (y_i - c) \| < \rho$ for all $x \in F_i$ and $k_i \ge N_i$ (i = 1, 2, ..., m).

Hence it follows that $f(F_i, z_k, \omega_k) - x_k \subseteq G$ for all $k \ge N_i$ (i = 1, 2), ..., m). Let $\hat{N} = \text{Max}\{N_i ; i = 1, 2, \ldots, m\}$. Then $f(B \cap K_b, z_k, \omega_k) - x_k \subseteq G$ for all $k \ge \hat{N}$. Again, by condition (ii), there exists $\hat{\hat{N}} \in N$ such that $m(f(B \cap K_b, z_k, \omega_k)) > m(B \cap K_b) - \frac{\varepsilon}{3}$ for all $k \ge \hat{\hat{N}}$. Equivalently, $m(f(B \cap K_b, z_k, \omega_k)) > m(B \cap K_b - c) - \frac{\varepsilon}{3}$ for all $k \ge \hat{\hat{N}}$, since m is translation invariant. Hence finally since m is monotone and subtractive,

$$\begin{aligned}
&m((B \cap K_b - c) \,\Delta\, \{f(B \cap K_b, z_k, \omega_k) - x_k\}) \\
&= m((B \cap K_b - c) \setminus \{f(B \cap K_b, z_k, \omega_k) - x_k\}) \\
&\quad + m(\{f(B \cap K_b, z_k, \omega_k) - x_k\} \setminus (B \cap K_b - c))
\end{aligned}$$

$$\leq m(G \setminus \{f(B \cap K_b, z_k, \omega_k) - x_k\}) - m(G \setminus \hat{B}) \text{ for all } k \geq \hat{N}.$$

$$< m(G) - m(\{f(B \cap K_b, z_k, \omega_k) - x_k\}) + m(G \setminus \hat{B}) \text{ for all } k \geq \hat{N}.$$

$$< m(G \setminus \hat{B}) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \text{ for all } k \geq N = \text{Max}\,\{\hat{N}, \hat{\hat{N}}\}.$$

Therefore, $\lim\limits_{\substack{x \to 0 \\ z \to c \\ \omega \to \omega_0}} m((B \cap K_b - c) \,\Delta\, \{f(B \cap K_b, z, \omega) - x\}) = 0.$

Hence the lemma.

Proof of the theorem. Since m is regular, without loss in generality we may assume that B is closed with $m(B) > 0$. Let $a \in \Phi\,(A)$, $b \in \Phi\,(B)$ and $c = b - a$. Now an open ball K_b can be so chosen that $b \in \Phi\,(B \cap K_b)$ and condition (iv) is satisfied. But then by the use of propositions I and II, $a \in \Phi\,(A \cap (B \cap K_b - c))$ and therefore $m\,(A \cap (B \cap K_b - c)) > 0$. Now on setting $q = p + 1$, by use of lemmas 1, 2 and 3 and the subadditivity of m, it follows that

$$\lim_{\substack{x \to 0 \\ u \to 0 \\ z^{(k)} \to c \\ \omega^{(k)} \to \omega_0}} m((A + u) \cap (B \cap K_b - c) \cap (\bigcap_{k=1}^{p+1} \{f(B \cap K_b, z^{(k)}, \omega^{(k)}) - x\}))$$

$= m\,(A \cap (B \cap K_b - c))$, where x, u, $z^{(k)}$ and $\omega^{(k)}$ $(k = 1,2,\ldots,p + 1)$ are independent continuous variables in R^n and Ω tending in the limit to 0, c and ω_0 independently of each other and sumultaneously.

Hence there exist an open ball in $\tilde{W}$ in Ω (with its centre at ω_0), open ball $\tilde{K}_0$ (with centre at the origin) and $\tilde{K}_c$ (with centre at the point $c = b - a$) such that for every x and u $(\in \tilde{K}_0)$, system of $p + 1$ vectors $z_1, z_2, \ldots, z_{p+1}$ $(\in \tilde{K}_c)$ and a system of $p + 1$ elements $\omega_1, \omega_2, \ldots, \omega_{p+1} (\in \tilde{W})$,

$$m((A + u) \cap (B \cap K_b - c) \cap (\bigcap_{k=1}^{p+1} \{f(B \cap K_b, z_k, \omega_k) - x\}))$$

$$> \frac{1}{2} m (A \cap (B \cap K_b - c)).$$

Let r_0, be the radius of $\tilde{K}_0$. Moreover $\mathscr{D}(\bar{\Lambda})$ being a bounded set has a finite diameter. We set K_0 to be the open ball with centre at the origin and radius equal to Min $\left\{ \frac{r_0}{\sup\{|x| ; x \in \bar{\Lambda}\}} , \frac{r_0}{\delta(\mathscr{D}(\bar{\Lambda}))+1} \right\}$, K_c and W to be open balls in Ω and R^n concentric to $\tilde{K}_c$ and $\tilde{W}$ and having half their radii.

Now let $x \in K_0$, sequences $\{\alpha_k\}_{k=1}^{\infty} \subseteq \Lambda \setminus \{o\}$, $\{z_k\}_{k=1}^{\infty} \subseteq K_c$ and $\{\omega_k\}_{k=1}^{\infty} \subseteq \Omega$. Clearly then there exist convergent subsequences $\{\alpha_{k_i}\}_{i=1}^{\infty}$ with first p terms as $\alpha_1, \alpha_1, \ldots, \alpha_p$; $\{z_{k_i}\}_{i=1}^{\infty}$ with first p trems as $z_1, z_2, \ldots, z_p$ and $\{\omega_{k_i}\}_{i=1}^{\infty}$ with first p terms as $\omega_1, \omega_1, \ldots, \omega_p$ converging (say) to the limits $\alpha \in \bar{\Lambda} \subseteq \mathscr{D}(\bar{\Lambda})$, $z' \in \bar{K}_c \subseteq \tilde{K}_c$ and $\omega' \in \bar{W} \subseteq \tilde{W}$. Now on setting $\lambda_k = \alpha_k - \alpha$ ($k = 1, 2, \ldots p$) and $\lambda_{k+1} = 0$, $z_{k+1} = z'$ and $\omega_{k+1} = \omega'$, we get

$$m((A + \alpha x) \cap (B \cap K_b - c) \cap (\bigcap_{k=1}^{p} \{(B \cap K_b, z_k, \omega_k) - \lambda_k x \} \cap$$

$$\{f (B \cap K_b, z', \omega')\})) > \frac{1}{2} m (A \cap (B \cap K_b, - c)) > 0.$$

Now a successive use of the first part of condition (ii), lemma 2 and some simple measure-theoritic computations yield that there exist subsequences $\{\alpha_{k_i}\}_{i=1}^{\infty}$ with first $\alpha_1, \alpha_1, \ldots, \alpha_p$; $\{z_{k_i}\}_{i=1}^{\infty}$ with first p terms as $z_1, z_2, \ldots, z_p$ and $\{\omega_{k_i}\}_{i=1}^{\infty}$ with first p terms as $\omega_1, \omega_1, \ldots, \omega_p$ such that

$$m((A + \alpha x) \cap (\bigcap_{k=1}^{\infty} \{f(B \cap K_b,\ z_{k_i},\ \omega_{k_i}) - \lambda_k\ x\ \}))$$

$$> \frac{1}{4}\ m\ (A \cap (B \cap K_b - c)) > 0, \text{ where } \lambda_{k_i} = \alpha_{k_i} - \alpha\ (i = 1,2,3, ..).$$

Equivalently, by translation invariance of m,

$$m((A \cap (\bigcap_{k=1}^{\infty} \{f(B \cap K_b,\ z_{k_i},\ \omega_{k_i}) - \alpha_{k_i}\ x\ \}))$$

$$> \frac{1}{4}\ m\ (A \cap (B \cap K_b - c)) > 0.$$

The remaining part is obvious.

Note 1.1. If we set $\Omega = R \backslash \{o\}$, $\omega_0 = 1$ and $f(x,y,\omega) = \frac{x-y}{\omega}$ $(x,y \in R^n,\ \omega \in \Omega)$, theorem 1 reduces to the following particular case.

Corollary 1.1. Let A, B ($\in \mathscr{L}$) with m(A), m(B) (> 0) and $\Lambda \subseteq R$ be any bounded open set. Then given any $p \in N$, for every $a \in \Phi$ (A), $b \in \Phi$ (B), we can determine an open interval I inR (with positive abscissae and 1 as an interior point), balls K_0 (with centre at the origin) and K_c (with centre at the point c = b − a) such that for every $x \in K_0$, sequences $\{\alpha_k\}_{k=1}^{\infty} \subseteq \Lambda \backslash \{o\}$, $\{z_k\}_{k=1}^{\infty} \subseteq K_c$ and $\{\beta_k\}_{k=1}^{\infty} \subseteq I$, there exist subsequence $\{\alpha_{k_i}\}_{i=1}^{\infty}$ whose first p terms are $\alpha_1, \alpha_1, \ldots, \alpha_p$; $\{z_{k_i}\}_{i=1}^{\infty}$ whose first p terms are $z_1, z_2, \ldots, z_p$ and $\{\beta_{k_i}\}_{i=1}^{\infty}$ whose first p terms are $\beta_1, \beta_1, \ldots, \beta_p$ respectively satisfying the following property : —

there exist vectors $a(x) \in A$ and $b_i\ (x) \in B$ (i = 1,2,3,...) such that : —

$$x = \frac{\frac{b_i(x)}{\beta_{k_i}} - (a(x) + \frac{z_{k_i}}{\beta_{k_i}})}{\alpha_{k_i}} \quad (i = 1,2,3,...)$$

By all means, the above carollary is far more general than theorem 1 of [1]. Firstly, it is more uniform in its presentation (its formulation does not require the initial choice of a convergent sequence $\{\alpha_k\}_{k=1}^{\infty}$). Secondly, it is a strong modification of the said result for here it is not necessary for the sequences $\{z_k\}_{k=1}^{\infty} \subseteq K_c$ and $\{\beta_k\}_{k=1}^{\infty} \subseteq I$ to have limit points at c (the centre of K_0) and at 1 respectively. Also it can be checked that all the results hitherto mentioned follows form the above corollary.

So long we have been considering measurable sets. We now proceed to formulate an analogous result for sets with Baire-property but this time with even greater generality. A set A is said to have the Baire-property (in short, $A \in \mathscr{B}$) if $A = G \Delta P$ where G is open and $P \in \mathscr{T}$.

As before, we consider Ω to be an arbitrary metric space and ω_0 a fixed element in Ω.

Let $f : R^n \times R^n \times \Omega \to R^n$ be a transformation staisfying the following auxilliary conditions :—

(i) $\lim\limits_{k\to\infty} f(x, y_k, \omega_k) = x - y$ for every x, y ($\in R^n$) and sequences $\{\omega_k\}_{k=1}^{\infty} \subseteq \Omega$ and $\{y_k\}_{k=1}^{\infty} \subseteq \Omega$ respectively such that $\lim\limits_{k\to\infty} \omega_k = \omega_0$ and $\lim\limits_{k\to\infty} y_k = y$.

(ii) If I be any interval, $P \in \mathscr{T}$ and $\{\omega_k\}_{k=1}^{\infty} \subseteq \Omega$ and $\{y_k\}_{k=1}^{\infty} \subseteq R^n$ are convergent sequences with $\lim\limits_{k\to\infty} \omega_k = \omega_0$, then there exists $N_0 \in N$ such that $f(I, y_k, \omega_k)$ is an interval and $f(P, y_k, \omega_k) \in \mathscr{T}$ for all $k \geq N_0$.

Example 1 and Example 2 illustrated above (with Ω in the second case representing the collection of all nxn non-singular matrices) exhibit two particular but non-trivial situations in which conditions (i′) and (ii′) stated above are satisfied.

The notion of density point defined above (Definition 1) is usually called metric density point. An exact Baire-category analogue of the metric density point is the notion of strong $\mathscr{T}$-density point. In [3], [9] definitions of this concept for sets with the Baire-property in the real line and in the euclidean plane are given. To meet the present requirements, we introduce here an extended version of this concept for sets in R^n.

Definition 2. A point $x \in R^n$ is called a point of strong $\mathscr{T}$-density of a set $A \in \mathscr{B}$ if

$$[-1.1]^n \setminus \liminf_{k\to\infty} (t_k^{(1)}, t_k^{(2)}, \ldots, t_k^{(n)})(A - x) \cap [-1.1]^n \in \mathscr{T} \text{ for every sequences}$$

$\left\{t_k^{(i)}\right\}_{k=1}^{\infty} \subseteq R^+$ such that $t_k^{(i)} \to \infty$ as $k \to \infty$ $(i = 1, 2, \ldots, n)$

For any $A \in \mathscr{B}$, we write $J(A) = \{x \in R^n$; x is a point of strong $\mathscr{T}$-density of $A\}$.

Likewise as in above, we may formulate (using definition 2 and some simple set-theoritic computations) the Baire-category analogues of propositions I and II stated above.

Proposition I. Let A, B ($\in \mathscr{B}$). Then $J(A \cap B) = J(A) \cap J(B)$

Proposition II. Let A, B ($\in \mathscr{B}$) ; $a \in J(A)$, $b \in J(B)$ and $c=b-a$. Then $a \in J(A \cap (B - C))$.

Theorem 2. Let A, B ($\in \mathscr{B}$) are of second category, $\Lambda \subseteq R$ be any bounded open set and $f : R^n \times R^n \times \Omega \to R^n$ be a transformation satisfying conditions (i′) and (ii′) stated above. Then given any $p \in N$, for every $a \in J(A)$, $b \in J(B)$, there exists an open ball K_0 (with centre at the origin) and K_c (with centre at the point $c=b-a$) such that for every $x \in K_0$, sequences $\{\alpha_k\}_{k=1}^{\infty} \subseteq \Lambda \setminus\{0\}$, $\{z_k\}_{k=1}^{\infty} \subseteq K_c$ and $\{\omega_k\}_{k=1}^{\infty} \subseteq W$, there exist vectors $a(x) \in A$, $b_k(x) \in B$ $(k = 1,2,3,\ldots)$ such that :—

$$x = \frac{f(b_k(x), z_k, \omega_k) - a(x)}{\alpha_k} \quad (k = 1,2,3,\ldots)$$

Proof of the theorem. By definition, $A = G_1 \Delta P_1$, $B = G_2 \Delta P_2$ where G_1, G_2 are non-empty open sets and P_1, P_2 ($\in \mathscr{T}$). Let $a \in J(A)$, $B \in J(B)$ and $c=b-a$. By proposition (ii′), $a \in J(A \cap (B - c))$.

$$\text{Therefore } [-1.1]^n \setminus \liminf_{k\to\infty} (t_k^{(1)}, t_k^{(2)}, \ldots, t_k^{(n)})(A \cap (B - c) - a) \cap [-1.1]^n \in \mathscr{T}$$

for every sequences $\left\{t_k^{(i)}\right\}_{k=1}^{\infty} \subseteq R^+$ such that $t_k^{(i)} \to \infty$ as $k \to \infty$ $(i = 1,2,\ldots,n)$.

Consequently, $A \cap (B - C)$ and therefore $G_1 \cap (G_2 - C)$ is a set of second category.

Let C_1 and C_2 be cubes (intervals in R^n whose all sides are of equal lengths) of equal volumes contained in G_1 and G_2 respectively. Let C^* be be a proper subcube of C_1 concentric with C_1. Now by using conditions (i′) and (ii′), we can select an open ball W in Ω (with centre at ω_0), open balls $\tilde{k}_0$ (with centre at the origin) and K_c (with centre at the point c=b–a) such that for every $u \in \tilde{k}_0$, $z \in K_c$ and $\omega \in W$ the inclusion.

$C_1 \cap \{f(C_2, z, \omega) - u\} \supseteq C^*$ holds. Let r_0 be the radius of $\tilde{k}_0$ and we set K_0 to be the open ball with centre at the origin and radius equal to $\dfrac{r_0}{\sup\{|x|\ ;\ x \in \bar{\Lambda}\}+1}$. Then for every $x \in K_0$, sequences $\{\alpha_k\}_{k=1}^{\infty} \subseteq \Lambda \setminus \{o\}$, $\{z_k\}_{k=1}^{\infty} \subseteq K_c$ and $\{\omega_k\}_{k=1}^{\infty} \subseteq W$, we have

$$C_1 \cap \left(\bigcap_{k=1}^{\infty} \{f(C_2, z_k, \omega_k) - \alpha_k x\}\right) \supseteq C^*$$

But then,

$$A \cap \left(\bigcap_{k=1}^{\infty} \{f(B, z_k, \omega_k) - \alpha_k x\}\right) \supseteq C_1 \cap \left(\bigcap_{k=1}^{\infty} \{f(C_2, z_k, \omega_k) - \alpha_k x\}\right)$$

$$\setminus P_1 \cup \left(\bigcup_{k=1}^{\infty} \{f(P_2, z_k, \omega_k) - \alpha_k x\}\right).$$

The right hand side of the above inclusion being a set of second category, it follows that $A \cap \left(\bigcap_{k=1}^{\infty} \{f(B, z_k, \omega_k) - \alpha_k x\}\right)$ is a set of second category and hence non-empty. The remaining part is obvious.

Note 2.1. If we set $\Omega = R \setminus \{0\}$, $\omega_0 = 1$ and $f(x,y,\omega) = \dfrac{x-y}{\omega}$ $(x,y \in R^n,\ \omega \in \Omega)$, then theorem 2 reduces to the following particular case.

Corollary 2.1. Let A, B ($\in \mathscr{B}$) are sets of second category and $\Lambda \subseteq R$ be any bounded open set. Then given any $p \in N$, for every $a \in J(A)$, $b \in J(B)$, we can determine an open interval I in R (with positive abscissae and 1 as an interior point), open balls K_0 (with centre at the origin) and K_c (with centre at the point $c = b - a$) such that for every $x \in K_0$, sequences $\{\alpha_k\}_{k=1}^{\infty} \subseteq \Lambda \setminus \{o\}$, $\{z_k\}_{k=1}^{\infty} \subseteq K_c$ and $\{\omega_k\}_{k=1}^{\infty} \subseteq W$, there exist vectors $a(x) \in A$, $b_k(x) \in B$ $(k = 1,2,3,\ldots)$ such that :—

$$x = \frac{f(b_k(x),\ z_k,\ \omega_k) - a(x)}{\alpha_k} \quad (k = 1,2,3,\ldots)$$

In [1], along with theorem 1, we have also formulated its exact Baire-category analogue (theorem 2 of [1]). On the same lines of reasoning (as stated at the end of corollary 1.1) corollary 2.1 may be viewed as more general and a strong modification of theorem 2 of [1].

It follows from the presentation of theorem 2 of this article that it concludes something far stronger than theorem 1, for here we need not require to restrict ourselves to subsequences. We now show that no parallal presentation is possible in the meaure-theoritic case. This is done through one counterexample which exhibits that the meaure analogue of corollary 2.1 is false. The following example is the one which fits our purpose.

Borwein and Ditor [2] showed that there exists a set $A \in \mathscr{L}$ and a monotonically decreasing sequences $\{d_k\}_{k=1}^{\infty}$ converging to zero such that for every $x \in A$, $x + d_k \notin A$ for infinitely many values of k. If now we set $\Omega = R \setminus \{0\}$, $\omega_0 = 1$, $f(x, y, \omega) = \frac{x-y}{\omega}$ $(x, y \in R^n, \omega \in \Omega)$ and $A = B$, $a = b$, $\alpha_k = d_k$, $\beta_k = 1$ and $z_k = 0$ for all $k \in N$, then it is evident from this example that corollary 2.1 in terms of measurable sets of positive measure never holds.

REFERENCES

1. S. Basu, Generalization of a theorem of S. Kurepa and its category analogue, Jour. Pure. Math. 12 (1995), 36-43.
2. D. Borwein and S. Z. Ditor, Translates of a sequence of sets of positive measure, Canad. Math. Bull. 21 (4) (1978), 497-498.
3. M. Balcerzak, E. Lazarow and W. Wilczynski, One and two-dimensional $\mathscr{J}$-densities and related kinds of continuity, Real. Anal. Ex. 13 (No. 1-2) (1987-88).

4. S. Kurepa, Note on the difference set of two sets in E^n, Glas. Mat. Astr 15(35) (1960), 99-105.
5. M. Pal, On an extension of a theorem of S. Kurepa, Glas. Mat. 20 (40) (1985), 327-335.
6. J. C. Oxtoby, Measure and Category, Springer-Verlag, 1980.
7. H. Wisdorm, Lectures in measure and integration, Van Nostrand, 1969, 116.
8. J. H. Williamson, Lebesgue integration, Holt, Rinehart and Winston, 1962.
9. W. Wilczynski, A category analogue of density topology, approximate continuity and approximate derivative, Real. Anal. Ex. 10(2) (1984-85) 241-265.

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SEMIGROUPS WHOSE NONEMPTY SUBSETS ARE ALL N-SUBSET

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1. INTRODUCTION

Let S be a semigroup. A nonempty subset A of S is called a N–subset if

$$(\forall\ x,\ y \in S_1)\ x\,A\,y \cap A \neq \phi \text{ implies } x\,A\,y \subseteq A.$$

If A is also a subsemigroup of S, then we call A a N–subsemigroup.

In [3], we have discussed the properties of N-subsets of S and given an explicit description of the π-group (that is an eventually regular semigroup with exactly one idempotent element or a group-bound semigroup) congruence on an eventually regular semigroup S using N-subsets of S. Especially, we got another description of the group congruence on a regular semigroup and proved that a full N-subsemigroup of S is the same as a full, unitary, self-conjugate and closed subsemigroup. So the N-subsets of a semigroup are useful. It is somewhat like an ideal of S. Therefore we define a congruence on S by a N-subset A of S as defing the Rees congruence on S by an ideal of S.

Suppose A is a N-subset of S. Let

$$Q_A = \{(x,y) \in S \times S \mid x = y \text{ or } x,y \in A\}$$

and ρ_A is the congruence generated by Q_A. Then ρ_A is the smallest congruence on S such that A is a congruence class of it[3].

It is natural to ask what is the relationship between two N-sets A and B when $\rho_A = \rho_B$. In order to see this, we give the following definition.

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Definition 1.1. Let $\{B_\alpha \mid \alpha \in X\}$ be a family of nonempty subsets of a semigroup S. A subset A of S is called an associate union of $\{B_\alpha \mid \alpha \in X\}$ if

(i) $A = \cup_{\alpha \in X} B_\alpha$

(ii) For different a and b in A, there exist $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ in X such that $\alpha \in B_{\alpha_0}$, $b \in B_{\alpha_l}$ and $B_{\alpha_i} \cap B_{\alpha_{i+1}} \neq \phi$ for i = 0,1,2,...,n.

Proposition 1.2. Let A and B be two N-subsets of a semigroup S which contains more than one element. Then the following are equivalent :

(1) $\rho_A = \rho_B$

(2) A is an associate union of $\{uBv \mid uBv \cap A \neq \phi,\ u,\ v \in S^1\}$ and B is an associate union of $\{sAt \mid sAt \cap B \neq \phi,\ s,\ t \in S^1\}$

Proof. (1) implies (2). Let $I = S^1BS^1$. Then for all $x \in A$, $A = x\rho_A = x\rho_B \subseteq x\rho_1$. Therefore $x\rho_1$ contains more than one element. Let $a \in x\rho_1$ and $a \neq x$. Then $(a,x) \in \rho_1 = Q_1$ and therefore $a, x \in I$. Hence $A \subseteq I$, which implies that for every $x \in A$, there exist $u, v \in S^1$ such that $x \in uBv$. Whence A is contained in the union of

$$\{uBv \mid \phi \neq A \cap uBv, \text{ where } u,\ v \in S^1\}$$

Assume $u, v \in S^1$ and $uBv \cap A \neq \phi$. Then there exists $b \in B$ such that $ubv \in A$. For any $b_1 \in B$, $(b, b_1) \in \rho_B = \rho_A$. Therefore $(ub_1v)\rho_A(ubv)$ and then $ub_1v \in (ub_1v)\rho_A = A$. Thus $uBv \subseteq A$. So that

$$A = \cup \{uBv \mid uBv \cap A \neq \phi,\ u,\ v \in S^1\}$$

To show the above union is associate, let $a_1, a_2 \in A$ and $a_1 \neq a_2$. Then $(a_1, a_2) \in \rho_A = \rho_B$, so that there exist $x_1, x_2, \ldots, x_n \in S$ such that $(x_i, x_{i+1}) \in Q_B^C$, $x_i \neq x_{i+1}$, $a_1 = x_1$, $a_n = x_n$, $i = 1, 2, \ldots, n-1$. And then there exist $p_i, q_i, u_i, v_i \in S^1$ such that $(p_i, q_i) \in Q_B$ and

$$x_i = u_ip_iv_i,\ x_{i+1} = u_iq_iv_i,\ i = 1, 2, \ldots, n.$$

Since $x_i \neq x_{i+1}$, we have that $p_i \neq q_i$, for $i = 1, 2, \ldots, n$, and then $p_i, q_i \in B$, so that $a_1 = x_1 = u_lp_lv_l \in u_lBv_l$, $a_2 = u_np_nv_n \in u_nBv_n$ and

$$x_i = u_ip_iv_i = u_{i-1}q_{i-1}v_{i-1} \in (u_iBv_i) \cap (u_{i-1}Bv_{i-1}) \neq \phi.$$

Since $(x_i, x_{i+1}) \in \rho_B = \rho_A$ for $i = 1, 2, \ldots, n-1$, then $x_1 = a_1 \in a_1\rho_A = A$. Therefore by the definition of ρ_A and $x_i \neq x_{i+1}$, we have $x_2, x_3, \ldots, x_n \in A$. Hence $x_i \in u_iBv_i \cap A \neq \phi$ which means that A is a associate union of $\{uBv \mid uBv \cap A \neq \phi,\ u,\ v \in S^1\}$. Similarly, we can prove that B is a associate union of $\{sAt \mid sAt \cap B \neq \phi,\ s,\ t \in S^1\}$.

(ii) implies (i). Suppose, conversely, that $(x,y) \in Q_A$. Then $x = y$ or $x,y \in A$. If $x = y$, then $x\rho_B y$. Otherwise, $x,y \in A$, by (ii), there exist a family $\{u_iBu_i \mid i = 1, 2, \ldots, n\}$ out of $\{uBv \mid uBv \cap A \neq \phi,\ u,\ v \in S^1\}$ such that $x \in u_1Bv_1$, $y \in u_nBv_n$ and $u_iBv_i \cap u_{i+1}Bv_{i+1} \neq \phi$, $i = 1, 2, \ldots, n-1$. Choose $x_i \in (u_iBv_i) \cap (u_{i+1}Bv_{i+1})$ for every $i = 1, 2, \ldots, n-1$. It is easy to see that $(x_i, x_{i+1}) \in \rho_B$ and thus $x\rho_B y$. Therefore $Q_A \subseteq \rho_B$. Hence $\rho_A \subseteq \rho_B$. Similarly, we have $\rho_B \subseteq \rho_A$. So $\rho_A = \rho_B$.

2. REGULAR SEMIGROUPS WHOSE NONEMPTY SUBSETS ARE ALL N-SUBSETS

Throughout this section, S will always be a regular semigroup.

E(S) denotes the set of idempotents of S. V(a) is the set of inverses of $a \in S$.

Lemma 2.1.[3] Let A be a N-subset of S. Then A is a N-subsemigroup if and only if A contains an idempotent of S.

Lemma 2.2. If every nonempty subset of S is N-subset, then the following hold:

(i) If there exist $a \in S$ and $e \in E(S)$ such that $a \neq e$ and $eae = e$, then $eSe = \{e\}$.

(ii) For every $a \in S$, $a \in V(a)$.

(iii) For every $a \in S$, $e \in E(S)$, $ea = e$ or a and $ae = a$ or e.

(iv) If S is a group, then $|S| \leq 2$.

Proof.

(i) Suppose there exist $a \in S$ and $e \in E(S)$ such that $a \neq e$ and $eae = e$. Then $A = S\backslash\{e\} \neq \phi$ and $eAe \not\subseteq A$. Since A is a N-subset, we have $eAe \cap A = e$. Hence $eAe = \{e\}$ and thus $eSe = \{e\}$.

(ii) For $a \in S$, if $a\ \overline{\in}\ V(a)$, then for every $a' \in V(a)$, $a \neq a'$. Let $A = S\backslash\{a\}$. Then $a = aa'a \in aAa \not\subseteq A$ and thus $aAa \cap A = \phi$ since A is a N-subset of S. So $aAa = \{a\}$. It is easy to see that $a\ \overline{\in}\ E(S)$. Thus $a'a \in A$ and then $a^2 = (aa')a = a(a'a) \in Aa = \{a\}$. Hence $a^2 = a \in E(S)$. The contradiction implies that $a \in V(a)$.

(iii) Let $a \in S$, $e \in E(S)$. If $a = e$, the result is immediate. Suppose $a \neq e$ and let $A = \{e,a\}$. Then $eA \neq A . \in \phi$. Thus $eA \subseteq A$. Therefore $ea = a$ or e. Similarly $ae = e$ or a.

(iv) Suppose S is a group containing more than two elements and let A be a subset of S such that it contains three elements including the identity e of S. Then by Lemma 2.1, A is a subsemigroup of S. From (2), for every $a \in A$, $a^2 = e$. Therefore A is a subgroup of S whose nonidentity elements are of order two. This is impossible since the order of A is three. The contradiction implies that $| S | \leq 2$.

Lemma 2.3. Suppose S is a completely simple semigroup with at least three elements and every nonempty subset of S is N-subset. Then S is a right zero semigroup or a left zero semigroup.

Proof. Since every completely simple semigroup can be written as a Rees metriix semigroup over a group. Let $S = M(I,G,\Lambda ; P)$. If $| I | = | \Lambda | = 1$, then S is a group. According to lemma 2.2 (4) and the assumption, this is impossible. Thus we can suppose $| \Lambda | > 1$. Let $\lambda, u \in \Lambda$ and $\lambda \neq u$. For any $i, j \in I$, let $e_{i\lambda}$, e_{ju} be the identities of the group $\mathscr{H}$-class $H_{i\lambda}$ and H_{ju}, respectively. Then $e_{i\lambda}e_{ju} = e_{iu} \in H_{iu}$. But from Lemma 2.2 (3), $e_{i\lambda}e_{ju} = e_{i\lambda}$ or e_{ju}. Therefore $i = j$ since no $\mathscr{H}$-class contains more than one idempotent. Which means that $| I | = 1$. So that $S = G \times \Lambda$ is a right group. Now, let $(e,\lambda) \in E(S)$. Then for every $(e,u) \neq (e,\lambda)$, $(e,\lambda)(e,u)(e,\lambda) = (e,\lambda)$. From Lemma 2.2 (2), $(e,\lambda) S^1(e,\lambda) = \{(e,\lambda)\}$, thus for any $g \in G$, $(e,\lambda)(g,u)(e,\lambda) = (e,\lambda)$ which implies that $g = e$. Hence $| G | = 1$ which means that S is a right zero semigroup. If $| I | > 1$, then we can similarly show that S is a left zero semigroup.

Lemma 2.4. Let S be a semilattice of completely regular semigroups $\{S_\alpha\}_{\alpha \in Y}$ ($| Y | > 1$) and every nonempty subset of S is a N-subset. Then S is a two-element semilattice.

Proof. Suppose $\alpha,\beta \in Y$ and $\alpha \neq \beta$. Let $e \in E(S_\beta)$, $f \in E(S_\alpha)$. Then $ef = e$ or f by Lemma 2.2 (3), and thus $\alpha\beta = \beta$ or α. So $\alpha<\beta$ or $\beta<\alpha$, which shows that Y is a chain. Assume $\beta<\alpha$ and $ef = e$. Then $efe = e$. Therefore $eSe = \{e\}$ by Lemma 2.2 (1). So that β is the smallest element of Y. For any $\gamma \in Y$, $\gamma \neq \alpha$, using a similar deduction, we have γ is also the smallest element of Y. Thus $\gamma = \beta$. Therefore $Y = \{\alpha,\beta\}$ and $S = S_\alpha \cup S_\beta$. Now, if $| S_\beta | > 1$, let $f \in E(S_\beta)$, $b \in S_\beta$ and $b \neq f$, then by Lemma 2.2 (3), we have $bf = f$ or b ; $fb = f$ or b. Suppose $fb = f$ and $A = \{e, f\}$ where $e \in E(S_\alpha)$, then $Ab = \{eb,f\} \cap A \neq \phi$. Thus $Ab \subseteq A$. But from Lemma 2.2 (3), $eb = b \neq e$. Hence $Ab \nsubseteq A$ contradicting to $Ab \subseteq A$. Suppose $fb = b$ and $B = \{e,b\}$, then $fB = \{f,b\} \cap B \neq \phi$. Therefore $fB \subseteq B$ which is impossible. Similarly, we can deduce a contradiction when $bf = b$ or f. So that $| S_\beta | = 1$.

Let $S_\beta = \{f\}$. Then f is the zero element of S. For any $a,b \in S_\alpha$, let $A = \{a, f\}$, $B = \{b, f\}$. Then $aB = \{ab, 2f\} \cap B \neq \phi$. Therefore $aB \subseteq B$. By Lemma 2.2 (3), $ab = b$. Similarly, from $Ab = \{ab, f\} \cap A \neq \Phi$ we have $Ab \subseteq A$ and then $ab = a$. Thus $a = b$. Hence $| S_\alpha | = 1$. Let $S_\alpha = \{e\}$. Then $S = \{e, f\}$, is a semilattice and $ef = fe = f$.

Lemma 2.5. If S is a left zero (right zero) semigroup, then every nonempty subset of S is a N-subset.

Proof. Suppose S is a left zero semigroup and A is a nonempty subset of S. For $x,y \in$ S, if xA$y \cap$ A $\neq \phi$, then there exists $a \in$ A such that $xay = x \in$ A. But xA$y = \{x\} \subseteq$ A. Then A is a N-subset of S. Similarly, the lemma holds for right zero semigroups.

Theorem 2.6. Every nonempty subset of S is N-subset if and only if S is one of the following semigroups : Group ($|$ S $| \leq 2$) ; semilattice ($|$ S $| = 2$) ; left zero semigroups ; right zero semigroup.

Proof. Necessarity. If $|$ S $| = 1$, S is a one-element group. If $|$ S $| = 2$ and S is not a band, then S is a regular semigroup with only one idempotent and hence a group. If S is a band having two elements, then S is either a semilattice, or a left (right) zero band. If $|$ S $| > 2$ and let $a \in$ S, then by Lemma 2.2 (2), $a \in$ V(a) and then $a = a^3 = a^2$S $\cap$ Sa^2 which implies that S is completely regular. By Lemma 2.4, S is also completely simple, then by Lemma 2.3, S is either a left zero band or a right zero band.

3. SEMIGROUPS WHOSE NONEMPTY SUBSETS ARE N-SUBSETS

Throughout this section, S will be an arbitary semigroup. Reg S denotes the set of regular elements of S.

Lemma 3.1. If all nonempty subsets of S are N-subsets, then

(i) For every a $\in$ S, $a^2 \in$ E(S).

(ii) Reg S is an ideal of S.

(iii) (S\Reg S)^2 = Reg S = S^2 = {0}

Proof.

(i) For any $a \in$ S, let A = $\{a, a^2\}$. Then aA = $\{a^2, a^3\} \cap$ A $\neq \Phi$. Therefore aA $\subseteq$ A and then $a^3 = a$ or a^2. So that $a^4 = a^2 \in$ E (S).

(ii) Let $\rho_{\text{Reg S}}$ be the congruence defined by Reg S as in section 1 and φ : S $\longrightarrow$ S/$\rho_{\text{Reg S}}$ be the cannonical homomorphism. By (i), we know that S is eventually regular. Then S/$\rho_{\text{Reg S}}$ is also eventually regular. For every regular element $\bar{a}$ of S/$\rho_{\text{Reg S}}$, there exists $a \in$ Reg S such that $a\varphi = \bar{a}$ (see [2]). Since Reg S is a congruence class of $\rho_{\text{Reg S}}$, there exists $e \in$ E (S) such that (Reg S) φ = Reg(S/$\rho_{\text{Reg S}}$) = $e\varphi = \bar{e}$ and Reg S = $\bar{e}\varphi^{-1}$.

For every $x \in$ S, $x^2 \in$ E(S) and $\bar{x}^2 = (x\varphi)^2 = \bar{e}$. Since $\bar{x}^3 = \bar{x}^2\dot{\bar{x}} = \bar{e}\bar{x} = \bar{e}^4x = \bar{x}^3\bar{x}^3\bar{x}^3$ we know that $\bar{x}^3$ is a regular element of S/$\rho_{\text{Reg S}}$. Therefore $\bar{x}^3 = \bar{e} = \bar{e}\bar{x} = \bar{x}\bar{e}$, which implies hat $\bar{e}$ is a zero element of S/$\rho_{\text{Reg S}}$. So Reg S = Kerφ is an ideal of S.

Suppose $x \in S$ is not regular, then $x^2 = f \in E(S)$. For any $e \in E(S)$, let $A = \{x,e\}$. Then $x\rho_A e$ and $f = x^2\rho_A e$ and thus $f \in A$. Therefore $e = f$. So that Reg S is a regular subsemigroup of S with exactly one idempotent e and thus a group.

Suppose $x,y \in S\backslash\text{Reg } S$. Then $x^2 = y^2 = e$. Let $A = \{x,y,e\}$. Then $x\rho_A e\rho_A y$ and then $(xy)\,\rho_A y^2 = e$. So that $xy \in A$. If $xy = x$, then $x = xy = (xy)y = xy^2 = xe \in \text{Reg } S$. If $xy = y$, then $y = xy = x(xy) = x^2y = ey \in \text{Reg } S$. These are impossible. So $xy = e$ which implies that $(S\backslash\text{Reg } S)^2 = \{e\}$.

If $|\text{ Reg } S\ | \geq 3$, choose $a \in \text{Reg } S$, $a \neq e$ and let $b \in \text{Reg } S\backslash\{e,a\}$. Then $a = \{e,a,b\}$ is a N-subsemigroup of S by Lemma 2.1 and the assumption. So that $ab \in A$, but $ab \neq a$, $ab \neq b$, we have $ab = e$. Then $b = a^{-1} = a$ by Lemma 2.2 (2). However $b \neq a$. The contradiction implies that $|\text{ Reg } S\ | \leq 2$. Assume $\text{Reg } S = \{e,a\}$. Now, let $x \in S\backslash\text{Reg } S$, then $xa \in \text{Reg } S$ since Reg S is an ideal of S, so that $xa = e$ or a. If $xa = e$, then $xe = x(xa) = x^2a = ea = a$. Let $A = \{x,e\}$, then $xA = \{e,a\} \cap A \neq \varnothing$ and then $xA \subseteq A$. Therefore $a = e$. If $xa = a$, let $B = \{x,a\}$, then $xB = \{e,a\} \cap B \neq \varnothing$, so that $xB \subseteq B$ and then $a = e$. Thus $\text{Reg } S = \{e\}$. By (ii), e is the zero of S and write it by 0, so that $S^2 = \{0\}$.

Theorem 3.2. Every nonempty subset of S is N-subset if and only if $S^2 = \{0\}$.

Proof. By Lemma 3.1, we need only to show the "if". Suppose $S^2 = \{0\}$ and A is a nonempty subset of S. If $0 \in A$, then A is an ideal of S and certainly a N-subset. If $0 \notin A$, for any $x,y \in S^1$, suppose $xAy \cap A \neq \Phi$. Then $x = y = 1$, so that $xAy = A \subseteq A$.

REFERENCES

1. J. M., Howie. An Introduction to Semigroup Theory, Academic Press, London, 1976.
2. P. M., Edward. Congruences and Green's relations on eventually regular semigroups. J. Austral. Math. Soc. (SeriA) 43 (1987). 64-69.
3. YuFen Zhang and D. C. Wang. π-Group congruence on an eventually regular semigroups (to appear).

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BIONOMIC EQUILIBRIUM AND OPTIMIZATION OF REVENUE

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ABSTRACT : In this paper, we consider joint harvesting of a one prey one predator model, with special reference to the existence of feasible and partially feasible bionomic equilibria and optimization of the total discounted revenue. It is proved that in some cases, there exists such a range of the threshold values of cost per unit effort c that only feasible bionomic equilibria can occur, and hence in such cases a feasible and a partially feasible bionomic equilibria do not exist simultaneously. But it is also shown that there are some cases where depending upon different ranges of threshold values of the cost per unit effort the value of Sup (E_{max}) for feasible equilibria is equal to the value of E_{max} at the partially feasible equilibrium, where E is the total effort exerted.

To find the optimal value of the revenue, we consider two types of depreciations of which one is the standard depreciation equal to $e^{-\delta t}$, $\delta > 0$. It is shown that under the other type of depreciation, optimization can be done under a weaker transversality condition. Lastly, all the results are verified by special choice of parameters.

Key words : Feasible/partially feasible bionomic equilibrium, threshold values of cost per unit effort, optimization of revenue.

INTRODUCTION

It was shown, for the first time, by D. K. Bhattacharya and S. Begum (1996) that the concepts of feasible and partially feasible bionomic equilibria could be applied to find out the threshold value/values of the cost per unit effort c and the maximum value of the effort E_{max} in models of two species systems under joint harvesting. It was shown that threshold values of c and E_{max} could be determined in terms of given parameters only. They denoted by $L = 0$ and $\pi = 0$, the locus of dynamic equilibrium and the zero profit line respectively, and defined feasible and partially feasible bionomic equilibria as follows :

A feasible bionomic equilibrium was defined to be the point of intersection (x,y) of $L = 0$ and $\pi = 0$ if $x > 0$, $y > 0$ and the total effort E at (x,y) remained positive.

It was denoted by (x_∞, y_∞). A partially feasible bionomic equilibrium was defined as the point of interesction of L = 0 with one of the co-ordinate axes, provided the non-zero co-ordinate of that point was positive and both E and π were positive at that point. It was denoted by $(x_\infty,0)$ or $(0,y_\infty)$ according as L = 0 intersected the x-axis or the y-axis respectively.

It was pointed out that, if c was not kept under proper threshold value/values, exploitation would become impossible, and if E_{max} was not known, calculation of optimal revenue would be meaningless, as the control set $V_t = [0, E_{max}]$ would remain unknown. In the above paper, joint harvesting of the following two species system was considered -(i) Lotka-Volterra model of one prey and one predator, (ii) Logistic growth model of two species ecologically independent and ecologically dependent seperately. It was found that when $(x_\infty,0)$ or $(0,y_\infty)$ was partially feasible, E_{max} remained independent of the choice of c whereas when (x_∞, y_∞) was feasible, E_{max} was dependent on the choice of c.

Again, in the existing literature whenever the optimization of the revenue of two species system was done under the depreciation $e^{-\delta t}$, $\delta > 0$, using Pontryagin's maximum principle, it was always observed that the auxilliary functions $\lambda_i(t)$, i = 1,2 of the corresponding Hamiltonian satisfied a stronger transversality condition, viz. "$e^{\delta t}\lambda_i(t)$ tends to a finite quantity as $t \to \infty$" for which no mathematical justification appeared to be known [Arrow, K. J. and M. Kurz, 1970]. Therefore, it was an open problem whether optimization could be done with a weaker type of transversality condition. Lastly, it also remained open whether optimization could be done in case of partial feasibility of $(x_\infty,0)$ or $(0,y_\infty)$ when the fishery could be viable with one species only.

We consider a model as considered by Goh, 1970 (Example 3.32, page 52). We limit our discussions in three sections and an appendix is also included. Section 1 contains a single theorem, which studies feasibility of bionomic equilibria (x_∞,y_∞) and partial feasibility of $(x_\infty,0)$ and $(0,y_\infty)$ under different restrictions on the given parameters. Section 2 contains two theorems of which the first one deals with optimization of the revenue under the standard depreciation $e^{-\delta t}$, $\delta > 0$, whereas the second one considers similar result by using a new type of depreciation, which depends on the given prices and catchability coefficients of the biomasses. Section 3 includes some discussions on the results obtained in sections 1 and 2 used in connection with optimization of the revenue. The appendix contains the detailed proofs of all the three theorems. Further, in the appendix, all the results of the three theorems are verified by special choice of the parameters of the given model.

1. THRESHOLD VALUES OF THE COST PER UNIT EFFORT AND THE MAXIMUM VALUE OF THE EFFORT.

Theorem 1.1. Let the exploited one prey one predator model be given by

$$\frac{dx}{dt} = x(r - ux - \alpha y - q_1 E),$$

$$\beta r > us \qquad \text{(I)}$$

$$\frac{dy}{dt} = y(-s + \beta x - vy - q_2 E),$$

with any one of the following restictions

(a) (i) $\alpha q_2 > vq_1$, $(\beta q_1 + uq_2)\, p_2 q_2 > (\alpha q_2 - vq_1)\, p_1 q_1$

(ii) $\alpha q_2 > vq_1$, $(\beta q_1 + uq_2)\, p_2 q_2 < (\alpha q_2 - vq_1)\, p_1 q_1$

(b) $\alpha q_2 < vq_1$

where x and y are the biomasses ; r is the growth rate of the prey x ; s is the rate at which the predator y dies out in the absence of the prey ; α is the amount of prey consumed by unit predator in unit time and α/β is the fraction of the energy consumed which goes into predator reproduction ; u, v are the density-dependent mortalities of x and y respectively and q_1, q_2 are their catchability coefficients.

Under both the restrictions (a) (i) and (b), (x_∞, y_∞) and $(x_\infty, 0)$ are feasible. Moreover, (x_∞, y_∞) is feasible if and only if

(i)
$$\frac{p_1 q_1 (s q_1 + r q_2)}{(\beta q_1 + u q_2)} < c < \frac{p_1 q_1 (rv + \alpha s) + p_2 q_2 (\beta r - us)}{(uv + \alpha\beta)} \qquad \text{... (1.1)}$$

In this case, E_{max} is given by
$$x_\infty = \bar{x} + \frac{(\alpha q_2 - v q_1)}{(uv + \alpha\beta)} E_{max} \qquad \text{... (1.2)}$$

where
$$x_\infty = \frac{-c(\alpha q_2 - v q_1) + p_2 q_2 (s q_1 + r q_2)}{p_2 q_2 (\beta q_1 + u q_2) - p_1 q_1 (\alpha q_2 - v q_1)} \qquad \text{... (1.3)}$$

$$\bar{x} = \frac{(rv + \alpha s)}{(uv + \alpha\beta)} \qquad \text{... (1.4)}$$

Further
$$\sup (E_{max}) = \frac{(\beta r - us)}{(\beta q_1 + u q_2)} \qquad \text{... (1.5)}$$

(ii) $(x_\infty, 0)$ is a partially feasible bionomic equilibrium if and only if

$$0 < c < \frac{p_1 q_1 (s q_1 + r q_2)}{(\beta q_1 + u q_2)} \qquad \text{... (1.6)}$$

In this case,
$$E_{max} = \frac{(\beta r - us)}{(\beta q_1 + u q_2)} \qquad \text{... (1.7)}$$

Again, under the restriction (a) (ii), only (x_∞, y_∞) is a feasible bionomic equilibrium if and only if

$$\frac{p_2q_2(\beta r - us) + p_1q_1(rv + \alpha s)}{(uv + \alpha\beta)} < c < \frac{p_1q_1(sq_1 + rq_2)}{(\beta q_1 + uq_2)} \qquad \ldots (1.8)$$

In this case, E_{max} is given by (1.2), (1.3) and (1.4). Further sup $(E_{max}) = \dfrac{(\beta r - us)}{(\beta q_1 + uq_2)}$

2. OPTIMIZATION OF REVENUE

Theorem 2.1. Let the exploited system be given by (I). Let the total discontinued revenue be given by

$$J = \int_0^\infty e^{-\delta t}\,(p_1\,q_1\,x + p_2\,q_2\,y - c)\,E\,(t)\,dt,\ \delta > 0,\ E(0) = E_0 \qquad \ldots(A)$$

When (x_∞, y_∞) is bionomically feasible, the optimal path is given by

$$(p_1\,q_1\,x + p_2\,q_2\,y - c)\,\{\delta^2 + (ux + vy)\delta + (uv + \alpha\beta)xy\} - [\{(\delta + vy)p_1q_1 + p_2q_2\beta y\}$$
$$(rx - \alpha xy - ux^2) + \{(\delta + ux)p_2q_2 - p_1q_1\alpha x\}\,(-sy + \beta xy - vy^2)] = 0 \qquad \ldots (2.1)$$

The optimal biomasses x_δ, y_δ are determined by the intersection of (2.1) and

$$L : (\beta q_1 + uq_2)x + (\alpha q_2 - vq_1)y - (sq_1 + rq_2) = 0 \qquad \ldots (2.2)$$

The optimal effort E^* is given by $x_\delta = \dfrac{(rv + \alpha s) + (\alpha q_2 - vq_1)E^*}{(uv + \alpha\beta)}$ ... (2.3)

When $(x_\infty, 0)$ is a partially feasible bionomic equilibrium, the optimal path is given by

$$(p_1\,q_1\,x + p_2\,q_2\,y - c)\,\{\delta^2 + (ux + vy)\delta + (uv + \alpha\beta)xy\} - \{p_1\,q_1^2\,x\,(\delta + vy)$$
$$+ p_2\,q_1\,q_2\,\beta xy + p_2\,q_2^2\,y\,(\delta + ux) - p_1\,q_1\,q_2\,\alpha xy\}\,\frac{(\beta r - us)}{(\beta q_1 + uq_2)} = 0 \qquad \ldots (2.4)$$

The optimal biomasses x_δ and y_δ are given by the intersection of (2.2) and (2.4). The optimal effort E^* is obtained from (2.3).

Theorem 2.2. Let the exploited system be given by (I). Let the total discounted revenue be given by

$$J = \int_{t_0}^\infty \frac{(p_1q_1 + p_2q_2y - c)}{t^{\delta+2}}\,E(t)dt,\ \delta>0,\ E(t_0) = E_0, (p_1q_1x + p_2q_2y - c) = O(t^\delta) \qquad \ldots(B)$$

When (x_∞, y_∞) is bionomically feasible, the optimal path is given by

$$(p_1 q_1 x + p_2 q_2 y - c)(uv + \alpha\beta) - \{(p_2 q_2 \beta + p_1 q_1 v)(r - \alpha y - ux)$$
$$+ (p_2 q_2 u - p_1 q_1 \alpha)(-s + \beta x - vy)\} = 0 \qquad \dots (2.5)$$

The optimal biomasses x_δ and y_δ are determined by the intersection of (2.5) and (2.2). The optimal effort E* is determined from (2.3).

When $(x_\infty, 0)$ is a partially feasible bionomic equilibrium, the optimal path is given by

$$(p_1 q_1 x + p_2 q_2 y - c)(\alpha\beta + uv)(\beta q_1 + u q_2) - [q_1(p_2 q_2 \beta + p_1 q_1 v)$$
$$+ q_2(p_2 q_2 u - p_1 q_1 \alpha)](\beta r - us) = 0 \qquad \dots (2.6)$$

The optimal biomasses x_δ and y_δ are determined by the intersection of (2.2) and (2.6) and the optimal effort E* is determined from (2.3).

3. DISCUSSIONS

(i) Optimization of the revenue is possible even in the case of partially feasible bionomic equilibrium, where the fishery remains viable with one species only.

(ii) The depreciation $\frac{1}{t^{\delta+2}}$ is preferred to $e^{-\delta t}$ due to the following reasons :

(a) Apart from economic justifications, the depreciation $e^{-\delta t}$ is introduced in calculating the discounted revenue J, in order to assure the convergence of the integral J. Naturally the case $\delta = 0$ remains indecisive, as the corresponding integral J may be divergent, whereas for the depreciation $\frac{1}{t^{\delta+2}}$ no such problem arises when $\delta = 0$, because the integral of J remains convergent in this case.

(b) When the depreciation is taken as $e^{-\delta t}$, the auxilliary function $\lambda_i(t)$, $i = 1,2$ of the corresponding Hamiltonian for J are always found to satisfy the transversality condition $e^{\delta t}\lambda_i(t) \to$ finite quantity as $t \to \infty$ [4.12 (a) and 4.12 (b), Appendix]. This is a very strong transversality condition as this cannot hold unless $\lambda_i(t) = O(e^{-\delta t})$. However, if the depreciation $\frac{1}{t^{\delta+2}}$ is used, then it is revealed from [4.17 and 4.18, Appendix] that $\lambda_i(f)$ satisfy a weaker type of transversality condition.

(c) From (2.1) and (2.4), it follows that the optimal path depends on the choice of δ, if $e^{-\delta t}$ is taken as the depreciation whereas from (2.5) and (2.6) it is seen that the optimal path remains independent of δ when $\frac{1}{t^{\delta+2}}$ is the depreciation. As the value of E_{max} and consequently the control set is independent of δ, so in the former case, it may not be possible to verify whether the optimal effort E^* belongs to the control set, by taking some value of δ at random. In case it happens that for some particular value of δ, $E^* \notin V_t$, then corresponding $e^{-\delta t}$ can not be taken as the depreciation. Hence the introduction of arbitrary $e^{-\delta t}$ is questionable, whereas no such problem arises in the latter case.

4. APPENDIX

Proof of theorem 1.1

Revenue function is given by $\pi = (p_1q_1x + p_2q_2y - c)\,E$ ... (4.1)

Dynamic equilibrium of the unexploited system is $(\bar{x},\bar{y})$ where

$$\bar{x} = \frac{(rv+\alpha s)}{(uv+\alpha\beta)},\ \bar{y} = \frac{(\beta r-us)}{(uv+\alpha\beta)}.$$

Dynamic equilibrium of the exploited system is $((\bar{x})_e, (\bar{y})_e)$, where

$$(\bar{x})_e = \bar{x} + \frac{(\alpha q_2 - vq_1)E}{(uv+\alpha\beta)},\ (\bar{y})_e = \bar{y} - \frac{(\beta q_1 + uq_2)E}{(uv+\alpha\beta)} \quad \text{... (4.2)}$$

Line of dynamic equilibrium is given by,

$L : (\beta q_1 + uq_2)x + (\alpha q_2 - vq_1)y - (sq_1 + rq_2) = 0$ ... (4.3)

So the point of intersection of $\pi = 0$ and $L = 0$ is given by $(\bar{x},\bar{y})$, where

$$\bar{x} = \frac{-c(\alpha q_2 - vq_1) + p_2q_2(sq_1 + rq_2)}{p_2q_2(\beta q_1 + uq_2) - p_1q_1(\alpha q_2 - vq_1)},\ \bar{y} = \frac{c(\beta q_1 + uq_2) - p_1q_1(sq_1 + rq_2)}{p_2q_2(\beta q_1 + uq_2) - p_1q_1(\alpha q_2 - vq_1)} \quad \text{...(4.4)}$$

Now for the cases (a) (i) (fig.1) and (a) (ii) (fig.2) it is noted that

$$(\bar{x})_e > \bar{x},\ (\bar{y})_e < \bar{y},\ (\bar{x})_e > 0, \quad \forall\ E > 0,$$

$(\bar{y})_e > 0$, for $E < \frac{(\beta r + us)}{(\beta q_1 + uq_2)}$. So $(\tilde{x}, \tilde{y})$ will be bionomically feasible and will be denoted by (x_∞, y_∞), if $E < \frac{(\beta r + us)}{(\beta q_1 + uq_2)}$ and if $\tilde{x} > 0$, $\tilde{y} > 0$, $\tilde{x} > \bar{x}$, $\tilde{y} < \bar{y}$, the condition for which are

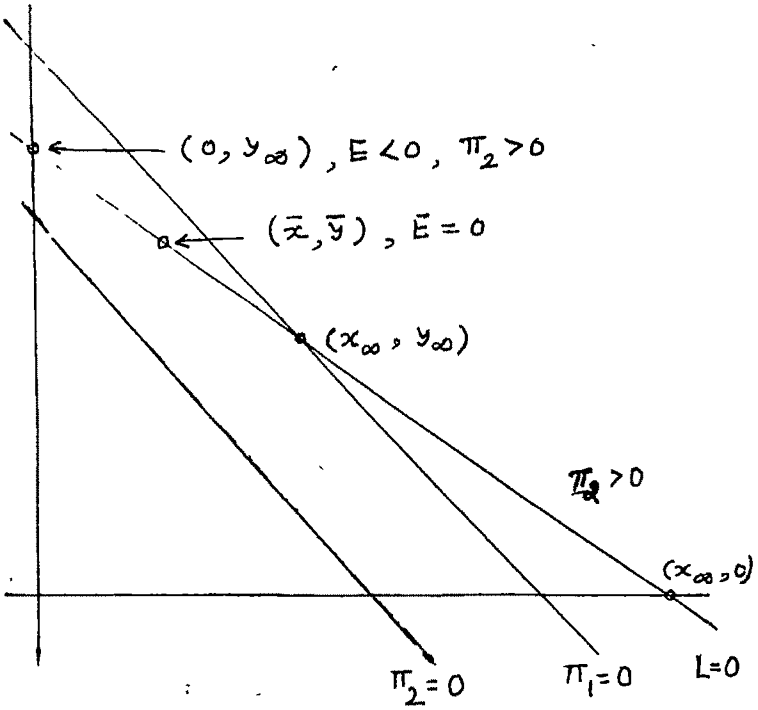

fig. 1.

Case (a) (i) $c<\dfrac{p_2q_2(sq_1+rq_2)}{(\alpha q_2-vq_1)}$, for $\tilde{x}>0$ ...(4.5)

$c>\dfrac{p_1q_1(sq_1+rq_2)}{(\beta q_1+rq_2)}$, for $\tilde{y}>0$ ...(4.6)

$c<\dfrac{p_2q_2(\beta r-us)+p_1q_1(rv+\alpha s)}{(uv+\alpha\beta)}$, for $\tilde{x}>\bar{x}$ and $\tilde{y}<\bar{y}$ ...(4.7)

Case (a)(ii) $c>\dfrac{p_2q_2(sq_1+rq_2)}{(\alpha q_2-vq_1)}$, for $\tilde{x}>0$ ...4.5)(a)

$c<\dfrac{p_1q_1(sq_1+rq_2)}{(\beta q_1+uq_2)}$, for $\tilde{y}>0$ ...(4.6)(a)

$c>\dfrac{p_2q_2(\beta r-us)+p_1q_1(rv+\alpha s)}{(uv+\alpha\beta)}$, for $\tilde{x}>\bar{x}$ and $\tilde{y}<\bar{y}$ ...(4.7)(a)

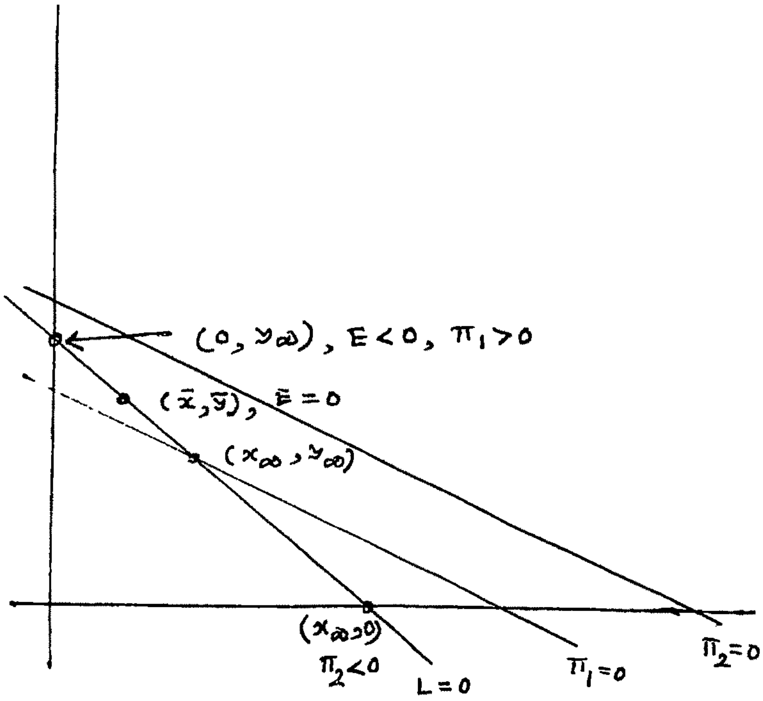

fig. 2.

As $\dfrac{p_2q_2(sq_1+rq_2)}{(\alpha q_2-vq_1)} \gtrless \dfrac{p_2q_2(\beta r-us)+p_1q_1(rv+\alpha s)}{(uv+\alpha\beta)}$ according as case (a) (i) and case (a) (ii) holds, so it is concluded that under (a)(i), (x_∞, y_∞) is feasible if and only if (1.1) holds and under (a)(ii), (x_∞, y_∞) is feasible if and only if (1.8) holds. Now for case (b) (fig.3), it is seen that $(\bar{x})_e<\bar{x}$, $(\bar{y})_e<\bar{y}$, $(\bar{x})_e>0$ if $E<\dfrac{(rv+\alpha s)}{(vq_1-\alpha q_2)}$ and $(\bar{y})_e>0$ if $E<\dfrac{(\beta r-us)}{(\beta q_1+uq_2)}$. As $\tilde{x}>0$, so $(\tilde{x}, \tilde{y})$ will be bionomically feasible and will be denoted by (x_∞, y_∞) if

$$E < \min\left(\frac{rv+\alpha s}{vq_1-\alpha q_2}, \frac{\beta r-us}{\beta q_1+uq_2}\right)$$ and if $\tilde{y}>0$, $\tilde{x}<\bar{x}$, $\tilde{y}<\bar{y}$, the conditions for which are

$$c>\frac{p_1q_1(sq_1+rq_2)}{(\beta q_1+uq_2)}, \text{ for } \tilde{y}>0 \qquad \ldots (4.6)(b)$$

$$c<\frac{p_2q_2(\beta r-us)+p_1q_1(rv+\alpha s)}{(uv+\alpha\beta)},\ \text{for}\ \tilde{x}>\bar{x}\ \text{and}\ \tilde{y}<\bar{y} \qquad ...(4.7)(b)$$

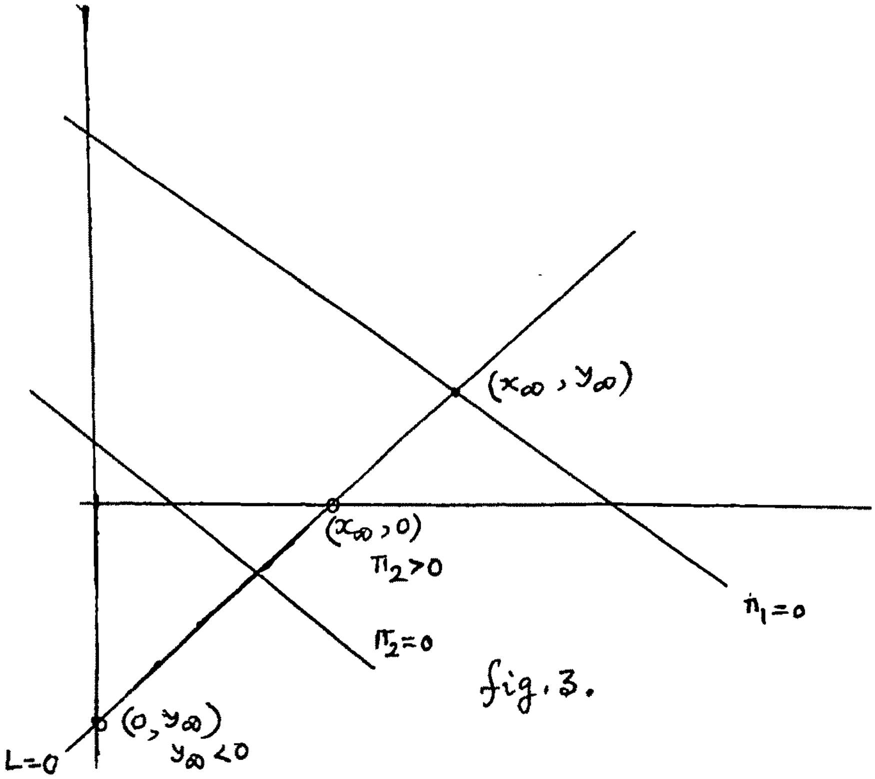

fig. 3.

So, under case (b), (x_∞, y_∞) is feasible if and only if (1.1) holds. Further, in all cases, E_{max}'s are determined by (1.2), (1.3) and (1.4). Also E_{max} depends on values of c given by (1.1) for the cases (a)(i) and (b) and values of c given by (1.8) for the case (a)(ii). In the cases (a)(i) and (a)(ii)

$$E<\frac{\beta r-us}{\beta q_1+uq_2},\ \text{but in case (b)}\ E<\min\left(\frac{\beta r-us}{\beta q_1+uq_2},\frac{rv+\alpha s}{vq_1-\alpha q_2}\right).$$

As $\frac{\beta r-us}{\beta q_1+uq_2}<\frac{rv+\alpha s}{vq_1-\alpha q_2}$, in case (b), so $E<\frac{\beta r-us}{\beta q_1+uq_2}$ in all the cases (a)(i), (a)(ii) and (b). Again in all the cases $(0, y_\infty)$ is not partially feasible. In fact, in case (b), $(0, y_\infty)$ is not feasible because $y_\infty<0$. In other cases, $(0, y_\infty)$ is not feasible, as $E<0$ at $(0, y_\infty)$. Lastly, to consider partial feasibility of $(x_\infty, 0)$, we note that the point of intersection of $\pi = 0$ and $L = 0$ with the X-axis are $\left(\frac{c}{p_1q_1}, 0\right)$ and $\left(\frac{sq_1+rq_2}{\beta q_1+uq_2}, 0\right)$. Hence in case (a)(i) and case (b), $(x_\infty, 0)$ is partially feasible if and only if (1.6) holds. But in case (a)(ii) $(x_\infty, 0)$ is not partially feasible as at the point where $L = 0$ intersects the X-axis, π is found to be negative (fig. 2). Further, in cases (a)(i) and (b), E_{max} is given by (1.7) when $(x_\infty, 0)$ is partially feasible. Moreover this value of E_{max} is the same as $\sup(E_{max})$ when (x_∞, y_∞) is feasible. This completes the proof of theorem 1.1.

Proof of theorem 2.1. For convenience, we write $F(x,y) = rx - ux^2 - \alpha xy$ and $G(x,y) = -sy + \beta xy - vy^2$. Then the Hamiltonian corresponding to (I) and (A) is given by

$$H = e^{-\delta t}\,(p_1q_1x + p_2q_2y - c)\,E(t) + \lambda_1(t)\,[F(x,y) - q_1Ex] + \lambda_2(t)[G(x,y) - q_2Ey]$$

$$= E(t)[e^{-\delta t}\,(p_1q_1x + p_2q_2y - c) - \lambda_1q_1x - \lambda_2q_2y] + \lambda_1(t)F(x,y) + \lambda_2(t)G(x,y) \quad \ldots (4.8)$$

where $\lambda_1(t)$, $\lambda_2(t)$ are the adjoint variables to be determined suitably.

Adjoint equations for $\lambda_1(t)$ and $\lambda_2(t)$ are given by

$$\frac{d\lambda_1}{dt} = -\frac{p_1q_1E}{e^{\delta t}} + \lambda_1(t)\,[q_1E - Fx] - \lambda_2\,(t)Gx$$

$$\frac{d\lambda_2}{dt} = -\frac{p_2q_2E}{e^{\delta t}} - \lambda_1Fy + \lambda_2\,(t)[q_2E - Gy]$$

Again, for the equilibrium solution,

$$E = \frac{F(x,y)}{q_1x} = \frac{G(x,y)}{q_2y}$$

$$\text{i.e., } E = \frac{r-\alpha y-ux}{q_1} = \frac{-s+\beta y-vy}{q_2} \quad \ldots (4.9)$$

$$\text{So } \frac{d\lambda_1}{dt} = -\frac{p_1q_1E}{e^{\delta t}} + \lambda_1ux - \lambda_2\,\beta y \quad \ldots (4.10)$$

$$\frac{d\lambda_2}{dt} = -\frac{p_2q_2E}{e^{\delta t}} + \lambda_1\alpha x + \lambda_2\,vy \quad \ldots (4.11)$$

From (4.9) and (4.10) we get,

$$\lambda_1(t) = A_1e^{m_1t} + B_1e^{m_2t} + \frac{[\delta+vy)p_1q_1+p_2q_2\beta y]E(t)e^{\delta t}}{\delta^2+(ux+vy)\delta+(uv+\alpha\beta)xy]}$$

and

$$\lambda_2(t) = A_2e^{m_1t} + B_2e^{m_2t} + \frac{[\delta+vy)p_2q_2-p_1q_1\alpha x]E(t)e^{\delta t}}{\delta^2+(ux+vy)\delta+(uv+\alpha\beta)xy]}$$

where $$m_1, m_2 = \frac{(ux+vy) \pm \sqrt{(ux+vy)^2 - 4(uv+\alpha\beta)xy}}{2} \qquad \ldots (4.12)$$

Now if we choose $A_1 = B_1 = 0$ and $A_2 = B_2 = 0$, then

$$\lambda_1(t)\, e^{\delta t} \rightarrow \frac{[\delta + vy)p_1q_1 + p_2q_2\beta y]E_{\max}}{\delta^2 + (ux+vy)\delta + (uv+\alpha\beta)xy]} \text{ as } t \rightarrow \infty \qquad \ldots (4.12)(a)$$

and $$\lambda_2(t)\, e^{\delta t} \rightarrow \frac{[(\delta + ux)p_2q_2 - p_1q_1\alpha x]}{\delta^2 + (ux+vy)\delta + (uv+\alpha\beta)xy]} \text{ as } t \rightarrow \infty \qquad \ldots (4.12)(b)$$

Taking H to be maximum for some $E \in V_t$ (control set), we have $\frac{\partial H}{\partial E} = 0$.

So, $(p_1 q_1 x + p_2 q_2 y - c) - e^{\delta t} (\lambda 1\ q1\ x + \lambda 2\ q2\ y) = 0$. Thus when (x_∞, y_∞) is bionomically feasible, the optimal path is given by

$$(p_1q_1x + p_2q_2y - c)$$

$$- \frac{[q_1x\{(\delta+vy)p_1q_1 + p_2q_2\beta y\} + q_2y\{(\delta+ux)p_2q_2 - p_1q_1\alpha x\}]E_{\max}}{\delta^2 + (ux+vy)\delta + (uv+\alpha\beta)xy} = 0, \qquad \ldots (4.13)$$

Now under (4.8), (4.12) reduces to

$$(p_1q_1x + p_2q_2y - c)\ [\delta^2 + (ux + vy)\delta + (uv + \alpha\beta)xy]$$

$$- [q_1x\{(\delta + vy)p_1q_1 + p_2q_2\beta y\}\frac{(r - \alpha y - ux)}{q_1}$$

$$+ q_2y\{(\delta + vy)p_2q_2 - p_1q_1\alpha x\}\ \frac{(-s + \beta y - vy)}{q_2}] = 0$$

After proper simplification, we see that the final optimal path is given by (2.1).

When $(x_\infty, 0)$ is partially feasible, we put $E_{\max}\ \frac{\beta r - us}{\beta q_1 + uq_2}$ in (4.13), and find that the optimal path is given by (2.4). The rest of the proof is a routine one.

Proof of theorem 2.3

We write F(x,y) and G(x,y) as in theorem 2.2 Then the Hamiltonian corresponding to (I) and (B) is given by

$$H = \frac{(p_1q_1x + p_2q_2y - c)E(t)}{t^{\delta+2}} + \lambda_1(t)[F(x,y) - q_1Ex] + \lambda_2(t)[G(x,y) - q_2Ey]$$

$$= E(t)\ [\frac{(p_1q_1x + p_2q_2y - c)E(t)}{t^{\delta+2}} - \lambda_1q_1x - \lambda_2q_2y] + \lambda_1(t)F(x,y) + \lambda_2(t)G(x,y), \qquad \ldots (4.14)$$

where $\lambda_1(t)$, $\lambda_2(t)$ are the adjoint variables to be determined suitably. Adjoint equaltions for $\lambda_1(t)$ and $\lambda_2(t)$ are given by

$$\frac{d\lambda_1}{dt} = -\frac{p_1q_1E}{e^{\delta+2}} + \lambda_1(q_1E - Fx) - \lambda_2Gx$$

$$\frac{d\lambda_2}{dt} = -\frac{p_2q_2E}{e^{\delta+2}} - \lambda_1Fy + \lambda_2(q_2E - Gy).$$

Now using (4.8) we get

$$\frac{d\lambda_1}{dt} = -\frac{p_1q_1E}{e^{\delta+2}} + \lambda_1ux - \lambda_2\beta y \qquad \dots (4.15)$$

$$\frac{d\lambda_2}{dt} = -\frac{p_2q_2E}{e^{\delta+2}} + \lambda_1\alpha x + \lambda_2vy \qquad \dots (4.16)$$

Solving (4.13) and (4.14) we get,

$$\lambda_1(t) = A_1'e^{m_1t} + B_1'e^{m_2t} + \frac{1}{xy(\alpha\beta+uv)}\left[\frac{p_2q_2\beta y + p_1q_1vy}{t^{\delta+2}} + \frac{1}{t^{\delta+3}}(\dots) + \frac{1}{t^{\delta+4}}(\dots) + \dots\right]E(t)$$

$$\text{and } \lambda_2(t) = A_2'e^{m_1t} + B_2'e^{m_2t} + \frac{1}{xy(\alpha\beta+uv)}\left[\frac{p_2q_2ux - p_1q_1\alpha x}{t^{\delta+2}} + \frac{1}{t^{\delta+3}}(\dots) + \frac{1}{t^{\delta+4}}(\dots) + \dots\right]E(t)$$

where m_1 and m_2 are given by (4.12). Now if we choose $A_1' = B_1' = 0$ and $A_2' = B_2' = 0$, then

$$\lambda_1(t)\,t^{\delta+2} \to \frac{(p_2q_2\beta y + p_1q_1vy)E_{\max}}{xy(\alpha\beta+uv)} \text{ as } t \to \infty \qquad \dots (4.17)$$

and

$$\lambda_2(t)\,t^{\delta+2} \to \frac{(p_2q_2ux - p_1q_1\alpha x)E_{\max}}{xy(\alpha\beta+uv)} \text{ as } t \to \infty \qquad \dots (4.18)$$

Taking H to be maximum for some $E \in V_t$ (control set), $\frac{\partial H}{\partial E} = 0$. When (x_∞, y_∞) is bionomically feasible, the optimal path is given

$$(p_1q_1x + p_2q_2y - c) - \frac{q_1x(p_2q_2\beta y + p_1q_1vy)E_{\max}}{xy(\alpha\beta+uv)} - \frac{q_2y(p_2q_2ux - p_1q_1\alpha x)E_{\max}}{xy(\alpha\beta+uv)} = 0. \qquad \dots (4.19)$$

Using (4.18), we find that the optimal path is given by (2.5). When $(x_\infty, 0)$ is partially feasible, we put $E_{\max} = \frac{\beta r - us}{\beta q_1 + uq_2}$ in (4.19), and find that the required optimal path is given by (2.6). The rest of the proof is a routine one.

Special choice of parameters.

Case (a) (i)

$\alpha = 3,\ r = 2,\ u = 1,\ q_1 = 1,\ p_1 = 2.$

$\beta = 2,\ s = 1,\ v = 2,\ q_2 = 1,\ p_2 = 5.$

For feasible (x_∞, y_∞), $2 < c < \frac{29}{8}$, let $c = 3$. Then $E_{max} = \frac{5}{13}$, $Sup(E_{max}) = 1$, so $E_{max} < Sup(E_{max})$, this verifies theorem 1.1

To verify theorem 2.1, let us choose $\delta = 2$, then the optimal path is

$$8xy^2 + 32x^2y + 8x^2 + 40y^2 - 43xy - 6x + 18y - 12 = 0 \qquad \ldots (4.20)$$

and $L : 3x + y - 3 = 0$, i.e., $y = 3(1 - x)$ ... (4.21)

Therefore, (x_δ, y_δ) is obtained by the intersection of (4.20) and (4.21). Thus $x_\delta = 0.9$, $y_\delta = 0.3$ and finally $E^* = 0.2$. So, $E^* < E_{max} < Sup(E_{max})$.

For theorem 2.2, the optimal path is, $-30x + 74y = 49$,

so $x_\delta = 0.9$. $y_\delta = 0.3$ and finally $E^* = 0.2$. Hence $E^* < E_{max} < Sup(E_{max})$.

For partially feasible $(x_\infty, 0)$, $E_{max} = Sup(E_{max}) = \dfrac{\beta r - us}{\beta q_1 + uq_2} = 1$. Also $0 < c < 2$, let $c = 1.5$.

For theroem 2.1, the optimal path is,

$$40xy^2 + 16x^2y + 4x^2 + 10y^2 - 16xy - 1.5x + 3y - 1.5 = 0 \qquad \ldots (4.22)$$

Therefore, (x_δ, y_δ) is obtained by the intersection of (4.22) and (4.21). Thus $x_\delta = 0.94$, $y_\delta = 0.18$ and finally $E^* = 0.52$. Hence $E^* < E_{max}$.

For theroem 2.2, the optimal path is $16x + 40y - 21 = 0$. So $x_\delta = \frac{95}{104}$, $y_\delta = \frac{27}{104}$,

$E^* = \frac{4}{13}$. Therefore, $E^* < E_{max}$.

Case (a) (ii)

$r = 1,\ \alpha = 2,\ u = 1,\ q_1 = 1,\ p_1 = 4,$

$s = 1,\ \beta = 2,\ v = 1,\ q_2 = 1,\ p_2 = 1.$

For feasibility of (x_∞, y_∞) $\frac{13}{5} < c < \frac{8}{3}$, let $c = 2.63$. Then $E_{max} = 0.15$,

$Sup(E_{max}) = 0.67$, so $E_{max} < Sup(E_{max})$, this verifies theorem 1.1

To verify theorm 2.1, we choose $\delta = 1$, then the optimal path is given by

$10xy^2 + 40x^2y + 8x^2 + 2y^2 - 15.15xy - 2.63x - 0.63y - 2.63 = 0$

As L is given by $y = 2 - 3x$, so optimal biomass x_δ is given by

$$30x_\delta^3 - 31.45x_\delta^2 + 15.04x_\delta - 4.11 = 0. \text{ Thus } x_\delta = 0.59 \text{ (approx).}$$

Now $0.59 = \frac{3}{5} + \frac{E^*}{5}$, hence E^* is negative. So E^* does not exist when $\delta = 1$.

This indicates the drawback of the use of arbitrary $e^{-\delta t}$ as the standard depreciation, as mentioned in (c) of the discussion (art. 3).

Similarly if $\delta = 2,3$ etc, E^* does not exist.

Now if we choose $\delta = 0.1$, then the optimal path is,

$10xy^2 + 40x^2y + 0.8x^2 + 0.2y^2 - 25.05xy - 0.263x - 0.353y - 0.0263 = 0$

and L : $3x + y = 2$.

So optimal biomass x_δ is given by

$-30x^3 - 37.75x^2 + 12.064x - 0.0677 = 0$. Thus $x_\delta = 0.62$ (approx), $y_\delta = 0.14$ (approx).

Now. $0.62 = \frac{3}{5} + \frac{E^*}{5}$, hence $E^* = 0.1$, therefore $E^* < E_{max} < Sup(E_{max})$.

To verify theorem 2.2, we find that the optimal path is $40x + 10y - 26.15 = 0$.

As L is given by $y = 2 - 3x$, so optimal biomass x_δ is given by

$x_\delta = 0.615$. Hence $E^* = 0.075$. So $E^* < E_{max} < Sup(E_{max})$. This verifies theorem 2.2

Case (b)

$r = 1,\ \alpha = 2,\ u = 1,\ q_1 = 1,\ p_1 = 1,$

$s = 1,\ \beta = 2,\ v = 1,\ q_2 = 1,\ p_2 = 1,$

For feasibility of $(x_{\infty 5555}, y_\infty)$ $\frac{2}{3} < c < 1$, let $c = 0.75$, then $E_{max} = \frac{1}{4}$, Sup $(E_{max}) = \frac{1}{3}$, so $E_{max} < Sup(E_{max})$, this verifies theorem 1.1

To verify theorm 2.1, we take $\delta = 2$, then the optimal path is

$4x^2y + 4xy^2 + 2x^2 + 4y^2 - 1.5xy + 0.25x + 3.5y - 1.5 = 0$. As L is given by $y = 3x - 2$, x_δ is determined by $48x^3 - 22.5x^2 - 18.25x + 7.5 = 0$. But we cannot get any positive solution for x_δ.

So E^* cannot be calculated. However, other values of δ may be tried.

To verify theorem 2.2, we find that the optimal path is given by $8x + 8y = 7$.

As $A : y = 3x - 2$, so the optimal biomass x_δ is given by

$x_\delta = \frac{23}{32}$. So $E^* = \frac{1}{8}$. Thus $E^* < E_{max} < Sup(E_{max})$.

When $(x_\infty, 0)$ is feasible, we have $0 < c < \frac{2}{3}$, let $c = 0.5$. Here $E_{max} = Sup(E_{max}) = \frac{1}{3}$.

To verify theroem 2.1, we take $\delta = 2$, then the optimal path is,

$12xy^2 + 12x^2y + 3x^2 + 6y^2 - xy + 0.5x + y - 1.5 = 0$. As $L : y = 3x - 2$, so the optimal biomass x_δ is given by $144x^3 - 114x^2 - 24.5x + 24.5 = 0$. In this case, we cannot get any positive value of x_δ. So E^* cannot be calculated. However, other values of δ may be tried.

To verify theorem 2.2, the optimal path is given by $3x + 3y - 2.5 = 0$.

As $L : y = 3x - 2$, so the optimal biomass x_δ is given by

$x_\delta = \frac{17}{24}$. So $E^* < \frac{1}{6}$. But $E_{max} = \frac{1}{3}$. So $E^* < E_{max}$ This verifies theorem 2.2.

REFERENCES

1. K. J. Arrow and M. Kurz, Public investment, the rate of return and optimal fiscal policy, Johns Hopkins press (Baltimore, 1970).
2. D. K. Bhattacharya, S. Begum, A note on bionomic equilibrium of two species system I, Mathematical Biosciences 135 (1996), 111-127 ; Vol. 151, 1998, pp 229.
3. K. S. Chaudhury, A bioeconomic model of harvesting a multispecies fishery— Eco. modelling 32 (1986), 267-279.
4. K. S. Chaudhury. Bioeconomic exploitation of Lotka Volterra prey-predator system Cal. Math. Soc. (India), 83 (1991) 175-186.
5. C. W. Clark, Math. Bioeconomics : The optimal management of renewable resources, Wiley Eastern, New York, 1976, 1990.
6. C. W. Clark, Bioeconomic modelling and fisheries management, Willy Eastern, New York, 1985.
7. G. F. Gause, La Theorie Mathematique de la Lutte pour la vie Hermann, 1935.
8. P. A. Larkin, Exploitation in a type of predator prey relationship, Jour. fisheries research Board (Canada) 23 (1996) 349-356.

9. M. Mesterton-Gibbons, On the optimal policy for combined harvesting of independent species-Natural res. model 2, no. 1 (1987) 109-134.

10. M. Mesterton-Gibbons, On the optimal policy for combined harvesting of independent species-Natural res. model 3, no. 1 (1988) 63-68.

11. Parish and Saila, Interspecific competition, predation and species diversity, J. Theo. Biol. 27 (1970) 207-220.

12. D. L. Ragozin, Harvest policies and nonmarket valuation in a predator prey system–J. of Environmental Eco. and management-12, (1985), 155-168.

13. F. M. Wilkes, Capital budgeting techniques, Wiley, New York, NY, 1977.

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REFERENCES

1. N. L. Alling, Foundations of Analysis on Surreal Number Fields, North-Holland Publishing Co., 1987.
2. E. Hewitt, Rings of continuous functions I, Trans. Amer. Math. Soc. 64(1948), 54-99.

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ON A GENERALIZATION OF CARLEMAN'S INTERGRAL INEQUALITY

BICHENG YANG AND LOKENATH DEBNATH

ABSTRACT : A generalization of Carleman's integral inequality with a best possible constant factor is made by introducing two parameters a and b

1991 AMS Subject Classification Code : 26D15

Key words and phrases : Hardy's inequality, Carleman's inequality, and geometric mean inequality

1. INTRODUCTION

If $\lambda_n > 0$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$, $a_n \geq 0$ $(n \in N)$, and $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$, then

$$\sum_{n=1}^{\infty} \lambda_n \left(a_1^{\lambda_1} a_2^{\lambda_2} \ldots a_n^{\lambda_n}\right)^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \lambda_n a_n, \qquad \ldots (1.1)$$

where the constant factor e is the best possible. This is called Hardy's inequality (see Hardy [1]).

When $\lambda_n \equiv 1$, (1.1) reduces to Carleman's inequality (see Carleman [2]) in the form

$$\sum_{n=1}^{\infty} (a_1 a_2 \ldots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n. \qquad \ldots (1.2)$$

The corresponding Carleman's integral inequality (see Knopp [3]) is stated as follows :

If $f \in L[0,\infty)$, $f(t) \geq 0$, and $0 < \int_0^\infty f(t)dt < \infty$, then

$$\int_0^\infty \exp \{\frac{1}{x}\int_0^x \ln f(t)\, dt\}dx < e\int_0^\infty f(t)\, dt, \quad \ldots (1.3)$$

where the constant factor e is still best possible.

In recent years, Yang and Dabnath [4, 5] proved two distinct strengthened versions of (1.2) by estimating the weight coefficient $\left(1+\frac{1}{n}\right)^n$. Similarly, some two distinct strengthened versions of (1.1) are studied by Yang [6, 7]. In this paper, we consider a generalization of inequality (1.3) by introducing two parameters a and b. More precisely, we prove the following theorem :

Theorem 1.1. If $a, b \in R$, $a < b$, $f \in L[a, b]$, $f(t) \geq 0$, and $0 < \int_a^b f(t)dt < \infty$, then

$$\int_a^b \exp \{\frac{1}{x-a}\int_a^x \ln f(t)dt\}dx < e \int_a^b [1-\frac{t-a}{b-a}] f(t)dt, \quad \ldots (1.4)$$

where the constant e is the best possible. In the limit b → ∞ with $0 < \int_a^\infty f(t)dt < \infty$, inequality (1.4) reduces to the form

$$\int_a^\infty \exp \{\frac{1}{x-a}\int_a^x \ln f(t)\, dt\}dx < e\int_a^\infty f(t)\, dt, \quad \ldots (1.5)$$

where the constant e is still the best possible.

2. PROOF OF THEOREM 1.1

To prove this theorem, we need the following geometric mean inequality of a function : If $f \in L[0, 1]$, $f(t) \geq 0$, and f is not a constant $c \geq 0$ a.e. in [0, 1], then [see Hardy et al. [10], (6.4.2) and (6.4.5)], we have

$$\exp [\int_0^1 \ln f(t)\, dt] < \int_0^1 f(t)dt \quad \ldots (2.1)$$

Remark 1. If $f = c \geq 0$ a. e. in $[0,1]$, then inequality (2.1) reduces to an equality.

Lemma 2.1. If $f \in L\,[a, b]$, $a < b$; $f\,(t) > 0$, and $\int_a^b f\,(t)\,dt > 0$, then

(i) $$\exp\,[\frac{1}{x-a}\int_a^x \ln f(t)\;dt] \leq \frac{e}{(x-a)^2}\int_a^x (t-a)\;f(t)dt, \qquad \ldots(2.2)$$

for any $x \in (a, b]$;

(ii) There exists $x_0 \in (a, b)$ such that, for any $x \in [x_0, b]$, (2.2) becomes a strict inequality.

Proof. We set $y = \frac{t-a}{x-a}$. Since

$$-\int_0^1 \ln y dy = \lim_{y\to 0^+} y\ln\,y + \int_0^1 yd\,\ln y = 1,$$

we have

$$\exp\,\{\frac{1}{x-a}\int_a^x \ln f(t)\;dt\} = \exp\int_0^1 \ln f\,((x-a)\,y + a\,)dy\}$$

$$= \exp\,\{\int_0^1 \ln[yf((x-a)\,y + a)]\;dy\}\;\exp\{-\int_0^1 \ln y dy\}$$

$$= e.\exp\,\{\int_0^1 \ln[yf((x-a)\,y + a)]dy\}. \qquad \ldots\;(2.3)$$

In view of (2.1) and Remark 1, we obtain

$$\exp\,\{\int_0^1 \ln[yf((x-a)\,y + a)]\;dy\} \leq \int_0^1 yf((x-a)\,y + a)dy$$

$$= \{\frac{1}{(x-a)^2}\int_a^x (t-a)f(t)dt.$$

In view of (2.3), inequality (2.2) is valid.

Since $f(t) \geq 0$, and $\int_a^b f\,(\mathrm{t})\,dt > 0$, then, it is obvious that there exists $x_0 \in (a, b)$, such that for any $x \in [x_0, b]$, $\int_a^x f(t)dt > 0$. For any $x \in [x_0, b]$, in (2.3), if $y = f((x - a) + a) = C(\geq 0)$, a.e. in $y \in [0, 1]$, then we have the equivalent form $f(t) = C(x - a) / (t - a)$ a.e. in $[x, a]$. Since $f(t)$ is independent of x, it follows that $f(t) = 0$ a.e. in $[a,x]$. It contradicts that $\int_a^x f(t)dt > 0$, $x \in [x_0, b]$. Hence, for any $x \in [x_0, b]$, $y f((x - a)y + a) \neq C$. a.e. in $y \in [0, 1]$. Using (2.1) and (2.3), we find

$$\exp\left\{\frac{1}{x-a}\int_a^x \ln f(t)\,dt\right\} = e\,.\exp\left\{\int_0^1 \mathrm{In}[y f((x - a)y + a)]\,dy\right\}$$

$$< e\int_0^1 y f((x - a)y + a)dy$$

$$= \frac{e}{(x-a)^2}\int_a^x (t - a)\,f(t)dt.$$

Proof of the theorem. We have, by the lemma 2.1,

$$\int_a^b \exp\left\{\frac{1}{x-a}\int_a^x \ln f(t)\,dt\right\}dx < e\int_a^b \frac{1}{(x-a)^2}\int_a^x (t - a)\,f(t)dtdx\}$$

$$= e\int_a^b \left[\int_t^b \frac{1}{(x-a)^2}dx\right](t - a)\,f(t)dt$$

$$= e\int_a^b \left(\frac{1}{t-a} - \frac{1}{b-a}\right)(t - a)\,f(t)dt$$

$$= e\int_a^b \left(1 - \frac{t-a}{b-a}\right)(t - a)\,f(t)dt$$

Inequality (1.4) is valid. In the limit as $b \to \infty$, we find (1.5).

For fixed $\varepsilon \in (0, 1)$, define the function f_ε as $f_\varepsilon(t) = \frac{1}{(t-a)^{1-\varepsilon}}$, $t \in (a, b)$. We have

$$\int_a^b \exp\{\frac{1}{x-a}\int_a^x \ln f_\varepsilon(t)\ dt\}dx = \int_a^b \exp\{-\frac{1-\varepsilon}{(x-a)}\int_a^x \ln(t-a)dt\}dx$$

$$= \int_a^b \exp\{-(1-\varepsilon)[\ln(x-a)-1]\}dx$$

$$= e^{1-\varepsilon}\int_a^b \exp\{-(1-\varepsilon)\ln(x-a)\}dx$$

$$= e^{1-\varepsilon}\int_a^b \frac{1}{(t-a)^{1-\varepsilon}}dt = e^{1-\varepsilon}\int_a^b f_\varepsilon(t)\ dt. \qquad \dots (2.4)$$

If there exist $a, b (a < b)$ such that the constant e in (1.4) is not the best possible, then there exists a constant where $k < e$, that makes

$$\int_a^b \exp\{\frac{1}{x-a}\int_a^x \ln f(t)\ dt\}dx < k\int_a^b \left[1-\frac{t-a}{b-a}\right] f(t)dt \le k\int_a^b f(t)dt.$$

In particular, by (2.4) we obtain

$$e^{1-\varepsilon}\int_a^b f_\varepsilon(t)\ dt = \exp\{\frac{1}{x-a}\int_a^x \ln f_\varepsilon(t)\ dt\}dx < k\int_a^b f_\varepsilon(t)dt$$

Since $0 < \int_a^b f_\varepsilon(t)\ dt < \infty$, then we have $e^{1-\varepsilon} < k$. This contradicts that fact that $e^{1-\varepsilon} \to e\ (> k)\ (\varepsilon \to 0^+)$. Hence for any fixed parameters a, $b (a < b)$, the constant e in (1.4) is the best possible.

If there exists a fixed parameter a, such that the constant e in (1.5) is not the best possible, then there exists $K < e$, that makes

$$\int_a^\infty \exp\{\frac{1}{x-a}\int_a^x \ln f(t)dt\}dx < K\int_a^\infty f(t)dt. \qquad \dots (2.5)$$

We next fix $b(>a)$. Then for any $f \in L\,[a, b]$, $f(t) \geq 0$, and $0 < \int_a^b f(t)dt < \infty$, we set

$$\tilde{f}(t) = f(t), \text{ for } t \in [a, b] \; ; \text{ and } \tilde{f}(t) = 0, \text{ for } t \in (b, \infty),$$

by (2.5), we have

$$\int_a^\infty \exp\{\frac{1}{x-a}\int_a^x \ln \tilde{f}(t)dt\}dx < K \int_a^\infty \tilde{f}(t)dt.$$

It is equivalent to

$$\int_a^b \exp\{\frac{1}{x-a}\int_a^x \ln f(t)dt\}dx < K \int_a^b f(t)dt.$$

This contradicts the fact that e is the best possible in (1.4). Hence, we show that for any fixed parameter a, the constant e in (1.5) is still the best possible.

This proves the theorem.

Remarks 2. When $a = 0$, inequality (1.4) reduces to the from

$$\int_0^b \exp\{\frac{1}{x}\int_0^x \ln f(t)dt\}dx < e \int_0^b \left(1-\frac{t}{b}\right)f(t)dt \; (b > 0), \qquad \ldots (2.6)$$

and inquality (1.5) reduces (1.3). Inequalities (2.6) and (1.5) are distinct generalizations of (1.3) with single parameter, and ineqality (1.4) is a generalization of (1.3) with two parameters and a best constant factor.

REFERENCES

1. G. H. Hardy, Note on some points in the integral calculus (LX), *Messenger of Math.* **54** (1925), 150-156.
2. T. Carleman, Sur les functions quasi-analytiques, *Conferences faites au cinquieme congres des mathematiciens scandinaves* (Helsingfors, 1923), 181-196.
3. K. Knopp, Ueber Reihen mit positiven Gliedern, *Journ. London Math. Soc.* **3** (1928), 205-211.

4. Bicheng Yang and L. Debnath, Some inequalities involving the constant e, and an application to Carleman's inequality, *Journal of Mathematical Analysis and Applications* **223** (1998), 347-353.
5. Bicheng Yang and Li Dachao, A strengthened Carleman's inequality, *Journal of Mathematics for Technology* **14,1** (1998), 130-133.
6. Bicheng Yang, A refinement on Hardy-Carleman's inequality, *Journal of Nanjing University, Mathematical Biquarterly* **14,2** (1997), 293-297.
7. Bicheng Yang, On Hardy's inequality, *Journal of Mathematical Analysis and Applications,* to appear (1999).
8. Bicheng Yang, Zeng Zhouhua, and Lokenath Debnath, On new Generalization of Hardy's integral inequality, *Journal of Mathematical Analysis and Applications* **217** (1998), 321-327.
9. Bicheng Yang, On Hilbert's integral inequality, *Journal of Mathematical Analysis and Applications* **220** (1998), 778-785.
10. G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities,* Cambridge University Press, Cambridge, 1952.

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STRONGLY SOBER θ-CONTINUOUS FUNCTIONS

S. JAFARI AND T. NOIRI

ABSTRACT : The object of this paper is to introduce and investigate a new class of functions called strongly sober θ-continuous functions.

(1991) AMS Mathematics Subject Classification Code : 54C08

Key words and phrases : Semi-open, θ-open, s-continuous, almost s-continuous, strongly sober θ-continuous, strongly sober θ-closed graphs.

0. INTRODUCTION

In 1963, Levine [10] defined a function f : X → Y to be semi-continuous if $f^{-1}(V)$ is semi-open in X for every open set V of Y. Since then, the functions have been extensively investigated. Cameron and Woods [2] and Abd El-Monsef et al. [1] defined a function f : X → Y to be s-continuous and strongly semi-continuous, respectively, if $f^{-1}(V)$ is open in X for every semi-open set V of Y. Recently, Noiri et al. [14] have defined and investigated the notion of almost s-continuous functions which is weaker than that of s-continuous functions. The purpose of the present paper is to introduce and investigate the notion of strongly sober θ-continuous functions which is stronger than that of s-continuous functions.

1. PRELIMINARIES

Throughout the present paper, X and Y are topological spaces. Let A be a subset of X. We denote the interior and the closure of a set A by Int (A) and Cl (A) respectively. A subset A is said to be **semi-open** [11] if A ⊂ Cl (Int (A)). The complement of a semi-open set is called **semi-closed** [4]. The intersection of all semi-closed sets containing A is called the **semi-closure** [4] of A and is denoted by sCl(A). The **semi-interior** of A is defined by the union of all semi-open stes contained in A and is denoted by sInt(A). The family of all semi-open sets of X is denoted by SO(X). We set SO(X, x) = $\{U | x \in U \in So(X)\}$. The following notions are due to Veličko [16] : A point $x \in X$ is said to be a θ–**cluster point** of a subset A of X if $Cl(U) \cap A \neq \phi$ for every open subset U of X containing x. The set of all θ-cluster points of A is called the **θ-closure** of A and is denoted by $Cl_\theta(A)$. A subset A is called **θ-closed** if $A = Cl_\theta(A)$. The complement of a θ-closed set is called **θ-open**. The set $\{x \mid x \in U \subset Cl(U) \subset A$ for some open set U of X$\}$ is called the **θ-interior** of A and is denoted by $Int_\theta(A)$.

Definition 1.1. A function $f : X \to Y$ is said to be **strongly sober θ-continuous** (briefly st. s. θ-c.) if for each $x \in X$ and each $V \in SO(Y, f(x))$, there exists an open set U in X containing x such that $f(Cl(U)) \subset V$.

Definition 1.2. A function $f : X \to Y$ is said to be **s-continuous** [2] (resp. **almost s-continuous** [14]) if for each $x \in X$ and each $V \in SO(Y, f(x))$, there exists an open set U in X containing x such that $f(U) \subset V$ (resp. $f(U) \subset sCl(V)$).

Remark 1.1. We have the following implications for a function : strongly sober θ-continuity implies s-continuity and s-continuity implies almost s-continuity. However, the converses are not ture. The function f in [14, Example 5.4] is s-continuous but it is not strongly sober θ-continuous at $a \in X$. The function f in [14, example 5.3] is almost s-continuous. But it is not continuous and hence not s-continuous.

2. CHARACTERIZATIONS OF st.s.θ-c. FUNCTIONS

In this section, we obtain several characterizations of strongly sober θ-continuous function.

Theorem 2.1. The following are equivalent for a function $f : X \to Y$:

(1) f is st.s.θ-c. ;

(2) $f^{-1}(V) \subset int_\theta (f^{-1}(V))$ for every $V \in SO(Y)$;

(3) the inverse image of a semi-open set of Y is θ-open ;

(4) the inverse image of a semi-closed set of Y is θ-closed.

(5) $f([A]_\theta) \subset sCl(f(A))$ for every subset A of X ;

(6) $[f^{-1}(B)]_\theta \subset f^{-1}(sCl(B))$ for every subset B of Y.

Proof. The proof is obvious and is thus omitted.

Recall that a filter base $\Im$ is called **θ-convergent** [16] (resp. **semi convergent**) to a point x in X if for any open set (resp. semi-open set) U containing x, there exists F $\in \Im$ such that $F \subset Cl(U)$ (resp. $F \subset U$).

Theorem 2.2. A function f : X → Y is st.s. θ-c. if and only if for each point x $\in$ X and each filter base $\Im$ in X θ-converging to x, the filter base $f(\Im)$ is semi convergent to f(x).

Proof. Necessity. Let x $\in$ X and $\Im$ be any filter base in X θ-converging to x. Since f is st.s. θ-c., then for any V $\in$ SO(Y, f(x)), there exists an open set U in X containing x such that $f(Cl(U)) \subset V$. Since $\Im$ is θ-converging to x, there exists an F $\in \Im$ such that $F \subset Cl(U)$. Hence $f(F) \subset V$. This shows that $f(\Im)$ is semi-convergent to f(x).

Sufficiency. Let x $\in$ X and V be any semi-open set is Y containing f(x). Let set $\Im = \{Cl(U) \mid x \in U$ and U is open is X$\}$, then $\Im$ is a filter base θ-converging to x. This means that there exists an open set U in $\Im$ such that $f(Cl(U)) \subset V$.

It should be noted that as a consequence of [16, Lemma 3], the family of all θ-open sets in a space (X, τ) establishes a topology for X which is denoted by τ_θ. We denote sometimes (X, τ_θ) by X_θ for the sake of simplicity.

Theorem 2.3. A function f : (X, τ) → (Y, σ) is a st.s.θ-c. if and only if $f : (X, \tau_\theta) \to (Y, \sigma)$ is s-continuous.

Proof. Let V be a semi-open set in (Y, σ). By Theorem 2.1, $f^{-1}(V)$ is θ-open in (X, τ). It follows that $f^{-1}(V)$ is open in (X, τ_θ). This shows that $f : (X, \tau_\theta) \to (Y, \sigma)$ is s-continuous.

Conversely, let V be a semi-open set is (Y, σ). Then $f^{-1}(V)$ is θ-open in (X, τ). Hence f is st.s.θ-c.

Theorem 2.4. Let X be a regular space. A function f : X → Y is st.s.θ-c. if and only if f is s-continuous.

Proof. Suppose that f is s-continuous. Let x $\in$ X and V $\in$ SO(Y, f(x)). By the s-continuity of f, there is an open set U in X containing x such that $f(U) \subset V$ Since X is regular, there is an open set W containing x such that $Cl(W) \subset U$. It follows that $f(Cl(W)) \subset f(U) \subset V$. This shows that f is st.s.θ-c.

Definition 2.1. A space X is said to be **semi-regular** [8] if for each semi-closed set A and each point x $\in$ X − A, there exist disjoint U, V $\in$ SO(X) such that x $\in$ U and $A \subset V$.

Dorsett [8] has shown that X is semi-regular if and only if for each point $x \in X$ and each $V \in SO(X, x)$, there exists $U \in SO(X, x)$ such that $x \in U \subset sCl(U) \subset V$. Recall that a subset A is said to be **semi-regular** [7] if it is semi-open and semi-closed. The family of all semi-regular stes of X is denoted by SR(X).

Lemma 2.1 (Di Maio and Noiri [7]). If $A \in SO(X)$, then $sCl(A) \in SR(X)$.

Theorem 2.5. Let Y be a semi-regular space. A function $f : X \to Y$ is st.s.θ-c. if and only if f is almost s-continuous.

Proof. Suppose that f is almost s-continuous. Let $V \in SO(Y)$ and $x \in f^{-1}(V)$. Since Y is semi-regular, there exists $W \in SO(Y)$ such that $f(x) \in W \subset sCl(W) \subset V$ [8, Theorem 2.1]. By Lemma 2.1, $sCL(W) \in SR(X)$ and by Theorem 3.3 of [14], there exists an open set U of X containing x such that $f(Cl(U)) \subset sCl(W)$. Therefore, we obtain $x \in U \subset Cl(U) \subset f^{-1}(V)$ and hence $x \in Int_\theta (f^{-1}(V))$. It follows from Theorem 2.1 that f is st.s.θ-c.

3. COMPOSITIONS OF st.s.θ-c. FUNCTIONS

Definition 3.1. A function $f : X \to Y$ is said to be

(1) **θ-continuous** [9] if for each $x \in X$ and each open set V of Y containing f(x), there exists an open set U of X containing x such that $f(Cl(U)) \subset Cl(V)$,

(2) **irresolute** [5] if $f^{-1}(V) \in SO(X)$ for each $V \in SO(Y)$.

Theorem 3.1. If $f : X \to Y$ is θ-continuous and $g : Y \to Z$ is st.s.θ-c., then $g \circ f : X \to Z$ is st.s.θ-c.

Proof. Let $x \in X$ and $W \in SO(Z, (g \circ f)(x))$. Since g is st.s.θ-c., there exists an open set V in Y containing f(x) such that $g(Cl(V)) \subset W$. Since f is θ-continuous, there exists an open set U in X containing x such that $f(Cl(U)) \subset Cl(V)$. This shows that $(g \circ f)(Cl(U)) \subset W$. Therefore, $g \circ f$ is st.s.θ-c.

Corollary 3.1. If $f : X \to Y$ and $g : Y \to Z$ are st.s.θ.c., then $g \circ f : X \to Z$ is st.s.θ.c.

Proof. Every st.s.θ-c. fuction is continuous and hence θ-continuous. Thus the proof is obvious by Theorem 3.1.

Theorem 3.2. If $f : X \to Y$ is st.s.θ-c. and $g : Y \to Z$ is irresolute, then $g \circ f : X \to Z$ is st.s.θ-c.

Proof. Let $x \in X$ and $W \in SO(Z, (g \circ f)(x))$. Since g is irresolute, there exists a semi-open set $V \in SO(Y, f(x)$ such that $g(V) \subset W$. By the hypothesis, there exists an open set U in X containing x suh that $f(Cl(U)) \subset V$. Hence $(g \circ f)(Cl(U)) \subset g(V) \subset W$. This shows that $g \circ f$ is is st.s.θ-c.

Theorem 3.3. If $f : X \to Y$ is a surjective open and closed function and $g : Y \to Z$ is a function such that $g \circ f : X \to Z$ is st.s.θ-c, then g is st.s.θ-c.

Proof. Suppose that $y \in Y$ and $y = f(x)$. Let $W \in SO(Z, g(y))$. Since $g \circ f$ is st.s.θ-c., there exists an open set U in X containing x such that $(g \circ f)(Cl(U)) \subset W$. Since f is open and closed, $f(U)$ is an open set in Y containing y and $Cl(f(U) \subset f(Cl(U))$. Therefore, we have $g(Cl(f(U))) \subset g(f(Cl(U)) \subset W$. This shows that g is st.s.θ-c.

4. STRONGLY SOBER θ-CLOSED GRAPHS

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) \mid x \in X\} \subset X \times Y$ is called the **graph** of f and is denoted by $G(f)$.

Definition 4.1. The graph $G(f)$ of a function $f : X \to Y$ is said to be **strongly sober θ-closed** (briefly st.s.θ-closed) if for each $(x, y) \in X \times Y - G(f)$, there exist an open set U in X containing x and $V \in SO(Y, y)$ such that $[Cl(U) \times V] \cap G(f) = \phi$.

Lemma 4.1. The graph $G(f)$ of $f : X \to Y$ is st.s.θ-closed in $X \times Y$ if and only if for each point $(x,y) \in X \times Y - G(f)$, there exist an open set U in X containing x and $V \in SO(Y,y)$ such that $f(Cl(U)) \cap V = \phi$.

Proof. It follows immediately from Definition 4.1.

A space X is said to be **semi** $-T_2$ [12] if for each pair of distinct points x and y in X, there exists $U \in SO(X, x)$ and $V \in SO(X, y)$ such that $U \cap V = o$.

Theorem 4.1. If $f : X \to Y$ is almost s-continuous and Y is *semi*–T_2, then $G(f)$ is st.s.θ-closed in $X \times Y$.

Proof. Let $(x, y) \in X \times Y - G(f)$. It follows that $f(x) \neq y$. Since Y is *semi* – T_2, there exist $V \in SO(Y, f(x))$ and $W \in SO(Y, y)$ such that $V \cap W = \phi$; hence $sCl(V) \cap W = \phi$. Since f is almost s-continuous, there exists an open set U in X containing x such that $f(Cl(U)) \subset sCl(V)$ [14, Theorem 3.3]. Therefore, $f(Cl(U)) \cap W = \phi$ and by Lemma 4.1, $G(f)$ is st.s.θ-closed in $X \times Y$.

A subset A of a space X is said to be **semi-compact relative to** X [5] (resp. **quasi H-closed relative to** X [15]) if for every cover $\{V_\alpha \mid \alpha \in \nabla\}$ of A by semi-open sets (resp. open sets) of X, there exists a finite subset ∇_o of ∇ such that $A \subset \cup \{V_\alpha \mid \alpha \in \nabla_o\}$ (resp. $A \subset \cup \{Cl(V_\alpha) \mid \alpha \in \nabla_o\}$).

A space X is said to be **semi-compact** [3] (resp. **quasi H-closed** [15]) if X is semi-compact relative to X (resp. quasi H-closed relative to X).

Theorem 4.2. If f : X → Y has a st.s.θ-closed graph,then the inverse image of a set K semi-compact relative to Y is θ-closed in X.

Proof. Let K be a set semi-compact relative to Y. Let $x \notin f^{-1}(K)$. For each y $\in$ K, (x, y) $\notin$ G(f) and by Lemma 4.1, there exist an open set U (y) in X containing x and V(y) $\in$ SO(Y, y) such that $f(Cl(U(y))) \cap V(y) = \phi$. By the hypothesis, there exists a finite subset $K_o \subset K$ such that $K \subset \cup \{V(y) \mid y \in K_o\}$. Set $U(x) = \cap \{U(y) \mid y \in K_o\}$. Then U(x) is an open set containing x and $Cl(U(x)) \cap f^{-1}(K) = \phi$. It follows that $x \notin [f^{-1}(K)]_\theta$. This shows that $f^{-1}(K)$ is θ-closed in X.

Theorem 4.3. If f : X → Y has a st.s.θ-closed graph and Y is semi-compact, then f is st.s.θ-c.

Proof. Suppose that W is any semi-closed set of Y. Since Y is semi-compact, then W is semi-compact relative to Y. By Theorem 4.2, $f^{-1}(W)$ is θ-closed in X. It follows from Theorem 2.1 that f is st.s.θ-c.

Corollary 4.1. Let Y be semi-compact and $semi - T_2$. Then the following are equivalent for a function f : X → Y :

(1) f is almost s-continuous ;

(2) G(f) is st.s.θ-closed ;

(3) the inverse image of a set K semi-compact relative to Y is θ-closed in X ;

(4) f is st.s.θ-c.

Proof. This is an immediate consequence of Theorems 4.1, 4.2 and 4.3.

5. SOME OTHER PROPERTIES

A space X is said to be **semi** $- T_0$ [1] if for each pair of distinct points x and y in X, there exists either U $\in$ SO(X, x) not containing y or V $\in$ SO(X,y) not containing x.

Theorem 5.1. If f : X → Y is a st.s.θ-c. injection and Y is $semi - T_0$, then X is Hausdorff.

Proof. By the injectivity of f, it follows that $f(x) \neq f(y)$ for any distinct points x and y in X. Since Y is *semi* $- T_0$, there exists either $V \in SO(Y, f(x))$ not containing f(y) or $W \in SO(Y, f(y))$ not containing f(x). If the former is the case, then there exists an open set U in X containing x such that $f(Cl(U)) \subset V$. It follows that $f(y) \notin f(Cl(U)$ and therefore X-Cl(U) is an open set containing y. If the latter is the case, then we obtain a similar result. This means that X is Hausdorff.

Theorem 5.2. If $f : X \rightarrow Y$ is s.t.θ-c. and K is quasi H-closed relative to X, then f(K) is semi-compact relative to Y.

Proof. Let $\{V_i \mid i \in I\}$ be a cover of f(K) by semi-open sets of Y. For each $k \in K$, there exists $i(k) \in I$ such that $f(k) \in V_{i(k)}$. Since f is st.s.θ-c., there exists an open set U_k in X containing k such that $f(Cl(U_k)) \subset V_{i(k)}$. Since $\{U_k \mid k \in K\}$ is a cover of K by open sets of X, there exists a finite subset K_0 of K such that $K \subset \cup \{Cl(U_k) \mid k \in K_0\}$. Hence we have

$$f(K) \subset \cup \{f(Cl(U_k)) \mid k \in K_0\} \subset \cup \{V_{i(k)} \mid k \in K_0\}.$$

This shows that f(K) is semi-compact relative to Y.

Theorem 5.3. If $f : X \rightarrow Y$ is st.s.θ-c. and A is any subset of X, then the restriction $f \mid A : A \rightarrow Y$ is st.s.θ-c.

Proof. Let $x \in A$ and $V \in SO(Y, f(x))$. Since f is st.s.θ-c., then there exists an open set U in X containing x such that $f(Cl(U)) \subset V$. Set $U_\alpha = U \cap A$. Then U_α is an open set of A containing x and $Cl_A(U_\alpha) = Cl(U_\alpha) \cap A$. We have

$$(f \mid A)\ (Cl_A(U_\alpha)) = f(Cl_A(U_\alpha)) \subset f(Cl(U) \cap A) \subset f(Cl(U)) \subset V.$$

Hence $f \mid A$ is st.s.θ-c.

Theorem 5.4. Let $f_i : X_i \rightarrow Y_i$ be a function for each $i \in I$ and $f : \Pi X_i \rightarrow \Pi Y_i$ a function defined by $f(\{x_i\}) = \{f_i(x_i)\}$ for each $\{x_i\} \in \Pi X_i$. If f is st.s.θ-c., then f_i is st.s.θ-c. for each $i \in I$.

Proof. Let f be st.s.θ-c. and $V_i \in SO(Y_i)$. The set $V = V_i \times \Pi\{Y_j \mid j \in I - \{i\}\}$ is semi-open in ΠY_i [13, Theorem 2] and therefore by Theorem 2.1, $f^{-1}(V) = f_i^{-1}(V_i) \times \Pi\{X_j \mid j \in I - \{i\}\}$ is θ-open in ΠX_i. Hence $f_i^{-1}(V_i)$ is θ-open by [11, lemma 2]. This shows that f_i is st.s.θ-c. for each $i \in I$.

Theorem 5.5. If a function $f : X \rightarrow \Pi Y_i$ is st.s.θ-C. and $p_i : \Pi Y_i \rightarrow Y_i$ is the projection, then the composition $p_i \circ f : X \rightarrow Y_i$ is st.s.θ-c.

Proof. Let $U_i \in SO(X_i)$ for a fixed i ∈ I. Then $p_i^{-1}(U_i) \in SO(\Pi X_j)$ [13, Theorem 2]. Since f is st.s.θ-c., then by Theorem 2.1 $f^{-1}(p_i^{-1}(U_i)) = (p_i \circ f)^{-1}(U_i)$ is θ-open is X. This shows that $p_i \circ f : X \to Y_i$ is st.s.θ-c.

REFERENCES

1. M. E. Abd El-Monsef, R. A. Mahmoud and A. A. Nasef, **Strongly semi-continuous functions**, Arab J. Phys. Math. (to appear).
2. D. E. Cameron and G. Woods, **s-continuous and s-open mappings** (preprint).
3. D. A. Carnahan, **Some Properties Related to Compactness in Topological Spaces,** Ph.D. Thesis, University of Arkansas, 1973.
4. S. G. Crossley and S. K. Hildebrand, **Semi-closure,** Texas J. Sci 22 (1971), 99-112.
5. S. G. Crossley and S. K. Hildebrand, **Semi-topological properties**, Fund. Math. 74 (1972), 233-254.
6. G. Di Maio and T. Moiri, **Weak and strong forms of irresolute functions**, Third National Conference on Topology (Italian) (Trieste, 1986). Rend. Circ. Palermo (2) Suppl. No 18 (1988), 255-273.
7. G. Di Maio and T. Noiri, **On s-closed spaces,** Indian J. Pure Appl. Math. 18(1987), 226-233.
8. C. Dorsett, **Semi-regular spaces**, Soochow J. Math. 8(1982), 45-53.
9. S. V. Fomin, **Extensions of topological spaces**, Ann. of Math. 44 (1943), 471-480.
10. N. Levine, **Semi-open sets and semi-continuity in topological spaces**, Amer. Math. Monthly 70 (1963), 36-41.
11. P. E. Long and L. L. Herrington, **Strongly θ-continuous functions**, J. Korean Math. Soc. 18(1981), 21-28.
12. S. N. Maheshwari and R Prasad, **Some new separation axioms**, Ann. Soc. Sci. Bruxelles 89(1975), 395-402.
13. T. Noiri, **On semi-continuous mappings**, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 54 (1973), 210-214.

14. T. Noiri, B. Ahmad and M. Khan, **Almost s-continuous functions**, Kyungpook Math. J. 35 (1995), 311-322.

15. J. Porter and J. Thomas, **On H-closed and minimal Hausdorff spaces**, Trans. Amer. Math. Soc. 138 (1969), 159-170.

16. N. V. Veličko. **H-closed topological spaces**, Amer. Math. Soc. Transl. (2) 78 (1968), 103-118.

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LATTICE OF IDEMPOTENT SEPARATING CONGRUENCES IN A CONVENTIONAL SEMIGROUP

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ABSTRACT : In this paper we introduce the concept of regular, normal subset of a regular semigroup, give an alternate characterisation of the maximum idempotent separating congruence in a conventional semigroup and finally describe the lattice of the idempotent separating congruences in a conventional semigroup.

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1. INTRODUCTION

A regular semigroup S is called conventional if the set E of idempotents of S is self-congugate, that is $aEa' \subseteq E$ for each $a \in S$ and for each inverse a' of a. Conventional semigroups were first developed by Masat in [4]. In [4, proposition, P-398] he showed that the class of orthodox semigroups $\underset{\neq}{\subset}$ the class of conventional semigroups $\underset{\neq}{\subset}$ the class of regular semigroups In [5] Masat characterised the maximum idempotent separating congruence μ on a conventional semigroup In this paper firstly we give an alternate characterisation of μ and secondly we give a description of the lattice of idempotent separating congruences on a conventional semigroup, thus generalizing the result of Feigenbaum [2] from orthodox to conventional semigroups. Throughout this paper we use the notation of Clifford and Preston [1]

2. PRELIMINARIES

In this section we define some terms and give some results which we use in the rest of this paper.

Definition 2.1. A subset B of a regular semigroup S is called regular if for every $x \in B$ there exists $x' \in V(x)$ such that $x' \in B$.

Definition 2.2. A subset N of a regular semigroup S is called normal if for a, b $\in$ S, $a' \in V(a)$, $b' \in V(b)$, $aa' = bb'$, $a'a = b'b$ and $ab', a'b \in N$ imply $aNb' \subseteq N$ and $a'Nb \subseteq N$.

Definition 2.3. In a regular semigroup S for any two idempotents e and f of S the set $S(e,f) = \{g^2 = g \in S : ge = g = fg \text{ and } egf = ef\}$ is called the sandwich set of e and f in that order.

Lemma 2.4. In a regular semigroup S if $a' \in V(a)$, $b' \in V(b)$ then $b'ga' \in V(ab)$ for all $g \in S(a'a, bb')$.

Lemma 2.5. [3 ; Proposition II.4.1.] Let a, b be two elements of a regular semigroup S. Then $(a, b) \in \mathcal{H}$ iff there exist $a' \in V(a)$, $b' \in V(b)$ such that $aa' = bb'$ and $a'a = b'b$. Actually if $(a, b) \in \mathcal{H}$ then for all $a' \in V(a)$ there exists $b' \in V(b)$ such that $aa' = bb'$ and $a'a = b'b$.

Lemma 2.6. In a conventional semigroup S the set E of idempotents of S is a regular normal subset of S.

Proof. Let $a, b \in S$, $a' \in V(a)$, $b' \in V(b)$ be such that $aa' = bb'$, $a'a = b'b$ and $ab' \in E$ and $a'b \in E$. We show that $aEb' \subseteq E$ and $a'Eb \subseteq E$. Let $e \in E$ then $aeb' = aa'aeb'bb' = bb'aea'ab' = bb'bb'aea'ab'ab' = ba'ab'\,(aea')\,ab'ab' \in E$. Thus $aEb' \subseteq E$. Similarly $a'Eb \subseteq E$. Also $e \in V(e) \cap E$. Thus E is a regular normal subset of S.

3. THE LATTICE OF IDEMPOTENT SEPARATING CONGRUENCES

In [5] Masat characterised the maximum idempotent separating congruence μ on a conventional semigroup S as $\mu = \{(a, b) \in SxS$: there are inverses a′ of a and b′ of b such that $a'ea = b'eb$ and $aea' = beb'$ for all $e \in E\}$.

Here we present an alternate characterisation of μ. For this we define centralizer of E denote by C(E) as $C(E) = \{\, x \in S : (x, e) \in \mu$ for some $e \in E\}$.Unless otherwise stated in this section S denotes a conventional semigroup.

Theorem 3.1. Let $\delta = \{(a,b) \in S \times S$: there exist $a' \in V(a)$, $b' \in V(b)$ for which $aa' = bb'$, $a'a = b'b$ and $ab', a'b \in C(E)\}$. Then $\delta = \mu$.

Proof. Let $(a, b) \in \mu$. Then as μ is an idempotent separating congruence, $\mu \subseteq \mathcal{H}$. Hence by Lemma 2.5 there exist $a' \in V(a)$, $b' \in V(b)$ such that $aa' = bb'$, $a'a = b'b$. Also $(ab', bb') \in \mu$, $(a'a, a'b) \in \mu$ where bb′ and a′a belong to E. Therefore $ab', a'b \in C(E)$. So $(a, b) \in \delta$.

Conversely let $(a, b) \in \delta$. Then there are inverses $a' \in V(a)$, $b' \in V(b)$ such that $aa' = bb'$, $a'a = b'b$, and $ab', a'b \in C(E)$. So $a \mathcal{H} b$, $a' \mathcal{H} b'$ which give $ab' \mathcal{H} bb'$. Since $ab' \in C(E)$, we have $(ab', bb') \in \mu$. Therefore $a\mu = a\mu\,(a'a)\mu = a\mu\,(b'b)\mu = (ab')\mu\, b\mu = (bb')\mu\, b\mu = b\mu$. So $(a, b) \in \mu$. Hence $\mu = \delta$.

Lemma 3.2. $C(E)$ is a regular normal subset of S.

Proof. Let $a \in C(E)$, then $(a, e) \in \mu$ for some $e \in E$. So $(a, e) \in \mathcal{H}$ Hence there exists $a' \in V(a) \cap H_a$ such that $aa' = a'a = e$. Also $(a'a, a'e) \in \mu$ implies $(e,a') \in \mu$. Thus $a' \in C(E)$. Next let $a,b \in S$, $a' \in V(a)$, $b' \in V(b)$, $aa' = bb'$, $a'a = b'b$, and ab', $a'b \in C(E)$. Then by Theorem 3.1 $(a, b) \in \mu$. So $(axb', bxb') \in \mu$ for all $x \in C(E)$. But $x \in C(E)$ implies that $(x, e) \in \mu$ for some $e \in E$. So $(bxb', beb') \in \mu$ and consequently $(axb', beb') \in \mu$ where $beb' \in E$ (since S is conventional). Therefore $axb' \in C(E)$. Similarly $a'xb \in C(E)$ for all $x \in C(E)$. So $C(E)$ is a regular normal subset of S.

Lemma 3.3. Let $A = \{a \in S : \text{there exists } a' \in V(a) \text{ for which } aea' = a'ea = aa'ea'a \text{ for all } e \in E\}$. Then $A = C(E)$.

Proof. Let $a \in C(E)$. Then $(a, f) \in \mu$ for some $f \in E$. Since $\mu \subseteq \mathcal{H}$ there exists $a' \in V(a) \cap H_a$ such that $aa' = a'a = f$. Now $(a,f) \in \mu$ implies that $(a'a, a'f) \in \mu$, that is, $(a', f) \in \mu$. For each $e \in E$ we have $(aea', fef) \in \mu$ and $(a'ea, fef) \in \mu$ which implies that $aea' = a'ea = fef = aa'ea'a$. So $a \in A$.

Conversely, if $a \in A$ then there esists $a' \in V(a)$ such that $aea' = a'ea = aa'ea'a$ for all $e \in E$. So we have $aa' = a\,(a'a)a' = aa'\,(a'a)\,a'a = aa'a'a = aa'\,(aa')\,a'a = a'\,(aa')\,a = a'a$.

Now $aea' = aa'eaa'$ and $a'ea = aa'eaa'$ for all $e \in E$ imply that $(a, aa') \in \mu$. So $a \in C(E)$.

Lemma 3.4. Let K ($\supseteq E$) be a normal subset of a regular semigroup S. For $a, b \in S$ if there exist $a' \in V(a)$, $b' \in V(b)$ such that $aa' = bb'$, $a'a = b'b$ and $a'b$, $ab' \in K$ then ba', $b'a \in K$ and ab^*, $a^*b \in K$ for all $a^* \in V(a)$ and $b^* \in V(b)$.

Proof. For any $a \in S$ and $a' \in V(a)$ we have $aa' = aa'$, $a'a = a'a$ and aa', $a'a \in K$. So $aKa' \subseteq K$ for all $a \in S$ and $a' \in V(a)$. Hence

$$ba' = bb'ba' = a(a'b)\,a' \in aKa' \subseteq K,$$

$$b'a = b'aa'a = b'(ab')\,b \in b'Kb \subseteq K,$$

$$ab^* = aa'ab^* = b(b'a)\,b^* \in bKb^* \subseteq K,$$

$$a^*b = a^*bb'b = a^*(ba')\,a \in a^*Ka \subseteq K.$$

Let $N_r = \{K \subseteq S : E \subseteq K \subseteq C(E)$, where K is a regular normal subset of S$\}$. Clearly E and C(E) belong to N_r. The following theorem gives a description of the lattice of idempotent separating congruences on a conventional semigroup.

Theorem 3.5. The map $K \to (K) = \{(a, b) \in S \times S :$ there exist $a' \in V(a)$, $b' \in V(b)$ for which $aa' = bb'$, $a'a = b'b$ and $ab', a'b \in K\}$ is a one to one order preserving map of N_r onto the set of all idempotent separating congruences on S.

Proof. First we show that if $K \in N_r$ then (K) is an idempotent separating congruence on S. Since $E \subseteq K$, (K) is reflexive. Moreover $K \subseteq C(E)$ implies that $(K) \subseteq \mu$ (by Theorem 3.1). So (K) is an idempotent separating relation. Also if $(a, b) \in (K)$ then by Lemma 3.4 $(b, a) \in (K)$. Hence (K) is symmetric. To prove that (K) is transitive, let $(a, b) \in (K)$, $(b, c) \in (K)$. Then there exist $a' \in V(a)$, $b', b^* \in V(b)$, $c^* \in V(c)$ such that $aa' = bb'$, $a'a = b'b$, $bb^* = cc^*$, $b^*b = c^*c$. But $(K) \subseteq \mu \subseteq \mathcal{H}$ implies that $a \mathcal{H} b \mathcal{H} c$ so that by Lemma 2.5 there exists $c' \in V(c)$ such that $aa' = bb' = cc'$ and $a'a = b'b = c'c$. Also $ab', a'b, bc', b'c \in K$ by Lemma 3.4. Therefore $ac' = aa'ac'cc' = ab'(bc')bb' \in K$ and $a'c = a'aa'cc'c = a'b(b'c)b'b \in K$ as K is normal subset of S. So $(a, c) \in (K)$. To show that (K) is compatible, let $(a, b) \in (K)$, $(c, d) \in (K)$. Since $K \subseteq C(E)$ we have $(K) \subseteq \mu$. So $(a, b), (c, d) \in \mu$. Hence there exist $a' \in V(a)$, $b' \in V(b)$, $c' \in V(c)$, $d' \in V(d)$ such that $aea' = beb'$, $a'ea = b'eb$, $cec' = ded'$, $c'ec = d'ed$ for all $e \in E$. Then as $\mu \subseteq \mathcal{H}$ it follows that $aa' = bb'$, $a'a = b'b$, $cc' = dd'$, $c'c = d'd$. By Lemma 2.4 $c'ga' \in V(ac)$ and $d'gb' \in V(bd)$ for all $g \in S(a'a, cc') = S(b'b, dd')$.

Now $(ac)(c'ga') = acc'ga' = aga' = bgb' = bdd'gb' = (bd)(d'gb')$ and $(c'ga')(ac) = c'ga'ac = c'gc = d'gd = d'gb'bd = (d'gb')(bd)$. Also $(ac)(d'gb') = aa'acd'gb' = aa'acc'cd'gb' = aa'agcc'cd'gb' = ag(cd')gb' \in K$ (as $cd' \in K$ by Lemma 3.4), and $(c'ga')(bd) = c'ga'bb'bdd'd = c'ga'bb'bgdd'd = c'g(a'b)gd \in K$ (as $a'b \in K$ by Lemma 3.4). Thus $(ac, bd) \in (K)$. Thus (K) is an idempotent separating congruence on S.

Now if ρ is an idempotent separating congruence on S then we define kernel of ρ denoted by $\ker \rho$ as $\ker \rho = \{a \in S : (a, e) \in \rho$ for some $e \in E\}$. We now show that $\ker \rho \in N_r$ and $(\ker\rho) = \rho$. To prove that $\ker \rho \in N_r$ let $a \in \ker \rho$ then $(a, e) \in \rho \subseteq \mathcal{H}$ for some $e \in E$. So there exists $a' \in V(a) \cap H_a$ such that $aa' = a'a = e$. Now $(a'a, a'e) \in \rho$ hence $(e, a') \in \rho$. Therefore $a' \in \ker \rho$. Thus $\ker \rho$ is regular subset of S.

Next let $a, b \in S$, $a' \in V(a)$, $b' \in V(b)$ such that $aa' = bb'$, $a'a = b'b$ and $a'b, ab' \in \ker \rho$. We show that axb' and $a'xb \in \ker \rho$ for all $x \in \ker \rho$. From the given condition $a \mathcal{H} b$ and $a' \mathcal{H} b'$. Therefore $ab' \mathcal{H} bb'$. As $ab' \in \ker \rho$, we get $(ab')\rho = (bb')\rho$. So $a\rho = a\rho(a'a)\rho = a\rho(b'b)\rho = (ab')\rho b\rho = (bb')\rho b\rho = b\rho$. Hence $(a, b) \in \rho$. So $(axb', bxb') \in \rho$ for all $x \in \ker \rho$. But $x \in \ker \rho$ implies that $(x, e) \in \rho$ for some $e \in E$. Hence $(bxb', beb') \in \rho$. Thus $(axb', beb') \in \rho$. But $beb' \in E$ (since S is conventional). So $axb' \in \ker \rho$. Similarly $a'xb \in \ker \rho$ for all $x \in \ker \rho$. So $\ker \rho$ is a normal subset of S. Also $E \subseteq \ker \rho \subseteq C(E)$. Therefore $\ker \rho \in N_r$.

Now we show that $(\ker \rho) = \rho$. For this let $(a, b) \in (\ker \rho)$ then there exist $a' \in V(a)$, $b' \in V(b)$ such that $aa' = bb'$, $a'a = b'b$ and $ab', a'b \in \ker \rho$. Therefore $a \mathcal{H} b$ and $a' \mathcal{H} b'$. Consequently $ab' \mathcal{H} bb'$. As $ab' \in \ker \rho$ we have $(ab') \rho = (bb') \rho$. Then $a\rho = a\rho\, (a'a)\, \rho = a\rho\, (b'b)\, \rho = (ab')\, \rho\, b\, \rho = (bb')\, \rho\, b\, \rho = b\, \rho$. That is $(a,b) \in \rho$. Next let $(a,b) \in \rho$ then $(a,b) \in \mathcal{H}$. So there exist $a' \in V(a)$, $b' \in V(b)$ such that $a'a = bb'$, $a'a = b'b$. Also $(ab', bb') \in \rho$ and $(a'a, a'b) \in \rho$. Therefore $ab', a'b \in \ker \rho$. Hence $(a, b) \in (\ker \rho)$. Thus $(\ker \rho) = \rho$. So the given map is onto. The given map is clearly order preserving. We now show that the given map is one to one. For this let $K, L \in N_r$ with $(K) = (L)$. Let $a \in K$, since $K \subseteq C(E)$, $a \in C(E)$. Therefore $(a, e) \in \mu$ for some $e \in E$. Since $\mu \in \mathcal{H}$ there exists $a' \in V(a) \cap H_a$ such that $aa' = a'a = e$. Now K is regular subset so there exists $a^* \in V(a) \cap K$. Then $a' = a'aa' = (a'a)\, a^*\, (aa') = ea^*e \in K$. Thus $a \mathcal{H} a'a$, $a(a'a) = a \in K$ and $a'\,(a'a) = a'\,(aa') = a' \in K$. Therefore, $(a,a'a) \in (K) = (L)$ so $a\,(a'a)^* \in L$ for all $(a'a)^* \in V(a'a)$ by Lemma 3.4. So in particular $a(a'a) \in L$ hence $a \in L$. Thus $K \subseteq L$. Similarly we can prove that $L \subseteq K$. Thus $K = L$. This completes the proof of the theorem.

REFERENCES

1. A. H. Clifford and G. B. Presteon, The Algebraic Theory of semigroups, I, A. M. S., Providence, 1961.
2. R. Feigenbaum, Kernels of orthodox semigroup homomorphisms, J. Austral. Math. Soc. 22 (Series A) (1976), 234-245.
3. J. M. Howie. An Introduction to Semigroup Theory, New York : Academic Press, 1976.
4. F. E. Masat, Right group and group congruences on a regular semigroup, Duke Math. J. 40(1973), 393-402.
5. F. E. Masat. Congruences on conventional semigroups, Czechoslovak Math. J. 31(106) (1981), 199-205.

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DYNAMIC OPTIMIZATION OF BEVERTON-HOLT DENSITY DEPENDENT CONTINUOUS MODEL

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ABSTRACT : In this paper we, we consider the dynamic optimization of the density dependent continuous, age structured model as introduced by Beverton and Holt in the year 1957. We divide our analysis into two parts—(i) Single cohrt analysis and (ii) Multiple cohort analysis. In the former case, considering the nonzero cost, we show that dynamic optimization consists of a combination of singular and bang-bang control, and that the corresponding results of Clark, C. W. [5], in the density independent model, follow as a limiting case. In the latter analysis also, nonzero cost is considered, but recruitment is taken to be idenpendent of the stock size. It is shown that the optimal time of harvesting can not be obtained, if the standard depreciation $e^{-\delta t}$, $\delta > 0$ is used, to calculate the net discounte revenue from the observed one. It is also shown that, proper depreciation can be chosen which differs from model to model, and which enables us to obtain the optimal time of hasvesting without any difficulty.

Key Words : Dynamic optimization, single cohort, multiple cohort, age structured model, Beverton-Holt continuous density dependent model.

1. INTRODUCTION

Beverton and Holt [1] considered static optimization of the density independent continuous age structured model given by

$$(1/N)\ (dN/dt) = -\ MN(t) \qquad \ldots \text{(i)}$$

where M was the natural mortality rate and N(t) was the fish population size at time t. It was C.W. Clark [5] who considered dynamic optimization of this density independent model in the single cohort as well as in the multiple cohort case under the hypothesis that cost of fishing was zero and that recruitment was independent of the stock size.

Clark did not analyze the case of nonzero cost of fishing but made the conjecture that optimal policy would be a pulse fishing policy for this case (and nonselective gear). Hanneson [8] examined, through simulation, the case in which fishing costs were nonzero, and concluded that pulse fishing was slightly better than stationary fishing policy tried. Although Beverton and Holt examined the effects of various density dependent mechanisms on optimal policy obtained from their dynamic pool model, analytical solutions of this problem did not include density dependence. However, they proposed the following density-dependent model given by

$$(1/N)\ (dN/dt) = -\ (\mu_1 + \mu_2 N) \qquad \ldots\ \text{(ii)}$$

μ_1 being the natural mortality coefficient and μ_2 being the density dependent mortality coefficient.

So far as the other types of density dependent age structured continuous models (population dependent and/or food dependent) were concerned, several important works were done even considering stock-recruitment relationship (Pope [9], Getz [7], Botsford [3] and many others). But no analysis was done with Beverton-Holt density dependent model (II), even in the static case.

In this paper, in order to keep parity with the Logistic model, we rewrite (II) in the following form

$$(1/N)\ (dN/dt) = -\ M[1 + N(t)/K],\ N(0) = R\ \text{(initial recruitment)} \qquad \ldots\ \text{(iii)}$$

where K is no longer the carrying capacity but is simply the reciprocal of the density dependent mortality coefficient.

We carry out the whole of our analysis in the single cohort and in the multiple cohort case following the corresponding analysis of Clark [5]. In the single cohort case, we verify our results by comparing with the results of Clark in the density independent situation, when $K \to \infty$. Since in the multicohort analysis, our results are obtained by taking a depreciation different from the standard one as used by Clark, so these results are not comparable with the results of clark, as $K \to \infty$.

Lastly, it is revealed from our discussion that it is not necessary to stick to the special form of w(t), the weight per unit biomass, viz. $w(t) = a(1 - be^{-ct})^3$; a, b, $c > 0$ [Bertalanffy function], but it is sufficient to consider any w(t) such that w(t) is monotonically increasing and $(\dot{w}/w)$ is monotonically decreasing.

1. Single cohort analysis in the density dependent case

Theorem 1.1. Let the exploited single species system be given by

$$(1/N)\ (dN/dt) = -\ M[1 + N(t)/K],\ qE(t),\ N(O) = R \qquad \ldots (1.1)$$

where E (t) is the effort at time t and q is the catchability coefficient. Then the no harvest curve B_0 (t) is given by

$$B_o\ (t) = \frac{KRw(t)\ -}{(K+R)e^{Mt} - R}$$

If the net discounted revenue be given by

$$PV = \int_0^{\infty} e^{-\delta t}\ [pqB(t) - c]\ E(t)\ dt,\ B(t) = N(t)\ W(t) \qquad \ldots (1.2)$$

where p is the price of unit biomass and c is the cost per unit effort, then the optimal biomass B(t) is given by $B^*\ (t) = N^*\ (t)\ w\ (t)$, where $N^*\ (t) = \frac{1}{4pqMw(t)}\ [-qK\{pMw(t) - (cM/qK) + \delta pw(t) - p\,\dot{w}(t)\} + \{(q^2K^2\ (pMw(t) + \delta\ pw(t) - (cM/qK) + \delta\ pw(t)\ -\ p\,\dot{w}(t))^2 + 8pqw(t)\ M\delta ck\}^{1/2}]$

The optimal value of the revenue is given by

$$(PV)^* = \int_0^{\infty} e^{-\delta t}\ [pqN^*\ (t)\ w(t) - c]\ E^*\ (t)\ dt$$

where E^* (t) can be calculated from

$$(1/N^*)\ dN^*/dt) = -\ M\ (1 + N^*/K) - qE^*\ (t).$$

Further, it is shown that, total harvesting is a combination of singular and bang-bang control.

Proof :

It follows easily that the solution of (III) is given by

$$N(t) = \frac{KR}{(K+R)e^{Mt} - R}$$

Hence, on no harvest curve is given by $B_0\ (t) = \frac{KRw(t)}{(K+R)e^{Mt} - R}$ ... (1.3)

Obviously, B_0 (t) exists finitely at t = 0 and B_0 (t) $\rightarrow$ 0 as t $\rightarrow \infty$. In fact, writing (1.3) as $[(K + R)e^{Mt} - R]\ B_0(t) = KRw(t)$, and differentiating with respect to t, we have

$$[(K + R)e^{Mt} - R]\ (dB_0(t)/dt) + M\ (K + R)e^{Mt}\ B_0(t) = KR\dot{w}(t) \qquad \ldots (1.4)$$

As $(\dot{w}/w)$ decreases with time, so does w. Hence it follows from (1.4) that B_0 (t) is maximum at $t = t_0$, if $(dB_0\ (t_0)/dt) = 0$. i.e. if

$$(\dot{w}/w) = \frac{M(K+R)e^{Mt_0}}{(K+R)e^{Mt_0} - R}$$

Otherwise, if there does not exist any t > 0 for which $(dB_0\ (t)/dt) = 0$, then $B_0(t)$ simply decreases with time.

To determine the nature of the optimal biomass curve B* (t), we consider the exploited model given by (1.1) and the net discounted revenue given by (1.2). In this case, the Hamiltonian is given by

$$H = e^{-\delta t}\ [pqN\ (t)\ w\ (t) - c]\ E\ (t) + \lambda\ (t)\ \{- MN\ (t)\ [1 + N\ (t)/K] - qE\ (t)\ N\ (t)\} \qquad \ldots (1.6)$$

So $-\ (\partial H / \partial N) = -\ e^{-\delta t}\ pqw\ (t)\ E\ (t) + \lambda\ (t)\ [M + (2MN\ (t) / K) + qE\ (t)]$

Assuming H to be maximum for some E (t) $\in V_t$ (control set), we have $(\partial H / \partial E) = 0$

$$\text{So } \lambda\ (t) = e^{-\delta t}\ [pw\ (t) - (c/qN\ (t)] \qquad \ldots (1.7)$$

By Pontryagin's principle of optimality, $(d\lambda / dt) = -\ (\partial H / \partial E)$.

$$\text{So } -\ \delta e^{-\delta t}\ [pw\ (t) - \frac{c}{qN(t)}] + e^{-\delta t}\ [p\dot{w}\ (t) + \frac{c\dot{N}(t)}{q(N(t))^2}]$$

$$= -\ e^{-\delta t}\ pqw\ (t)\ E\ (t) + \lambda\ (t)\ [M + (2MN/K) + qE\ (t)] \qquad \ldots (1.8)$$

Substituting in (1.8), the value of λ (t) given by (1.7), we get

$$2pqw\ (t)\ M\ (N(t))^2 + qK\ [pMw\ (t) - (cM/qK) + \delta pw\ (t) - p\dot{w}\ (t)]\ N\ (t) - \delta cK = 0 \qquad \ldots (1.9)$$

Hence optimal value N* (t) of N (t) is given by

$$N^*\ (t) = \frac{1}{4qpMw(t)}\ [-A\ (t) + \{(A\ (t))^2 + 8\ pqM\delta cKw(t))^{1/2}] \qquad \ldots (1.10)$$

where $$A(t) = qK[pMw(t) - (cM/qK) + \delta pw(t) - p\dot{w}\ (t)] \qquad \text{. . (1.11)}$$

So the optimal biomass is given by

$$B^*\ (t) = N^*\ (t)\ w\ (t)$$

$$= \frac{1}{4pqM}\ [-\ A\ (t) + \{(A\ (t))^2 + 8pqM\delta cKw\ (t)\}^{1/2}] \qquad \text{... (1.12)}$$

Obviously, B^* (t) has a maximum at $t = t'$ given by

$$p(M + d)\ w\ (t') = p\dot{w}\ (t') + (cM/qK) \qquad \text{... (1.13)}$$

and $$B^*\ (t') = \frac{1}{4pqM}\ [8pqM\delta cKw\ (t')\}^{1/2}] \qquad \text{... (1.14)}$$

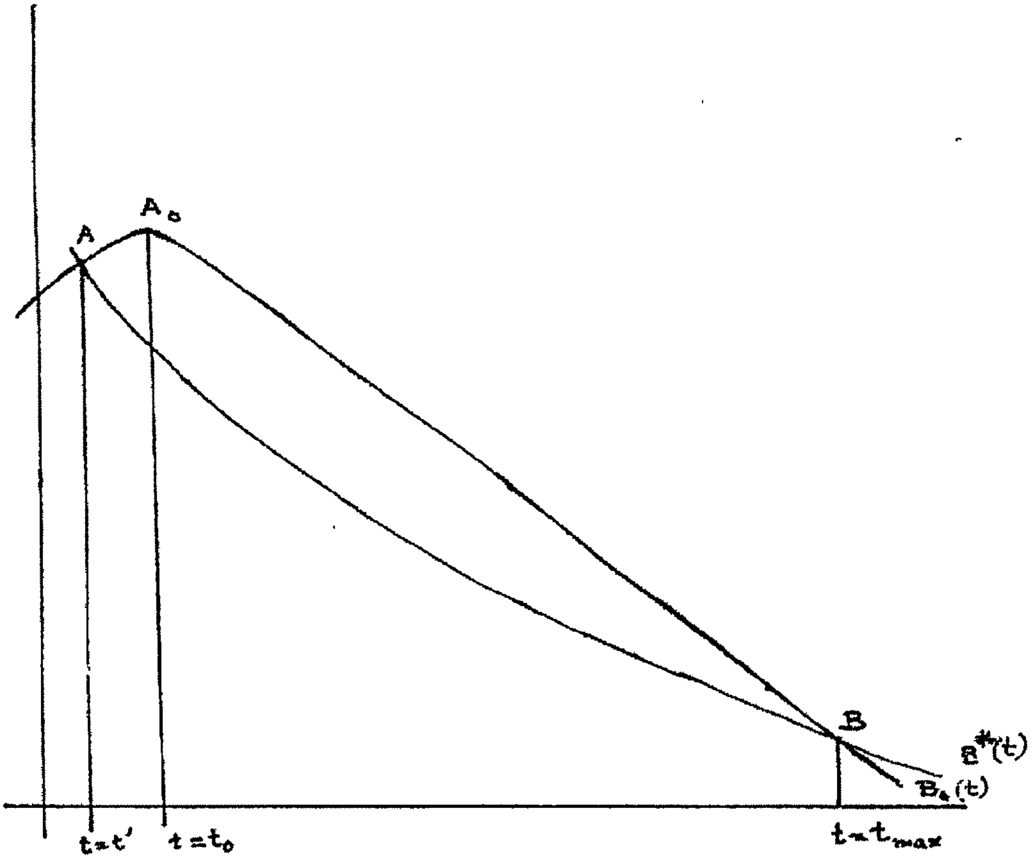

Fig. 1

Further, $B^*(t)$ is a monotone decreasing curve and $B^*(t)$ tends to a finite nonzero limit as $t \to \infty$. To show that for the single cohort model, optimization is a combination of singular and bang-bang control, we consider $B_0(t)$ and $B^*(t)$ together. (Fig. 1)

Since $B^*(t)$ is monotone decreasing and is maximum at $t = t'$, so there is no loss of generality in assuming that $E(t) = 0$, $t < t'$. Again as $t \to \infty$, $b_0(t) \to 0$, whereas $B^*(t)$ tends to a finite limit. Hence there is no loss of generality in assuming the point of interesction B of $B_0(t)$ and $B^*(t)$ as the point where the biomass attains its maximum age t_{max}. Naturally, $E(t) = 0$ for all $t > t_{max}$. So optimization takes place only for the singular control acting between A and B ($t' < t < t_{max}$).

2. MULTIPLE COHORT ANALYSIS IN THE DENSITY DEPENDENT CASE

Theorem 2.1. Let the state equation of the exploited nth cohort be

$$(dN_n/dt) = - MN_n/K] - K] - qE(t) N_n(t), \; N_n(n) = R, \; t \geq n \qquad \ldots (2.1)$$

Then the no harvest curve $B_n(t)$ for the nth cohort is given by

$$B_n(t) = \begin{cases} N_n(t)\, w(t-n), & t \geq n \\ 0 & , t < n \end{cases} \qquad \ldots (2.2)$$

where

$$N_n(t) = \frac{KR}{(K+R)e^{M(t-n)} - R} \qquad \ldots (2.3)$$

The total biomass at time t is given by $\sum_{n=1}^{[t]} B_n(t)$, and it exists for large values of t. But corresponding to the principal value of the revenue,

$$PV = \int_0^\infty e^{-\delta t} [pq \sum_{n=1}^{[t]} B_n(t) - c]\, E(t)\, dt, \; t \geq n \qquad \ldots (2.4)$$

the optimal time of harvesting does not exist.

Proof. (2.3) follows readily from (2.1). Next we show that $\sum_{n=0}^{\infty} B_n(t)$ converges so that $\sum_{n=1}^{[t]} B_n(t)$ exists for all t.

Now,

$$B_n(t) = \frac{KRw(t-n)}{(K+R)e^{M(t-n)}-R} \le \frac{KRw_\infty}{(K+R)e^{M(t-n)}-R} = \frac{c_1}{(K+R)e^{M(t-n)}-R} \quad \begin{bmatrix} c_1 = KRw_\infty \\ =\text{constant} \end{bmatrix}$$

So $\sum_{n=0}^{m} B_n(t) \le c_1 \sum_{n=0}^{m} \frac{1}{(K+R)e^{M(t-n)}-R} < c_1 \sum_{n=0}^{m} \frac{1}{(K+R)e^{M(t-n)}-(K+R)}$

$$= \frac{c_1}{(K+R)} \sum_{n=0}^{m} \frac{1}{e^{M(t-n)}-1} = c_2 \sum_{n=0}^{m} \frac{1}{e^{M(t-n)}-1} \quad \left[\ c_2 = \frac{c_1}{K+R} = \text{const}\right].$$

Now $\frac{1}{e^{M(t-n)}-1} < \frac{1}{e^{\alpha M(t-n)}-1}$ for some α, $0 < \alpha < 1$

So $\sum_{n=0}^{\infty} B_n(t) = \lim_{m \to \infty} \sum_{n=0}^{m} B_n(t) < c_2 \sum_{n=0}^{\infty} e^{-\alpha M(t-n)}$

But $\sum_{n=0}^{\infty} e^{-\alpha M(t-n)}$ is a convergent series. So $\sum_{n=0}^{\infty} B_n(t)$ converges.

Now we consider (2.1) and (2.3) and write the Hamiltonian as

$$H = e^{-\delta t}\left[pq \sum_{n=0}^{[t]} B_n(t) - c\right] E(t) + \sum_{n=0}^{[t]} \lambda_n(t) \{-MN_n(1 + N_n/K) - qE(t)\, N_n(t)\}$$

In order that H be maximized for some $E(t)\ \varepsilon\ V_t$ (control set), $(\partial H/\partial E) = 0$ and auxilliary functions $\lambda_n(t)$ should be such that $(d\lambda_n/dt) = -(\partial H/\partial N_n)$, and $\sum_{n=0}^{[t]} \lambda_n(t)\, N_n(t)$, $\sum_{n=0}^{[t]} \lambda_n(t)\, N_n^2(t)$ exists separately for moderately large but finite values of t. If possible, let us assume that such λ_n's exist.

$$\text{Now } (\partial H/\partial E) = 0 \Rightarrow \sum_{n=0}^{[t]} \lambda_n(t)\, N_n(t) = (1/q)\, e^{-\delta t} \left(pq \sum_{n=0}^{[t]} B_n(t) - c\right)$$

As the right hand side vanishes for moderately large values of t such that $\delta t > 1$, so $\sum_{n=0}^{[t]} \lambda_n (t)\, N_n(t) = 0$ for such positive integral values of t. So the question of maximization of H does not arise because Pontryagin's maximum principle is a necessary condition. Hence optimal time of harvesting does not exist.

3. DENSITY-DEPENDENT MULTIPLE COHORT MODEL AND PROPER DEPRECIATION

It is remarked that in the above ariticle, the depreciation $e^{-\delta t}$ caused the difficulty in calculating the optimal time of harvesting. Again, it is observed that introduction of $e^{-\delta t}$ in the integral giving the principal value of the revenue assures the convergence of the integral

$$PV = \int_0^\infty e^{-\delta t} \, [pq\ (t) \sum_{n=0}^{\infty} B_n (t) - c]\ E\ (t)\ dt$$

From this stand point, we may take $(1/t^{\delta + 2})$ as the depreciation, so that

$$PV = \int_0^\infty (1/t^{\delta + 2})\ [pq\ (t) \sum_{n=0}^{\infty} B_n (t) - c]\ E\ (t)\ dt,$$

where $$[pq \sum_{n=0}^{\infty} B_n (t) - c] = 0\ (t^{\delta}),\ \delta > 0$$

remains convergent even for $\delta = 0$.

We, therefore, consider the following

Theorem 3.1. Let the state equation of the nth cohort and $B_n(t)$ be given by (2.1) and (2.2) respectively. Let $B\ (t) = \sum_{n=0}^{\infty} B_n (t) = \sum_{n=0}^{[t]} B_n (t),\ t \geq n$.

Let $V(t) = pqB\ (t) - c$ and $V^H\ (t) = (dV/dt)$ when $E(t) = 0$.

Let $(pqB\ (t) - c)\ E\ (t) = o\ (t^{\delta}),\ \delta > 0$.

Let $$PV = \int_0^\infty (1/t^{\delta + 2})\ [pqB(t) - c]\ E(t)\ dt \qquad \ldots (3.1)$$

Then the optimal value t* is the solution of

$$-(\delta/t)\ V(t) + V^{H}(t) = (qM/K)\ t^{\delta+2} \sum_{n=0}^{\infty} \lambda_n\ N_n^2(t) \qquad . \ (3.2)$$

Proof. Form (2.1) and (3.1), we write the Hamiltonian as

$$H = (1/t^{\delta+2})\ [pqB\ (t) - c]\ E\ (t) - \sum_{n=0}^{\infty} \{MN_n\ (1 + (N_n/K)\}$$

$$- q \sum_{n=0}^{\infty} \lambda_n(t)\ N_n\ (t)\ E\ (t) \qquad \ldots (3.3)$$

Let H be maximum for some E ε V_t, then (∂ H/∂ E) = 0. So we have

$$\sum_{n=0}^{[t]} \lambda_n(t)\ N_n\ (t) = (1/t^{\delta+2})\ [pq \sum_{n=0}^{[t]} B_n(t) - c] \qquad \ldots (3.4)$$

As the right hand side of (3.4) exists for moderately large but finite values of t, so $\sum_{n=0}^{[t]} \lambda_n(t)\ N_n\ (t)$ exists at least for these values of t. Naturally, $\sum_{n=0}^{[t]} \lambda_n(t)\ N_n^2\ (t)$ exists. Moreover, if $\sum_{n=0}^{\infty} \lambda_n(t)\ N_n\ (t)$ converves, then by Abel's test, $\sum_{n=0}^{\infty} \lambda_n(t)\ N_n^2\ (t)$ converges, as (t) N_n (t) is nomotone and $\sum_{n=0}^{\infty} N_n(t)$ converges. Thus for workable purpose, E may be chosen so that (∂ H/∂ E) = 0.

Differentiating (3.4) with respect to t, we have

$$- (d + 2)\ t^{-(\delta+3)}\ [pqB\ (t) - c] + t^{-(\delta+2)}\ [pq\ \Sigma\ \frac{d}{dt}\ (N_n\ (t))\ w\ (t - n)$$

$$+ N_n\ (t)\ \dot{w}\ (t)] = q\ \Sigma\ (d\lambda_n\ (t)/dt)\ N_n(t) + q\ \Sigma\ \lambda_n\ (t)\ (dN_n\ (t)/dt) \ \ldots (3.5)$$

Again by Pontryagin's principle, $-(\partial H/\partial N_n) = (d\lambda_n/dt)$ ... (3.6)

But $(\partial H/\partial N_n) = -t^{-(\delta+2)}[pqB(t)\lambda_n(t)M[1 + (2N_n/K)] + q\lambda_n(t)E(t)$ ...(3.7)

Using (2.1), (3.6) and (3.7) and simplifying suitably, (3.5) is reduced to

$$-[(\delta+2)/t](pqB(t)-c) - pqM[B(t) + (1/K)\sum_{n=0}^{\infty} N_n^2 w(t-n)$$

$$+ pq\,w(t)\sum_{n=0}^{\infty} N_n(t) = (qM/K)\,t^{(\delta+2)}\sum_{n=0}^{\infty}\lambda_n(t)\,N_n^2(t) \quad \ldots (3.8)$$

where convergence of all the infinite series may be assured as above. However, as we are meaning $\sum\limits_{n=0}^{\infty}$ by $\sum\limits_{n=0}^{[t]}$, $t \geq n$, so there is no difficulty for (3.8) to exist for some t^* which may be large but finite. Using $V(t)$, and $V^H(t)$, (3.8) is reduced to (3.2).

4. DISCUSSION

(i) (1.12) gives the optimal biomass $B^*(t) = \dfrac{p^{-1}\delta c}{M+\delta-(\dot{w}/w)}$ as $K \to \infty$. This coincides with the corresponding result of Clark [5]. Again taking $K \to \infty$ in (1.13), we get the time t' when $B^*(t)$ is maximum and it is given $\dfrac{\dot{w}(t')}{w(t')} = M + \delta$. Thus $t' = t_\delta$ where t_δ is the asymptote to the density independent optimal biomass curve $B^*(t)$ of Clark [5]. This justifies taking $E(t) = 0$, $t < t'$. Thus $t = t'$ gives the mesh size parameter.

(ii) Although for the density dependent Beverton Holt model, with $e^{-\delta t}$ as the depreciation on the observed revenue, it is not possible to obtain the optimal time of harvesting, still Clark could successfully obtain the corresponding results in such a situation simply because $e^{\delta t}$ never occured explicitely in the final calculations.

(iii) Depreciation)$1/t^{\delta+2}$) works even for $\delta = 0$; this may not be true for $e^{-\delta t}$, because (2.4) may not give a convergent integral when $\delta = 0$.

REFERENCES

1. R. J. H. Beverton and S. J. Holt, on the dynamics of exploited fish population, Fish. Invert. London Ser. 2, (1957), 533.
2. L. W. Botsford and D. E. Wickham, Behaviour of age specific, density dependent models and the Northern California Dungeness Crab Fishery, Jour. Fish. Res. Board. Can. 24 (1973), 833-843.
3. L. W. Botsford, Optimal fishery policy for size-specific, density dependent population models, Jour. of Mathematical Biology 12 (1981), 265-293.
4. C. W. Clark, The economics of over exploitation, Science. 181 (1973) 630-634.
5. C. W. Clark, Mathematical Bioeconomics, The optimal management of renewable resources, p. 352, New York, John Wiley and Sons. Inc. 1976, 1989.
6. C. Clark, G. Edwards and M. Friendlander, Beverton, Holt model of a commercial fishery, Optical dynamics. J. Res. Board, Can. 30 (1973), 1629-1640.
7. W. M. Getz, Optimal harvesting of structured populations, Math. Biosci. 44 (1979) 269-291.
8. R. Hanneson, Fishery Dynamics : A North Atlantic Cod fishery, Can. Jour.Econ. 8 (1975), 151-173.
9. J. G. Pope, An investigation into the effects of variable rates of the exploitation of fishery resources, The mathematical theory of the dynamics of biological poplations. pp. 347, New York, Academic Press, (1973).
10. C. J. Walters, a generalised computer simulation model for fish population studies, Trans. Amer. Fish. Soc. 98 (1969) 505-512.

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ON θ-SOMEWHAT OPEN FUNCTIONS

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1. INTRODUCTION

Fomin [3] defined the concept of θ-continuity as follow and investigated several properties of such functions : A function $f : X \to Y$ is said to be θ-continuous if for each $x \in X$ and each open neighbourhood V of f(x), there exist an open neighbourhood U of x such that $f(Cl(U)) \subset Cl(V)$.

Gentry and Hoyle [4] have defined a function which is not local as follow : A function $f : (X, \tau) \to (Y, \sigma)$ is said to be somewhat continuous provided that if $V \in \sigma$ and $f^{-1}(V) \neq \varnothing$, then there is a $U \in \tau$ such that $U \neq \varnothing$ and $U \subset f^{-1}(V)$

Recently, Balaz [1] has defined almost and weak variants of somewhat continuous : A function $f : (X, \tau) \to (Y, \sigma)$ is said to be almost somewhat continuous (resp.weak somewhat continuous) (briefly, a.sw.c. and w.sw.c., respectively) provided that if $V \in \sigma$ and $f^{-1}(V) \neq \varnothing$, then there is a $U \in \tau$ such that $U \neq \varnothing$ and $U \subset f^{-1}(Int(Cl(V)))$ (resp. $U \subset f^{-1}(Cl(V))$).

Quite recently, in [5], θ-somewhat continuous function has been defined : A function $f : (X, \tau) \to (Y, \sigma)$ is said to be θ-somewhat continuous (briefly, θ-sw.c.) provided that if $V \in \sigma$ and $f^{-1}(V) \neq \varnothing$ then there is a $U \in \tau$ such that $U \neq \varnothing$ and $Cl(U) \subset f^{-1}(Cl(V))$.

The purpose of this paper is to introduce 'θ-somewhat open functions' and investigate the several properties of such functions and also compare the other weak forms of somewhat open functions.

Throughout the present paper, (X, τ) and (Y, σ) (or simply X and Y)will denote topological spaces. Let A be a subset of a topological space X. The closure of A and the interior of A are denoted by Cl(A) and Int(A), respectively.

2. θ-SOMEWHAT OPEN FUNCTIONS

Definition 2.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be θ-open if for each $x \in X$ and each neigborhood U of x, there exists a neighborhood V of $f(x)$ in Y such that $Cl(V) \subset f(Cl(U))$.

Definition 2.2. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be θ-somewhat open (resp. somewhat open [4], almost somewhat open, weak somewhat open), (briefly, θ-sw.o., sw.o., a.sw.o., w.sw.o., respectively), provided that if $U \in \tau$ and $U \neq \varnothing$ then there exists a $V \in \sigma$ such that $V \neq \varnothing$ and $Cl(V) \subset f(Cl(U))$ (resp. $V \subset f(U)$, $V \subset f(Int(Cl(U)))$, $V \subset f(Cl(U))$).

DIAGRAM

$$\begin{tikzcd}
\text{Open} \arrow[r, Rightarrow] \arrow[d, shift right, "/" marking] & \text{Somewhat open} \arrow[r, Rightarrow] \arrow[d, shift right, "/" marking] & \text{Almost somewhat open} \arrow[d, shift right, "/" marking] \\
\theta\text{-Open} \arrow[r, Rightarrow] \arrow[u, shift right, "/" marking] & \theta\text{-Somewhat open} \arrow[r, Rightarrow] \arrow[u, shift right, "/" marking] & \text{Weakly somewhat open} \arrow[u, shift right, "/" marking]
\end{tikzcd}$$

We conclude from the diagram above the follwoing implications due to examples 2.1-2.5 (below) :

(1) θ-openness implies θ-sw.o., but not conversely (Example 2.1)

(2) θ-sw.o. implies w.sw.o., but not conversely (Example 2.2, also 2.4)

(3) θ-sw.o. and openness are independent (Example 2.1, 2.4)

(4) θ-sw.o. and sw.o. are independent (Example 2.1, 2.2)

(5) θ-sw.o. and a.sw.o. are independent (Example 2.1, 2.2)

(6) θ-openness and openness are independent (Example 2.3, 2.4)

(7) θ-openness and sw.o. are independent (Example 2.2, 2.3).

Example 2.1. Let $X = \{a,b,c\}$ $\tau = \{X, \varnothing, \{b\}, \{c\}, \{b,c\}\}$ and $\sigma = \{X, \varnothing, \{a\}, \{b,c\}\}$. Let $f : (X,\tau) \to (X,\sigma)$ be the identity function. Then f is θ-sw.o., but it is neither θ-open, sw.o., open nor a.sw.o.

Example 2.2. Let $X = \{a,b,c\}$ $\tau = \{X, \varnothing, \{a\}, \{b,c\}\}$ and $\sigma = \{X, \varnothing, \{a\}, \{c\}, \{a,c\}\}$. Let $f : (X,\tau) \to (X,\sigma)$ be the identity. Then f is sw.o., but it is neither θ-sw.o., nor open.

Example 2.3. Let $X = \{a,b,c\}$ $\tau = \{X, \varnothing, \{a\}, \{b,c\}\}$ and $\sigma = \{X, \varnothing, \{a\}, \{c\}, \{a,c\}\}$. Let $f : (X,\tau) \to (X,\sigma)$ be the identity. Then $f^{-1} : (X,\sigma) \to (X,\tau)$ is θ-open, but it is neither sw.o. nor open.

Example 2.4. Let $X = \{a,b\}$, $\tau = \{X, \varnothing, \{a\}\}$. Let $Y = \{x,y,z\}$, $\sigma = \{Y, \varnothing, \{x\}, \{y\}, \{x,y\}\}$. Define $f : (Y,\tau) \to (Y,\sigma)$ by $f(a) = x$ and $f(b) = y$. Then f is open, but it is neither θ-sw.o. nor θ-open.

Example 2.5. Let $X = \{a,b,c\}$, $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a,c\}\}$ and $\sigma = \{X, \varnothing, \{a\}, \{b\}, \{a,b\}\}$. Define $f : (X,\tau) \to (X,\sigma)$ by $f(a) = c$, $f(b) = a$, $f(c) = b$. Then f is a.sw.o., but not sw.o.

Definition 2.3. [2]. A topological space X is said to be ;

(a) Quasi regular if for any nonempty open set $G \subset X$, there exist a nonempty open set D such that $Cl(D) \subset G$,

(b) Semi quesi regular if for any nonempty open set $G \subset X$, there exist a nonempty regular open set A such that $A \subset G$,

(c) Almost quasi regular if for any nonempty regular open set $A \subset X$, there exist a nonempty open set D such that $Cl(D) \subset A$.

Remark. A space X is quasi regular if and only if it is both semi quasi regular and almost quasi regular.

Theorem 2.1. Let $f : X \to Y$ be a function. Then the following statements hold :

(a) If Y is quasi regular space, then f is θ-sw.o. if and only if it is w.sw o.

(b) If X is almost quasi regular space, then f is a.sw.o. if and only if it is w.sw.o.

(c) If X is semi quasi regular space, then f is sw.o. if and only if it is a.sw.o.

Corollary 2.1. **(a)** If X is quasi regular space, then the following three concepts are equivalent :

(1) sw.o., **(2)** a.sw.o., **(3)** w.sw.o.

(b) If X and Y are quasi regular space, then the followwing four properties are equivalent :

(1) θ-sw.o., **(2)** sw.o., **(3)** a.sw.o., **(4)** w.sw.o.

3. PROPERTIES OF θ-SOMEWHAT OPEN FUNCTIONS.

Theorem 3.1. If $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\varphi)$ are θ-sw.o., then $gof : (X,\tau) \to (Z,\varphi)$ is θ-sw.o.

Proof. Since f is θ-sw.o., for each nonempty $U \in \tau$, there exists a nonempty $V \in \sigma$ such that $Cl(V) \subset f(Cl(U))$. Since g is also θ-sw.o., there exists a nonempty $W \in \varphi$ such that $Cl(W) \subset g(Cl(V))$. Thus $Cl(W) \subset g(Cl(V)) \subset g(f(Cl(U))) = (gof)(Cl(U))$.

Theorem 3.2 If $f : (X,\tau) \to (Y,\sigma)$ is θ-sw.o. and A is an openclosed subset of X, then the restriction $f|_A : (A, \tau_A) \to (Y, \sigma)$ is θ-sw.o.

Proof. For each nonepty $U_A \in \tau_A$, there exists $U \in \tau$ such that $U_A = U \cap A$. Since A is open in X, then $U_A \in \tau$. Since f is θ-sw.o., there exists nonempty $V \in \sigma$ such that $Cl(V) \subset f(Cl(U_A))$. Since $U_A \subset A$ and A is closed in X, we have $Cl(U_A) \subset Cl(A) = A$. Hence $Cl_A(U_A) = Cl(U_A) \cap A = Cl(U_A)$. Therefore, $Cl(V) \subset f(Cl_A(U_A)) = (f|_A)(Cl_A(U_A))$.

Remark. In Theorem 3.2, we can not remove the condition closed on A. In Example 2.1, take $A = \{b,c\}$, then $\tau_A = \{A, \emptyset, \{b\}, \{c\}\}$, $f|_A(Cl_A\{b\})) = f|_A(\{b\}) = \{b\}$, but $Cl(V) \not\subset \{b\}$ for each nonempty set $V \in \sigma$.

Theorem 3.3. If (X,τ) and (Y,σ) are topological spaces and A is dense in X and $f : (A, \tau_A) \to (Y, \sigma)$ is θ-sw.o., then any extension $F : (X, \tau) \to (Y, \sigma)$ is θ-sw.o.

Proof. For each nonempty $U \in \tau$, since A is dense in X, $\emptyset \neq U \cap A \in \tau_A$. Since f is θ-sw.o., there exists $V \in \sigma$ such that $Cl(V) \subset f(Cl_A(U \cap A))$. Hence $f(Cl_A(U \cap A)) = f(Cl(U \cap A) \cap A) = F(Cl(U \cap A) \cap A) \subset F(Cl(U))$. Thus, $Cl(V) \subset F(Cl(U))$. This shows that F is θ-sw.o.

Theorem 3.4. If (X, τ) and (Y, σ) are topological spaces and $X = A \cup B$ where A and B are subsets of X and $f : (X, \tau) \to (Y, \sigma)$ is a function such that the restrictions $f|_A$ and $f|_B$ are θ-sw.o., then f is θ-sw.o.

Proof. For each nonempty open set U in X, $U = X \cap U = (A \cup B) \cap U = (A \cap U) \cup (B \cap U)$. Suppose that $A \cap U \neq \emptyset$. Since $f|_A$ is θ-sw.o., for nonempty $A \cap U \in \tau_A$, there exist nonempty $V \in \sigma$ such that $Cl(V) \subset (f|_A)(Cl_A(A \cap U))$, but $(f|_A)(Cl_A(A \cap U)) = f(Cl(U \cap A) \cap A) \subset f(Cl(U))$. Thus $Cl(V) \subset f(Cl(U))$. If $B \cap U \neq \emptyset$, by using that $f|_B$ is θ-sw.o., we can similarly prove that f is θ-sw.o.

4. D-SPACES

Definition 4.1. A topological space X is said to be a D-space [8] if every nonempty open subset of X is dense in X.

Remark. 4.1. (1) By f^{-1} in Examples 2.1 and 2.2, θ-somewhat continuity and somewhat continuity are independent.

(2) In [4, Theorem 19], it is known that D-spaces are preserved somewhat continuous surjections.

Theorem 4.1. If X is a D-space and $f : X \to Y$ is a θ-somewhat continuous surjection, then Y is a D-space.

Proof. Since f is surjective, for each nonempty open set V in Y, $f^{-1}(V) \neq \emptyset$ Since f is θ-somewhat continuous, there exists a nonempty open set U in X such that $Cl(U) \subset f^{-1}(Cl(V)$. Since X is D-space $Cl(U) = X$. Hence $X \subset f^{-1}(Cl(V)$ and $Y = f(X) \subset f(f^{-1}(Cl(V)) = Cl(V)$. Therefore, Y is D-space.

Theorem 4.2. If Y is D-space and $f : X \to Y$ is a θ-sw.o. injection, then X is D-space.

Proof. Since f is injective, for each nonempty open set U in X, $f(U) \neq \emptyset$. since f is θ-sw.o., there exists a nonempty open set V in Y such that $Cl(V) \subset f(Cl(U))$. Since Y is D-space, $Cl(V) = Y$. Hence $Y \subset f(Cl(U))$. Since f is injective, $X = Cl(U) = f^{-1}(f(Cl(U)))$. Then, X is D-space.

Definition 4.2. A function $f : X \to Y$ is said to be almost feebly continuous [10] if, for every nonempty regular open set V of Y, $f^{-1}(V) \neq \emptyset$ implies $sInt(f^{-1}(V)) \neq \emptyset$ (where sInt is semi-interior).

Lemma 4.1. A function $f : X \to Y$ is almost feebly continuous if and only if it is almost somewhat continuous.

Proof. Necessity. Suppose that f is almost feebly continuous. For every nonempty regular open set V in Y, $f^{-1}(V) \neq \emptyset$ implies $sInt(f^{-1}(V)) \neq \emptyset$ Lemma 4 in [9] says that if $\emptyset \neq A \in SO(X)$, then $Int(A) \neq \emptyset$. Therefore, we have $\emptyset \neq Int(sInt(f^{-1}(V))) \subset Int(f^{-1}(V))$.

Sufficiency. Suppose that f is almost somwwhat continuous. For every nonempty regular open set V in Y, $f^{-1}(V) \neq \emptyset$ implies $Int(f^{-1}(V)) \neq \emptyset$. Then, we have $sInt(f^{-1}(V)) \supset Int(f^{-1}(V)) = Int(f^{-1}(Int(Cl(V)))) \neq \emptyset$.

Corollary 4.1. D-spaces are preserved under almost somewhat continuous surjections.

Proof. It is showed in [10, Theorem 4.4] that D-spaces are preserved under almost feebly continuous surjections.

Definition 4.3. A space X is θ-irreducible [7] if every pair of nonempty regular closed subsets of X has a nonempty intersection.

Remark 4.2. Every D-space is θ-irreducible, but the converse is false. The space (X,σ) in Example 4.1 (below) is θ-irreducible but not a D-space.

Theorem 4.3. If X is a D-space and $f : X \to Y$ is a weakly somewhat continuous surjection, then Y is θ-irreducible.

Proof. Suppose that Y is not θ-irreducible. Then there exist nonempty open subsets U and V of Y such that $Cl(U) \cap Cl(V) = \emptyset$. Then,

$$\emptyset \neq (f^{-1}(Cl(U))) \cap (f^{-1}(Cl(V))) \supset Int((f^{-1}(Cl(U))) \cap (f^{-1}(Cl(V)))).$$

Since f is a weakly somewhat continuous surjection, $f^{-1}(U) \neq \emptyset$ and $f^{-1}(V) \neq \emptyset$ implies $f^{-1}(Cl(U))) \neq \emptyset$ and $Int(f^{-1}(Cl(V))) \neq \emptyset$ respectively. Thus X is not D-space.

The following example shows that in the previous Theorem Y being θ-irreducible can not be replaced by a D-space.

Example 4.1. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{c\}, \{a, c\}, \{b, c\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then (X, τ) is a D-space and f is a weakly somewhat continuous surjection. However, (Y, σ) is not a D-space.

Theorem 4.4. If Y is a D-space and $f : X \to Y$ is a w.sw.o. injection, then X is θ-irreducible.

Proof. Suppose that X is not θ-irreducible. Then, there exists nonempty open subsets U and V of X such that $Cl(U) \cap Cl(V) = \emptyset$. Then $f(Cl(U)) \cap f(Cl(V)) = \emptyset$. Since f is w.sw.o., there exist nonempty open subsets G, H of Y such that $G \subset f(Cl(V))$ and $H \subset f(Cl(V))$. Hence $G \cap H = \emptyset$. Thus Y is not D-space.

Theorem 4.5. If X is a θ-irreducible space and $f : X \to Y$ is a θ-somewhat continuous surjection, then Y is θ-irreducible.

Proof. Suppose that Y is not θ-irreducible. There exist nonempty open sets U and V of Y such that $Cl(U) \cap Cl(V) = \emptyset$. Since f is a θ-somewhat continuous surjection, there exist nonempty open sets G and H of X such that $Cl(G) \subset f^{-1}(ClU))$ and $Cl(H) \subset f^{-1}(Cl(V))$. Therefore, we obtain $Cl(G) \cap Cl(H) \subset f^{-1}(Cl(U) \cap Cl(V)) = \emptyset$ and hence X is not θ-irreducible.

Theorem 4.6. If Y is a θ-irreducible space and $f : X \rightarrow Y$ is a θ-sw.o. injection, then Y is θ-irreducible.

Proof. Suppose that X is not θ-irreducible. Then, there exist nonempty open sets U and V of X such that $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \varnothing$. Since f is θ-sw.o., there exist nonempty open sets G and H of Y such that $\mathrm{Cl}(G) \subset f(\mathrm{Cl}(U))$ and $\mathrm{Cl}(H) \subset f(\mathrm{Cl}(V))$ Since f is injective, $\mathrm{Cl}(G) \cap \mathrm{Cl}(H) = \varnothing$ and hence Y is not θ-irreducible.

5. θ-OPEN FUNCTIONS

By the Examples 2.2 and 2.3, θ-openness and somewhat openness are independent. We obtain a result analogous to [4, theorem 20]. A topological space (X, τ) is said to be locally compact on a dense subset of X if there exist a dense subset D of X such that for each $p \in D$ there exists $U \in \tau$ containing p such that $\mathrm{Cl}(U)$ is compact.

Theorem 5.1. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a continuous, θ-open surjection. If (X, τ) is locally compact on a dense subset of X, then (Y, σ) is locally compact on a dense subset of Y.

Proof. Let (X, τ) be locally compact on a dense subset of X. Then there exists a dense subset D of X such that for each $x \in D$ there exists an open subset U of X containing x such that $\mathrm{Cl}(U)$ is compact. Since f is continuous surjection and $\mathrm{Cl}(D) = X$, $Y = f(X) = f(\mathrm{Cl}(D)) \subset \mathrm{Cl}(f(D))$. This shows that $f(D)$ is dense in Y.

Now, for every $y \in f(D)$, there exist $x \in D$ so that $f(x) = y$. By the hypothesis, there exists an open neighbourhood U of x such that $\mathrm{Cl}(U)$ compact. Since f is continuous, $f(\mathrm{Cl}(U))$ is compact. By the θ-openness of f, there exists an open neighbourhood V of $f(x) = y$ such that $\mathrm{Cl}(V) \subset f(\mathrm{Cl}(U))$. Here $\mathrm{Cl}(V)$ is closed in Y and then $\mathrm{Cl}(V)$ is compact. Thus, Y is locally compact on a dense subset $f(D)$ of Y.

Definition 5.1. Let G be a group and $\chi(G)$ be the set of open neighbourhoods of the identity e in G. Then, (G, τ) is said to be a θ-topological group [6] if the following conditions are satisfied :

(1) If $U \in \tau$ and $g \in G$, then both gU and Ug are open,

(2) For each $V \in \chi(G)$, there exists a $U \in \chi(G)$ so that $\mathrm{Cl}(U).\,\mathrm{Cl}(U) \subset \mathrm{Cl}(V)$,

(3) For each $V \in \chi(G)$, there exists a $U \in \chi(G)$ so that $i(\mathrm{Cl}(U)) \subset \mathrm{Cl}(V)$, where i denotes the inversion $i : G \rightarrow G$.

Theorem 5.2. Let $(X, \tau, \bullet)$ and (Y, σ, o) be θ-topological groups. If $f : (X, \tau, \bullet) \rightarrow (Y, \sigma, o)$ is a θ-somewhat open homomorphism, then f is θ-open.

Proof. Let e be the identity element of $(X, \bullet)$ and let e' be the identity element of (Y, o).

First, we show that for every open neighbourhood U of e, there exist an open neighbourhood V of $f(e) = e'$ such that $Cl(V) \subset f(Cl(U))$. Let U be a τ-open neighbourhood of e. Since $(X, \tau, \bullet)$ is θ-topological group, there exists a τ-open neighbourhood W of e such that $Cl(W) \bullet Cl(W)^{-1} \subset Cl(U)$. Since f is θ-somewhat open, there exists a nonempty $V \in \sigma$ such that $Cl(V) \subset f(Cl(W))$. Hence, there exist $y \in V$ and $x \in Cl(W)$ such that $f(x) = y$. Then $e' \in Voy^{-1} \in \sigma$. Since left and right translations are homeomorphism,

$$Cl(Voy^{-1}) = Cl(V)oy^{-1} \subset f(Cl(W))oy^{-1} \subset f(Cl(W)) \text{ of } (Cl(W)^{-1})$$

$$= f(Cl(W) \bullet Cl(W)^{-1}) \subset f(Cl(U)).$$

Secondly, we must show that f is θ-open. For every $x \in X$ and every open neighbourhood U of x, $U \bullet x^{-1}$ is an open neighbourhood of e. By the first result, there exists an open neighbourhood V of e' such that $Cl(V) \subset f(Cl(U \bullet x^{-1}))$. Then $V_0 f(x)$ is an open neighbourhood of f (x) and we obtain

$$Cl(Vof(x)) = Cl(V) \text{ o } f(x) \subset f(Cl(U \bullet x^{-1}) \text{ o } f(x) = f(Cl(U) \bullet x^{-1}) \text{ o } f(x)$$

$$\subset f(Cl(U)) \text{ o } f(x^{-1}) \text{ o } f(x) = f(Cl(U)).$$

This shows that f is θ-open.

REFERENCES

1. V. Balaz, On almost and weak forms of continuity of functions and multifunctions, Math. Slovaca 37 (1987), 53-61.
2. V. Balaz, Remark on almost and weak forms continuity of functions, 1989, (Submitted).
3. S. V. Fomin, Extension of topological spaces, Ann. of Math. 44 (1943), 471-480.
4. K. R. Gentry and H. B. Hoyle 111, Somewhat continuous functions, Czechoslovak Math. J. 21 (69) (1971), 5-12.
5. E. Hatir, The generalization of topological groups, ph. D. Thesis, University of Selçuk, Konya-Turkey, (1993).
6. E. Hatir and E. J. Beggs, Theta topological groups, Turkish J. Math. 18 (1994), 193-201.

7. T. Noiri, E. Hatir and Ş. Yüksel, Almost Semistrongly θ-continuous Functions, Res. Rep. Yatsushiro Nat. Coll. Tech. No. 17 (1994).

8. N. Levine, Dense topologies, Amer. Math. Monthly 75 (1968), 847-852.

9. T. Noiri, On semi-continuous mappings, Atti Accad Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 54 (1973), 210-214.

10. T. Noiri, Properties of Hyperconnected spaces, Acta Math. Hungar. 66(1995), 125-132.

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THE JACOBSON RELATIONAL RADICAL ON HEMIRINGS WITH INVOLUTION

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ABSTRACT : In this paper we give a useful characterisation for the "Jacobson relational radical" (defined by Lide Li) on a hemiring and obtain the analogues concerning quasi-regularity We apply these ideas to the study of hemirings with involution and obtain that for any semisimple hemiring R with an involution *, $D/\overline{\Delta}$ is isomorphic to the subdirect product of "*-primitive" hemirings, where $\overline{\Delta}$ is the least additively cancellative congruence on R.

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1. INTRODUCTION

A semiring is an algebraic system (S, +, •) such that (S, +) and (S, •) are semigroups and in which ring-like distributive laws hold. A hemiring is an additively commutative semiring with an additive neutral element zero (denoted by 0), which is also a multiplicative zero. A semimodule over a hemiring is defined in a natural way. But here irreducible semimodules (to be defined later) are not in general represented by ideals or even special type of ideals, as in the case of rings. However, it is possible to obtain a representation in terms of a special class of equivalence relations. In [3] Lide Li introduced the concept of a relational radical on a hemiring and obtain some interesting results analogous to the theory of rings.

Following his definition of the relational radical on a hemiring we obtain another useful characterisation for it (§ 1) and develop the analogues concerning quasi-regularity (§ 2), which certiainly remove the possibility of thinking about distinct right and left relational radicals on a hemiring, just as in the case of rings. In the final section (§ 3) we study some properties of hemirings with involution. We show that, if R is a semisimple hemiring with an involution *, then $R/\overline{\Delta}$ is isomorphic to the subdirect product of "*- primitive" hemirings, where $\overline{\Delta} = \{(x, y) \in R \times R : x + z = y + z$, for some $z \in R\}$ is the least additively cancellative congruence on R. We also establish the relation between the "*-primitive" hemirings and primitive hemirings.

1. PRELIMINARIES

Let R be a hemiring. A (right) R-semimodule M is a commutative additive semipgroup which has a zero element, together with a mapping from $M \times R$ into M (sending (m, r) to mr) such that,

(i) $m(r_1 + r_2) = mr_1 + mr_2$

(ii) $(m_1 + m_2)r = m_1r + m_2r$

(iii) $(mr_1)\, r_2 = m(r_1r_2)$

(iv) $m0 = 0r = 0$

for all m_i, $m \in M$ and r_i, $r \in R$, $i = 1, 2$. An equivalence relation σ on M is called right R-linear relation if for all x_i, $y_i \in M$, $i = 1, 2$, $r \in R$,

(i) $(x_1, y_1), (x_2, y_2) \in \sigma$ implies $(x_1 + x_2, y_1 + y_2) \in \sigma$ and

(ii) $(x, y) \in \sigma$ implies $(xr, yr) \in \sigma$.

A right R-linear relation σ on M is called cancellative if for all $x, y, z \in M$, $(x + z, y + z) \in \sigma$ implies $(x, y) \in \sigma$. Clearly R itself is a (right) R-semimodule. An equivalence relation σ on R is said to be right regular [3] if there is a pair $e_1, e_2 \in R$, $(e_1, e_2) \notin \sigma$ such that $(x + e_1 x, e_2 x) \in \sigma$ for all $x \in R$. A right regular relation is called right w-relation [3] if it is maximal in the set of all cancellative right R-linear relations on R. For any right w-relation σ on R,

$$(\sigma : R) = \{(x, y) \in R \times R : (ax, ay) \in \sigma \text{ for all } a \in R\}.$$

Let Σ denote the set of all right w-relations on R. Lide Li showed that

$$\bigcap_{\sigma\in\Sigma} (\sigma : R) = \bigcap_{\sigma\in\Sigma} \sigma$$

and he called it the (right) Jacobson relational radical (J_{rr}) of R. If $\Sigma = \Phi$, then $J_{rr} = R \times R$. As $(\sigma : R)$ is a cancellative (two-sided) R-linear relation for each $\sigma \in \Sigma$, J_n is also so. A (right) R-semimodule M is called additively cancellative if for any x, y,

$z \in M$, $x + z = y + z$ implies $x = y$. An additively cancellative (right) R-semimodule M with $M \neq \{0\}$ is called irreducible if for an arbitrarily fixed pair of $u_1, u_2 \in M$ with $u_1 \neq u_2$ and any $x \in M$, there exist $a_1, a_2 \in R$ such that $x + u_1a_1 + u_2a_2 = u_1a_2 + u_2a_1$. Let I be the set of all irreducible (right) R-semimodules. For any $M \in I$, $\sigma_M = \{(x, y) \in R \times R : mx = my \text{ for all } m \in M\}$ and for any $o \neq m \in M$,

$\sigma_m = \{(x, y) \in R \times R : mx = my\}$. We have $\sigma_m \in \Sigma$ [3].

Let M be a (right) R-semimodule and S be a subsemimodule of M. Let $\overline{S} = \{m \in M : m + x = y, x, y \in S\}$. Any subsemimodule S of M is called closed [2] if $\overline{S} = S$ and $\overline{S}$ is called the k-closure of S. It is easy to verify that if M is irreducible, then it has no closed subsemimodules other than $\{0\}$ and M itself.

Lemma 1.1. For any $M \in I$, $\sigma_M = (\sigma_m : R)$ for some $0 \neq m \in M$.

Proof. As $M \in I$, $MR \neq \{0\}$. Thus there exists $0 \neq m \in M$ such that $mR \neq \{0\}$. Let $(x, y) \in \sigma_M$ and $a \in R$. Then $max = may$. This implies $(ax, ay) \in \sigma_m$. Thus $(x, y) \in (\sigma_m : R)$. Conversely let $(x, y) \in (\sigma_m : R)$. Now $\overline{mR}$ is a closed subsemimodule of M and $mR \neq \{0\}$. Thus $\overline{mR} = M$ as M is irreducible. Let $u \in M$. Then there are $r_1, r_2 \in R$ such that $u + mr_1 = mr_2$. Also since $(x, y) \in (\sigma_m : R)$, $(r_1x, r_1y), (r_2x, r_2y) \in \sigma_m$ and so $mr_ix = mr_iy$, $i = 1,2$. This implies $ux = uy$ as M is additively cancellative. Thus $(x,y) \in \sigma_M$.

Remark 1.2. Let $M \in I$ and $m \in M$ be such that $\sigma_M = (\sigma_m : R)$. Lide Li [3] showed that R/σ_m is isomorphic to mR which is a subsemimodule of M and we have just seen that $\overline{mR} = M$. Thus any irreducible (right) R-semimodule M can be represented by σ_m for some $0 \neq m \in M$, as a k-closure of R/σ_m.

Henceforth we denote any σ-class of $r \in R$ by $[r]_\sigma$, for any equivalence relation σ on R and we write simply [r] where there is no chance of confusion. We know that for any $\sigma \in \Sigma$, R/σ is an irreducible (right) R-semimodule with the definition $[r]a = [ra]$ for all $r, a \in R$ [3].

Lemma 1.3. For any $\sigma \in \Sigma$, $(\sigma : R) = \sigma_{R/\sigma}$.

Proof. $(x, y) \in (\sigma : R)$

$\Leftrightarrow (rx, ry) \in \sigma$ for all $r \in R$

$\Leftrightarrow [rx] = [ry]$ for all $r \in R$

$\Leftrightarrow [r]x = [r]y$ for all $r \in R$

$\Leftrightarrow (x, y) = \sigma_{R/\sigma}$.

These two lemmas lead to another characterisation of J_{rr}, namely,

Theorem 1.4. If $I \neq \Phi$, $J_{lr} = \bigcap_{M \in I} \sigma_M$.

2. QUASI REGULARITY

Let R be a hemiring.

Definition 2.1. A pair of elements (x_1, x_2) in R is said to be right quasi-regular if there are a_1, a_2, t in R such that,

$$x_1 + a_1 + x_1a_1 + x_2a_2 + t = x_2 + a_2 + x_1a_2 + x_2a_1 + t.$$

Left quasi-regularity is defined similarly.

Definition 2.2. A cancellative right (left) R-linear relation σ on R is said to be right (left) quasi-regular relation if $(x,y) \in \sigma$ implies (x,y) is right (left) quasi-regular.

Simple application of Zorn's lemma yields the following :

Lemma 2.3. If ρ is a proper cancellative right regular relation on R, then there exists a right w-relation σ on R, which contains ρ.

Lemma 2.4. J_{ll} is a right quasi-regular relation.

Proof. Let $(x, y) \in J_{lr}$.

Let $\rho = \{(u, v) \in R \times R : u + a_1 + xa_1 + ya_2 + t = v + a_2 + xa_2 + ya_1 + t\ ;\ a_1, a_2, t \in R\}$. Clearly ρ is a cancellative right R-linear relation. Also $(r + xr, yr) \in \rho$ for all $r \in R$ with $a_1 = t = 0$ and $a_2 = r$. Now if $\rho \neq R \times R$, then $(x, y) \notin \rho$. This imples ρ is a proper cancellative right regular relation on R. Then, by lemma 2.3, there is a right w-relation σ on R contiaining ρ. But $(x, y) \in \sigma$ as $(x, y) \in J_{rr}$. This implies $\sigma = R \times R$, which is a contradiction. Thus $\rho = R \times R$. Then $(x, y) \in \rho$ and so (x, y) is right quasi-regular.

Let $\Delta = \{(x, x) : x \in M\}$, where M is a (right) R-semimodule. Iizuka [2] observed that for any irreducible (right) R-semimodule M, the only cancellative right R-linear relations are Δ and $M \times M$ itself.

Lemma 2.5. If ρ is a right quasi-regular relation on R, then $\rho \subseteq J_{rr}$.

Proof. Let, if possible, $\rho \not\subseteq J_{rr}$. Then, by theorem 1.3, $\rho \not\subseteq \sigma_M$ for some irreducible (right) R-semimodule M. This implies $\rho \not\subseteq \sigma_m$ for some $0 \neq m \in M$. Let

$$\gamma = \{(m_1, m_2) \in M \times M : m_1 + my = m_2 + mx,\ (x, y) \in \rho\}$$

Clearly γ is a cancellative right R-linear relation on M. Also as $\rho \not\subseteq \sigma_m$. $\gamma \neq \Delta$. This implies $\gamma = M \times M$, as M is irreducible. Then $(0, m) \in \gamma$ and so $my = m + mx$ for some $(x, y) \in \rho$. Since ρ is right quasi-regular, there are a_1, a_2, $t \in R$ such that,

$$x + a_1 + xa_1 + ya_2 + t = y + a_2 + xa_2 + ya_1 + t.$$

This with $my = m + mx$ gives $m = 0$, as M is additively cancellative, which is a contradiction.

We summarize the above lemmas in the following theorem.

Theorem 2.6. J_{rr} is a right quasi-regular relation on R and contains all right quasi-regular relations on R, that is, J_{rr} is the unique maximal right quasi-regular relation on R.

To show that J_{rr} is a left quasi-regular relation on R, we need the following lemma.

Lemma 2.7. If a pair of elements (x, y) in R is both left and right quasi-regular, i.e. if there are a_i, b_i , $t_i \in R$, $i = 1, 2$ such that

$$x + a_1 + xa_1 + ya_2 + t_1 = y + a_2 + xa_2 + ya_1 + t_1 \qquad \ldots (1)$$

and

$$x + b_1 + b_1x + b_2y + t_2 = y + b_2 + b_2x + b_1y + t_2 \qquad \ldots (2)$$

then $a_1 + b_2 + z = a_2 + b_1 + z$ for some $z \in R$.

Proof. Multiplying (1) by b_i from the left and (2) by a_i from the right, $i = 1, 2$ we have the following :

$$b_1x + b_1a_1 + b_1xa_1 + b_1ya_2 + b_1t_1 = b_1y + b_1a_2 + b_1xa_2 + b_1ya_1 + b_1t_1$$

$$b_2y + b_2a_2 + b_2xa_2 + b_2ya_1 + b_2t_1 = b_2x + b_2a_1 + b_2xa_1 + b_2ya_2 + b_2t_1$$

$$ya_1 + b_2a_1 + b_2xa_1 + b_1ya_1 + t_2a_1 = xa_1 + b_1a_1 + b_1xa_1 + b_2ya_1 + t_2a_1$$

$$xa_2 + b_1a_2 + b_1xa_2 + b_2ya_2 + t_2a_2 = ya_2 + b_2a_2 + b_2xa_2 + b_1ya_2 + t_2a_2$$

Adding the above equations we get,

$$b_1x + b_2y + ya_1 + xa_2 + t = b_1y + b_2x + xa_1 + ya_2 + t$$

$t \in R$. This with two given equations proves the lemma.

Theorem 2.3. J_{rr} is a left quasi-regular relation on R.

Proof. Let $(x, y) \in J_{rr}$. By right quasi-regularity of J_{rr} we have a_1, a_2, $t \in R$ such that $x + a_1 + xa_1 + ya_2 + t = y + a_2 + xa_2 + ya_1 + t$. Since $(x, y) \in J_{rr}$ and J_{rr} is a cancellative R-linear relation on R, $(a_1, a_2) \in J_{rr}$. Thus there are b_1, b_2, $r \in R$ such that,

$$a_1 + b_1 + a_1b_1 + a_2b_2 + r = a_2 + b_2 + a_1b_2 + a_2b_1 + r.$$

Then (a_1, a_2) is both left and right quasi-regular pair of elements in R. Therefore by Lemma 2.7. $x + b_2 + z = y + b_1 + z$ for some $z \in R$. Now routine calculations show that

$$x_1 + a_1 + a_1x + a_2y + s = y + a_2 + a_2x + a_1y + s,$$

where $$s = b_1 + a_1b_1 + a_2b_2 + a_1z + a_2z + r + z,$$

which implies (x, y) is left quasi-regular and this proves the theorem.

Remark 2.9. The above theorem and the left analogue of theorem 2.6 show that J_{rr} is contained in the (left) Jacobson relational radical. Similarly the (left) Jacobson relational radical is contained in J_{rr} and hence they are equal. This enables us to say that, the intersection of all right w-relations on a hemiring is equal to the intersection of all left w-relations on it.

3. HEMIRINGS WITH INVOLUTION

In this section, we discuss some properties for hemirings with involution analogous to some of the interesting results studied in [1].

Definition 3.1. An involution * of a hemirng R is a noe-to-one mapping of R onto iteslf such that,

$$\begin{aligned} &\text{(i)} \quad (x + y)^* &&= x^* + y^* \\ &\text{(ii)} \quad (xy)^* &&= y^* x^* \\ &\text{(iii)} \quad (x^*)^* &&= x \end{aligned}$$

for all $x, y \in R$.

Example 3.2. Let M_2 be the ring of all 2 × 2 matrices over the ring of integers. Let R be the subset of M_2 consisting of elements of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ where $a, c \geqslant 0$.

Then R is a hemiring with an involution $*$, defined by $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^* = \begin{pmatrix} c & b \\ 0 & a \end{pmatrix}$.

Definition 3.3. Let M be a (right) R-semimodule. $\sigma_{M*} = \{(x, y) \in R \times R : mx = my \text{ and } mx^* = my^* \text{ for all } m \in M\}$.

Definition 3.4. A (right) R-semimodule is called $*$-faithful if $\sigma_{M*} = \overline{\Delta}$.

Defineiton 3.5. A hemiring R is called $*$-primitive if there exists an irreducible (right) R-semimodule which is $*$-faithful.

Definition 3.6. A cancellative R-linear relation σ on R is called $*$-relation on R if $\sigma^* = \sigma$, where $\sigma^* = \{x^*, y^*) : (x, y) \in \sigma\}$.

If σ is a $*$-relation on R, then the involution induced in R/σ will still be denoted by $*$, i. e., $[r]_\sigma^* = [r^*]_\sigma$ for all $r \in R$.

Definition 3.7. A cancellative R-linear relation is called $*$-primitive if it is a $*$-relation on R and R/σ is $*$-primitive.

Clearly σ_{M^*} is a $*$-relation on R and by remark 2.9 J_{rr} is also a $*$-relation on R. Let I denote the set of all irreducible (right) R-semimodule.

Lemma 3.8. If $I \neq \Phi$, then $J_{rr} = \bigcap_{M \in I} \sigma_{M^*}$.

Proof. Let $J_* = \bigcap_{M \in I} \sigma_{M^*}$. Clearly $J_* \subseteq J_{rr}$. Let $(x, y) \in J_{rr}$. Then $(x^*, y^*) \in J_{rr}$, as J_{rr} is a $*$-relation on R. Thus for any $M \in I$, $(x, y), (x^*, y^*) \in \sigma_M$. This implies $(x, y) \in \sigma_{M^*}$. Therefore $(x, y) \in J_*$. Hence $J_{rr} = J_*$.

Lemma 3.9. σ_* is a $*$-primitive relation on R iff $\sigma_* = \sigma_{M^*}$ for some $M \in I$.

Proof. Let σ_* be a $*$-primitive relation on R. Then R/σ_* is a $*$-primitive hemiring with $*$-faithful irreducible (right) R/σ_* – semimodule M. By defining $mr = m\,[r]$, $m \in M$, $r \in R$, M becomes an irreducible (right) R-semimodule. Let $(x, y) \in \sigma_*$, then $(x^*, y^*) \in \sigma_*$ as σ_* is a $*$-relation on R. Thus $mx = m\,[x] = m\,[y] = my$ and $mx^* = m[x^*] = m\,[y^*] = my^*$ for all $m \in M$. This implies $(x, y) \in \sigma_{M^*}$. Again let $(x, y) \in \sigma_{M^*}$. Then $mx = my$ and $mx^* = my^*$ for all $m \in M$. This implies $m\,[x] = m\,[y]$ and $m\,[x]^* = m\,[x^*] = m\,[y^*] = m\,[y]^*$ for all $m \in M$. Therefore $[x] = [y]$ as M is a $*$-faithful (right) R/σ_* – semimodule. Thus $(x, y) \in \sigma_*$. Hence $\sigma_* = \sigma_{M^*}$.

Conversely, let $\sigma_* = \sigma_{M^*}$. for some $M \in I$. We define $m\,[r] = mr$, $m \in M$ $r \in R$. With this definition M becomes an irreducible (right) R/σ_*–semimodule. In fact if $(m_1, [r_1]) = (m_2, [r_2])$, then $m_1 = m_2$ and $(r_1, r_2) \in \sigma_* = \sigma_{M^*}$, which implies $m_1r_1 = m_2r_2$. Now let $m\,[x] = m\,[y]$ and $m\,[x]^* = m\,[y]^*$ for all $m \in M$. Then $mx = my$ and $mx^* = my^*$ for all $m \in M$. This implies $(x, y) \in \sigma_{M^*} = \sigma_*$. Then $[x] = [y]$. Therefore M is an irreducible $*$-faithful (right) R/σ_*–semimodule. Hence R/σ_* is $*$-primitive. This completes the proof.

A hemiring R is said to be semisimple if $J_{rr} = \overline{\Delta}$.

Theorem 3.10. If R is a semisimple hemiring with an involution $*$, then $R/\overline{\Delta}$ is isomorphic to a subdirect product of $*$-primitive hemirings.

Proof. Let λ denote the class of all $*$-primitive relations on R. We define

$$\psi : R/\overline{\Delta} \to \prod_{\sigma_* \in \lambda} R/\sigma_* \text{ by } \psi\left([r]_{\overline{\Delta}}\right) = \left\{[r]_{\sigma_*}\right\}_{\sigma_* \in \lambda}$$

Since each σ_* is cancellative two-sided R-linear relation on R, ψ is clearly a well-defined homomorphism. Let

$$\left\{[r_1]_{\sigma_*}\right\}_{\sigma_* \in \lambda} = \left\{[r_2]_{\sigma_*}\right\}_{\sigma_* \in \lambda}.$$

Then $(r_1, r_2) \in \bigcap_{\sigma_* \in \lambda} \sigma_* = J_{rr}$, by lemma 3.9. Also $J_{rr} = \overline{\Delta}$ as R is semisimple. Thus ψ is an isomorphism, as required.

A hemiring is called primitive if it has an irreducible semimodule M such that $\sigma_M = \overline{\Delta}$. Now we establish the relation between $*$-primitive hemirings and primitive hemirings.

Theorem 3.11. If R is a $*$-primitive hemiring, then R is either a primitive hemiring or there is an R-linear relation $\sigma \neq \overline{\Delta}$ on R such that $\sigma \cap \sigma^* = \overline{\Delta}$, R/σ is (right) primitive and R/σ^* is (left) primitive.

Proof. Let M be an irreducible $*$-faithful (right) R-semimodule. If $\sigma_M = \overline{\Delta}$, then R is primitive. Let $\sigma_M \neq \overline{\Delta}$. Let $\sigma = \sigma_M$.As before M becomes an irreducible (right) R/σ – semimodule by defining m [r] = mr. Also since $\sigma_M = \sigma$, M is faithful and so R/σ is a (right) primitive hemiring.

Now M is an irreducible (right) R – semimodule, then by lemma 1.1, $\sigma_M = (\sigma_m : R)$ for some $0 \neq m \in M$. We set $M' = R/\sigma_m^*$ (see definition 3.6). Then M' is a (left) R-semimodule with the definition $x\,[r]_{\sigma_m^*} = [xr]_{\sigma_m^*}$. Now

$$\begin{aligned}
(x, y) \in \sigma^* &\Leftrightarrow (x^*, y^*) \in \sigma = \sigma_M = (\sigma_m : R) \\
&\Leftrightarrow (rx^*, ry^*) \in \sigma_m \text{ for all } r \in R \\
&\Leftrightarrow (xr, yr) \in \sigma_m^* \text{ for all } r \in R \\
&\Leftrightarrow [xr]_{\sigma_m^*} = [yr]_{\sigma_m^*} \text{ for all } r \in R \\
&\Leftrightarrow x[r]_{\sigma_m^*} = y[r]_{\sigma_m^*} \text{ for all } r \in R \\
&\Leftrightarrow xm' = ym' \text{ for all } m' \in M'.
\end{aligned}$$

Thus we have $\sigma^* = \{(x, y) \in R \times R : xm' = ym' \in$ for all $m' \in M'\}$. Also routine verification shows that M′ is irreducible. Thus M′ is an irreducible faithful (left R/σ* – semimodule and hence R/σ* is (left) primitive.

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REFERENCES

1. W. E. Baxter, W. S. Martindale III, Rings with involution and polynomial identities, Canad. J. Math. 20 (1968), 465-473.
2. K. Iizuka, On the Jacobson radical of a semiring, Tôhoku Math. J. 11 (2) (1959), 409-421.
3. Lide Li, On the structure of hemirings, Simon Stevin, A Quarterly Journal of Pure and Applied Mathematics 58 (1-2) (1984), 91-113.

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IDEMPOTENT—SEPARATED GOOD CONGRUENCES ON AN IC QUASI—ADEQUATE SEMIGROUP

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ABSTRACT : In this paper idempotent-separated good congruences on an IC quasi-adequate semigroup are considered. Some characterizations and a construction are obtained.

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1. INTRODUCTION AND PRELIMINARIES

On a semigroup S, the relation $\mathcal{L}^*$ ($\mathcal{R}^*$) is defined by the rule that $a\mathcal{L}^* b$ ($a\mathcal{R}^*b$) if and only if the elements a, b of S are related by Green's relation $\mathcal{L}$ ($\mathcal{R}$) in some oversemigroup of S. Following Fountain [4], we say that a semigroup in which each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains at least one idempotent is *abundant*. When the idempotents commute in an abundant semigroup, then it is called an *adequate semigroup* [3]. An adequate semigroup S in which $eS \cap aS = eaS$ and $Se \cap Sa = Sae$ for any $e^2 = e, a \in S$ is termed a *type A semigroup* [3]. An abundant semigroup S is called *idempotent-connected* (for short, *IC*) [1] when for each element a of S and for some $a^+ \in R_a^* \cap E(S)$, $a^* \in L_a^* \cap E(S)$, there is a bijection $\theta : \langle a^+\rangle \to \langle a^*\rangle$ satisfying $xa = a(x\theta)$ for all $x \in \langle a^+\rangle$, where R_a^* $\left(L_a^*\right)$ is the $\mathcal{R}^*$ – ($\mathcal{L}^*$ –) class containing a, $E(S)$ denotes the set of idempotents of S and $\langle e\rangle$ ($e \in E(S)$) stands for the subsemigroup generated by the idempotents of $eE(S)e$.

As an analogue of orthodox semigroup in the range of abundant semigroup theory, El-Qallali and Fountain [2] introduced quasi-adequate semigroup, that is, abundant semigroup in which the idempotents form a subsemigroup, and described the structure of a class of quasi-adequate semigroups, so-called type W semigroups, by virtue of Hall's semigroups and their quotient semigroups for δ (for δ, see latter Lemma 1.4). Recently, the author [5] proved that any IC quasi-adequate semigroup is type W.

Recall from [1] that a homomorphism $\varphi : S \to T$ of semigroups is a *good homomophism* when for all elements a, b of S, we have that $a\mathcal{L}^*b$ implies that $a\varphi\mathcal{L}^* b\varphi$ and that $a\mathcal{R}^*b$ implies that $a\varphi\mathcal{R}^*b\varphi$. We say that a congruence ρ on S is a *good congruence* if the natural homomorphism $S \to S/\rho$ is good. As defined in [6], a congruence ρ on S is an *idempotent-separated* if for all $e, f \in E(S)$, $e\rho = f\rho$ implies that $e = f$.

In this paper we shall be concerned with idempotent-separated good congruences on an IC quasi-adequate semigroup. We obtain some characterizations and a construction of such congruences.

For the purpose, we need recall some known results used repeatedly without mentions in the sequel.

Lemma 1.1. [4]. Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent :

(i) $a\mathcal{L}^* b(a\mathcal{R}^*b)$;

(ii) for all $x, y \in S^{-1}$, $ax = ay$ $(xa = ya)$ if and only if $bx = by$ $(xb = yb)$.

As an easy but useful consequence, we have

Corollary 1.2. [4]. Let S be a semigroup and $e^2 = e$, $a \in S$. Then the following statements are equivalent :

(i) $a\mathcal{R}^* e(a\mathcal{L}^*e)$;

(ii) $a = ea$ $(a = ae)$ and for all $x, y \in S^{-1}$, $xa = ya$ $(ax = ay)$ implies that $xe = ye(ex = ey)$.

Obviously, $\mathcal{R}^*$ $(\mathcal{L}^*)$ is a left (right) congruence on S. On a general semigroup, $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$. But when a, b are regular elements, $a\mathcal{L}^*$ $(\mathcal{R}^*)b$ if and only if $a\mathcal{L}(\mathcal{R})b$ (see, [4]). As in [4], $\mathcal{H}^*$ is defined as $\mathcal{L}^* \cap \mathcal{R}^*$. So, when a, b are regular elements, $a\mathcal{H}^* b$ if and only if $a\mathcal{H}b$.

Lemma 1.3. [7]. Let S be an abundant semigroup. Then the following statements are equivalent :

(i) S is IC

(ii) for each element $a \in S$ two conditions hold : for some [all]$a^* \in L^*_a \cap E(S)$, $a^+ \in R^*_a \cap E(S)$

1. for all $e \in \omega(a^*)$, there exists $g \in \omega(a^+)$ such that $ae = ga$;

2. for all $f \in \omega(a^+)$, there exists $h \in \omega(a^*)$ such that $fa = ah$,

where $\omega(e)$ $(e \in E(S)) = \{f \in E(S) : f = fe = ef\}$.

Let B be a band. By [6, Theorem IV 3.1], B is a semilattice Y of rectangular bands E_α ($\alpha \in Y$). For $e \in B$, we write $E(e) = E_\alpha$ when $e\ E_\alpha$. And if $E_\alpha E_\beta \subseteq E_\beta$ we denote $E_\beta \leqslant E_\alpha$. It is obvious that for all $e, f \in B$, $E(e) = E(f)$ if and only if $E(e) \leqslant E(f)$ and $E(f) \leqslant E(e)$.

The next result is due to [5]. For completeness, we give its proof.

Lemma 1.4. Let S be an IC quasi-adequate semigroup. Then

$$\delta = \{(a, b) \in S \times S : (\exists\ e \in E(b^+), f \in E(b^*))\ a = ebf\} \text{ [2]}$$

is a good congruence on S.

Proof. We verify firstly the assert : if $e, f \in E(S)$ with $a = ebf$, then $E(a^+) \leqslant E(e)$ and $E(a^*) \leqslant E(f)$. Indeed, as $a = ebf$, $ca = a$ and $af = a$. Now $ea^+ = a^+$ and $a^*f = a^*$. It follows that $E(a^+) \leqslant E(e)$ and $E(a^*) \leqslant E(f)$.

From [2, Proposition 2.6], it suffices to show that δ is left and right compatible. Here we only prove the case for the left side. Let $a, b, c \in S$ and $a\delta b$. The for some $e \in E(b^+)$ and $f \in E(b^*)$, $a = ebf$. Thus

$$\begin{aligned} ca &= cebf = c \bullet c^* eb^* \bullet bf \\ &= c(c^* eb^* c^*)(b^* c^* eb^*).\ bf && (c^* eb^* \in E(c^* b^*)) \\ &= gcbhf && (g,h \in E(S)) \\ &= [g(cb)^+](cb)[(cb)^* hf] \end{aligned}$$

so that by the assert above, $E((ca)^+) \leqslant E(g(cb)^+)$ and $E((ca)^*) \leqslant E((cb)^* hf)$. Hence $E((ca)^+) \leqslant E((cb)^+)$ and $E((ca)^*) \leqslant E((cb)^*)$. On the other hand, since $a = ebf$. $b^+ ab^* = b$. Applying a similar discussion to $b = b^+ ab^*$, one can obtain that $E((cb)^+) \leqslant E((ca)^+)$ and $E((cb)^*) \leqslant E((ca)^*)$. Thus $E((ca)^+) = E((cb)^+)$ and $E((ca)^*) = E((cb)^*)$. Now $E((cb)^+) = E((cb)^* hf)$ and $E((cb)^+) = E(g((b)^+)$. Thus δ is left compatible, as required.

By virtue of Lemma 1.4 and [2, Proposition 2.6], we have

Corollary 1.5 Let S be an IC quasi-adequate semigroup. Then δ is the minimum adequate good congruence. Moreover, S/δ is a type A semigroup.

Lemma 1.6 **[2]** Let S be an IC quasi-adequate semigroup and $a, b \in S$.

(i) if $a\delta b$, then $E(a^+) = E(b^+)$ and $E(a^*) = E(b^*)$

(ii) if for some $e \in E(a^+)$ and $f \in E(a^*)$, $b = eaf$, then $b\mathcal{L}^* f$ and $b\mathcal{R}^* e$.

(iii) $\mathcal{H}^* \cap \delta = \tau_S$ (the identity mapping on S).

2. RESULTS

For brevity, in the entire section, we always assume that S is an IC quasi-adequate semigroup, and that E is the set of idempotents of S. For $a \in S$, a^+ (a^*) denotes an idempotent in $R_a^* \cap E$ $(L_a^* \cap E)$. In what follows, T stands for S/δ.

Theorem 2.1. Let ρ be a good congruence on S. Then ρ is idempotent-separated if and only if $\rho \subseteq \mathcal{H}^*$.

Proof. Assume that ρ is an idempotent-separated good congruence on S. Let $a, b \in S$ and $a\rho b$. Then, since ρ is good, $(a^+ \rho)\ \mathcal{R}\ (b^+ \rho)$ so that $a^+ \rho = (b^+ a^+) \rho, b^+ \rho = (a^+ b^+) \rho$. But ρ is idempotent-separated. So, $a^+ = b^+ a^+$ and $b^+ = a^+ b^+$, and hence $a^+ \mathcal{R} b^+$. Thus $a\mathcal{R}^*b$. A dual discussion shows that $a\mathcal{L}^*b$. Thus $a\mathcal{H}^*b$, giving $\rho \subseteq \mathcal{H}^*$.

Conversely, let $\rho \subseteq \mathcal{H}^*$. For all $e, f \in E$, if $e\rho = f\rho$, then $e\ \mathcal{H}^*\ f$ so that $e\ \mathcal{H}\ f$. But no $\mathcal{H}$-class contains more than one idempotent, so $e = f$. Thus ρ is idempotent-separated.

Lemma 2.2. Let ρ be an idempotent-separated good congruence on S. Then $\rho \circ \delta = \delta \circ \rho$.

Proof. Let $x, y \in S$ and $x\rho \circ \delta y$. Then there exists $z \in S$ such that $x\rho z$, $z\delta y$. The latter formula means that there are $e \in E(y^+)$, $f \in E(y^*)$ such that $z = eyf$. Now $x\rho eyf$. Hence

$$(y^+ xy^*)\ \rho\ (y^+ eyfy^*) = (y^+ ey^+)\ y\ (y^* fy^*) = y \qquad .. \ (1)$$

On the other hand, since $\rho \subseteq \mathcal{H}^*$ (Theorem 2.1), $x\mathcal{H}^* z$ so that $E(x^+) = E(z^+)$, $E(x^*) = E(z^+)$. Moreover, by Lemma 1.6, $E(y^+) = E(x^+)$ and $E(y^*) = E(x^*)$. Obviously, $y^+ \in E(x^+)$ and $y^* \in E(x^*)$. Thus $y^+ xy^*\ \delta x$ and so, by (1), $x\delta \circ \rho y$ so that $\rho \circ \delta \subseteq \delta \circ \rho$; The reverse inclusion may similarly be obtained. Therefore $\rho \circ \delta = \delta \circ \rho$.

It is well known that all congruences on S form a lattice under inclusion Naturally, one wants to know whether all good congruences constitute a sublattice. In other words, whether is the join $\rho \vee \sigma$ of good congruences ρ, σ on S still a good one? For the general case, we have not been able to prove that. But we have

Theorem 2.3. Let ρ be an idempotent-separated good congruence on S. Then $\rho \vee \delta$ is good.

Proof. Certainly, $\rho \vee \sigma$ is a congruence on S. To show Theorem 2.3, it suffices to prove that $\delta \vee \rho$ preserves $\mathcal{L}^*$-classes and $\mathcal{R}^*$-classes. Here we only verify the case for $\mathcal{L}^*$. To see this, let $a, b \in S$ and $a\ \mathcal{L}^*b$. Let $(x)\ \rho \vee \delta$, $(y)\ \rho \vee \sigma \in (S/\rho \vee \delta)^1$ $(x, y \in S)$. If

$$(ax)\ \rho \vee \delta = (ay)\ \rho \vee \delta \qquad \ldots \ (2)$$

then we have $u \in S$ such that $ax\rho u$, $u\delta ay$, since $\rho \vee \delta = \rho \circ \delta$ by Lemma 2.2. Hence there exist $e \in E((ay)^+)$, $f \in E((ay)^*)$ such that $eayf = u$. Since S is IC and $a^+ a^+ \in \omega(a^+)$, by Lemma 1.3, there exists $g \in \omega(a^*)$ such that

$$a^+ u = a^+ eayf = (a^+ ea^+)\, ayf = agyf$$

so that

$$(a\rho)(x\rho) = u\rho = (a^+ u)\,\rho = (agyf)\,\rho = (a\rho)\,(gyf)\,\rho.$$

Notice that ρ is good, so that $a\rho\mathcal{L}^* b\rho$, it follows from Lemma 1.1 that

$$(b\rho)(x\rho) = (b\rho)\,(gyf)\,\rho = (bgyf)\,\rho \qquad \dots (3)$$

On the other hand, $a\mathcal{L}^*b$ deduces that $b\mathcal{L}^*a^*$. By Lemma 1.3, there exists $h \in E$ such that $bgyf = hbyf$ since $g \in \omega(a^*)$. Now, by (3),

$$(bx)\,\rho = (hbyf)\,\rho = [(h(by)^+)\, byf]\,\rho$$

so that $(bx)\,\rho = [(h(by)^+]\,\rho\,(bx)\,\rho$. Since ρ is good, $(bx)\,\rho\,\mathcal{R}^*\,(bx)^+\rho$. It follows that

$$(bx)^+\rho = (h(by)^+)\,\rho.\,(bx^+)\,\rho = [h(by)^+\,(bx)^+]\,\rho,$$

giving $(bx)^+ = h\,(by)^+\,(bx)^+$ since ρ is idempotent-separated. Hence $E((bx)^+) \leqslant E(h\,(by)^+)$ and so $E((bx)^+) \leqslant E(by)^+$. Considering $ax\delta \circ \rho ay$, a similar discussion shows that $E((by)^+) \leqslant E((bx)^+)$. Thus $E((ax)^+) = E((by)^+)$ so that $E(h\,(by)^+) = E((by)^+)$. This means $h\,(by)^+ \in E((by)^+)$. In addition, since $\mathcal{L}^*$ is a right congruence, $a\mathcal{L}^*\, b$ admits that $ay\mathcal{L}^*by$ which implies $f \in E((by)^*)$. Therefore

$$bgyf = hbyf\ \delta\ by.$$

From this, together with (3), $(bx)\,\rho \circ \delta = (by)\,\rho \circ \delta$, that is,

$$(b)\,\rho \vee \delta \circ (x)\,\rho \vee \delta = (b)\,\rho \vee \delta \circ (y)\,\rho \vee \delta \qquad \dots (4)$$

We have now proved that (2) can imply (4). Similarly, (4) means (2). Thus $(a)\,\rho \vee \delta\ \mathcal{L}^*\ (b)\,\rho \vee \delta$. That $\rho \vee \delta$ preserves $\mathcal{L}^*$-classes follows.

Theorem 2.4. Let ρ, σ be idempotent-separated good congruences on S. Then $\rho = \sigma$ if and only if $\rho \vee \delta = \sigma \vee \delta$.

Proof. We only prove the "only if". Suppose that $\rho \vee \delta = \sigma \vee \delta$. Let $x, y \in S$ with $x\rho y$. Then, since ρ is an idempotent-separated good congruence, giving $\rho \subseteq \mathcal{H}^*$, $x\,\mathcal{H}^*\,y$. Also, $x\rho \vee \delta y$. Hence, by hypothesis, $x\sigma \vee \delta y$. By Lemma 2.2, $\sigma \vee \delta = \sigma \circ \delta$ so that there exists $z \in S$ such that $x\sigma z$, $z\delta y$. On the other hand, since $\sigma \subseteq \mathcal{H}^*$, $x\,\mathcal{H}^*z$. Thus $y\mathcal{H}^*z$ (since $x\mathcal{H}^*y$) and hence $(y,z) \in \mathcal{H}^* \cap \delta$. It follows from Lemma 1.6 that $y = z$. Therefore $x\sigma y$, showing $\rho \subseteq \sigma$. Similarly, we have $\sigma \subseteq \rho$. Thus $\rho = \sigma$, as required.

Definition 2.5. Let ρ be a congruence on T. On S. define : for all $a, b \in S$ $(a, b) \in \overline{\rho}$ if and only if $(a\delta, b\delta) \in \rho$.

It is easy to see that $\overline{\rho}$ is a congruence on S. Moreover, we may prove

Lemma 2.6. Let ρ be a good ceongruence on T. Then $\overline{\rho}$ is a good congruence on S.

Proof. Assume $a, b \in S$ and $a\ \mathcal{L}^*\ b$. Then since δ is a good congruence, $ad\mathcal{L}^*\ b\delta$ and so, since ρ is a good congruence on T, $(a\delta)\rho\mathcal{L}^*\ (b\delta)\rho$, from which, applying Lemma 1.1, it is routine to prove that $a\ \overline{\rho}\,\mathcal{L}^*\ b\,\overline{\rho}$. Thus $\overline{\rho}$ preserves $\mathcal{L}^*$-classes. Dually, we may prove that $\overline{\rho}$ preserves *$\mathcal{R}^*$-classes. Thus $\overline{\rho}$ is a good congruence on S.*

For any idempotent-separated good congruence ρ on S, we shall denote by ρ^* the congruence $\rho \vee \delta/\delta$ on T. Thus for all $a, b \in S$

$$(a\delta,\ b\delta) \in \rho^* \textit{ if and only if } (a,\ b) \in \rho \vee \delta.$$

Lemma 2.7. Let ρ be an idempotent-separated good congruence on S. Let $a, b \in S$ and $a\mathcal{H}^*b$. Then $(a\delta, b\delta) \in \rho^*$ if and only if $(a, b) \in \rho$.

Proof. It is sufficient to show that "only if". Assume $(a\delta, b\delta) \in \rho^*$, that is $(a, b) \in \rho \vee \delta$. But by Lemma 2.2, $\rho \vee \delta = \rho \circ \delta$. Thus there exists $z \in S$ such that $a\rho z$, $z\delta b$. On the other hand, by Theorem 2.1, $\rho \subseteq \mathcal{H}^*$ so that $a\mathcal{H}^*z$. From this, together with the fact : $a\mathcal{H}^*b$, $z\mathcal{H}^*b$. Thus $(z, b) \in \mathcal{H}^* \cap \delta$, giving $b = z$ by Lemma 1.6. Therefore $(a, b) \in \rho$.

Lemma 2.8. Let $a, b \in S$. If $(a\delta, b\delta) \in \mathcal{L}^*$ $(\mathcal{R}^*\ ;\ \mathcal{H}^*)$. Then there exists $x \in \delta b$ (the δ-class containing b) such that $(a, x) \in \mathcal{L}^*$ $(\mathcal{R}^*\ ;\ \mathcal{H}^*)$.

Proof. Here we only prove the case for $\mathcal{H}^*$. Let $(a\delta, b\delta) \in \mathcal{H}^*$. Then, since δ is a good congruence, $(a^*\ \delta)\ \mathcal{L}^*\ (a^*\ b^*)$ so that $a^*\ \delta = (a^*\ b^*)\delta$. It follows form Lemma1.6 that $E(a^*)\ E(a^*\ b^*)$ so that $E(a^*) \leqslant E(b^*)$. The same argument to $(b^*\ \delta^*) = (b^*\ a^*)\delta)$ shows that $E(b^*) \leqslant E(a^*)$. Thus $E(a^*) = E(b^*)$. Applying a dual dicsussion to a^+ and b^-, we obtain that $E(a^+) = E(b^+)$. From these, $a^+ \in E(b^+)$ and $a^* \in E(b^*)$ so that $(a^+\ ba^*)\ \delta b$. But, from Lemma 1.6, $a^+\ \mathcal{R}^*a^+ba^*$ and $a^+\mathcal{L}^*\,a^+\ ba^*$. Thus $a\mathcal{R}^*a^+\ ba^*$ and $a\mathcal{L}^*\ a^+ba^*$, and so $a\ \mathcal{H}^*a^+ba^*$, giving the proof.

Lemma 2.9. Let ρ be an idempotent-separated good congruence on S. Then ρ^* is an idempotent-separated good congruence on T.

Proof. At first, we prove that ρ^* is a good congruence on T. let $a, b \in S$ and $(a\delta)$ $\mathcal{L}^*$ $(b\delta)$. Then by Lemma 2.8, ther exists $d \in \delta_b$ such that $a\mathcal{L}^*\ d$. For all $x\delta, y\delta \in T^1$ $(x, y \in S^1)$, if

$$[(a\delta)\ (x\delta)]\rho^* = [(a\delta)\ (y\delta)]\rho^*$$

that is,

$$(ax)\rho \vee \delta = (ay)\ \rho \vee \delta,$$

then, since $\rho \vee \delta$ is a good congruence (Theorem 2.3), giving $[(a)\rho \vee \delta]\ \mathcal{L}^*\ [(d)\ \rho \vee \delta]$. we have

$$(dx)\ \rho \vee \delta = (dy)\ \rho \vee \delta,$$

that is,

$$[(dx)\delta]\ \rho^* = [(dy)\delta]\ \rho^*$$

and thus

$$[(b\delta)\ (x\delta)]\ \rho^* = [(b\delta)\ (y\delta)]\ \rho^*$$

The observation together with its dual gives that $(a\delta)\rho^*\ \mathcal{L}^*\ (b\delta)\ \rho^*$. This means that ρ^* preserves $\mathcal{L}^*$-classes. A dual discussion can show that ρ^* preserves $\mathcal{R}^*$-classes. Thus ρ^* is a good congruence on T.

We now prove that ρ^* is idempotent-separated. To see this, let $x, y \in E(T)$ and $(x, y) \in \rho^*$. Then by [7, Theorem 1.6], there exist $e, f \in E$ such that $x = e\delta$, $y = f\delta$. It follows from the fact : $(x, y) \in \rho^*$, that $(e, f) \in \rho \vee \delta = \rho \circ \delta$. Hence for some $z \in S$, $e\rho z$ and $z\delta f$. The latter formula deduces that $z \in E$ since δ is idempotentpure. On the other hand, since $\rho \subseteq \mathcal{H}^*$ (Theorem 2.1) $e\rho z$ admits that $e\mathcal{H}^*z$, so that $e = z$ since no $\mathcal{H}$-class contains more than one idempotent. Thus $e\delta f$, giving $x = y$. Therefore ρ^* is idempotent-separated.

Recall from [1] that on any IC quasi-adequate semigroup, the largest congruence μ contained in $\mathcal{H}^*$ is a good congruence. It is bovious from Theorem 2.1 that μ is the maximum idempotent-separated good congruence. The next theorem gives an approach of constructing idempotent-separated good congruence on S in terms of good congruence on T, contained in μ^*.

Theorem 2.10. Let S be an IC quasi-adequate semigroup and let ρ be an idempotent-separated good congruence on $T = S/\delta$, contained in μ^*. Then $\mathcal{H}^* \cap \overline{\rho}$ is an idempotent-separated good congruence on S.

Conversely, if σ is an idempotent-separated good congruence on S, then there is a unique idempotent-separated good congruence τ on T, contained in μ^*, such that $\sigma = \mathcal{H}^* \cap \overline{\tau}$.

Proof. Let ρ be an idempotent-separated good congruence on T. By Lemma 2.6, $\overline{\rho}$ is a good congruence on S. Set $\rho^{\pm\pm} = \overline{\rho} \cap \mu$. Then $\rho^{\pm\pm}$ is clearly a good congruence and is contained in $\mathcal{H}^* \cap \overline{\rho}$. By Theorem 2.1, $\rho^{\pm\pm}$ is an idempotent-separated good congruence on S. So, to show the direct part, it is sufficient to prove that $\rho^{\pm\pm} = \mathcal{H}^* \cap \overline{\rho}$. We need only verify that $\rho^{\pm\pm} \supseteq \mathcal{H}^* \cap \overline{\rho}$. To see this, let $(a, b) \in \mathcal{H}^* \cap \overline{\rho}$. Then $(a, b) \in \mathcal{H}^*$ and, since $\rho \subseteq \mu^*$, $(a\delta, b\delta) \in \mu^*$. Thus by Lemma 2.7, $(a, \mathrm{b}) \in \mu$. It follows that $(a, b) \in \mu \cap \overline{\rho}$, giving $\mathcal{H}^* \cap \overline{\rho} \subseteq \mu \cap \overline{\rho}$, as required.

Conversely, suppose that σ is an idempotent-separated good congruence on S. Obviously, $\sigma \subseteq \mu$. Then $\sigma^* = \sigma \vee \delta/\delta$ is, by Lemma 2.9, an idempotent-separated good congruence on T and is contained in $\mu^* = \mu \vee \delta/\delta$. Now, by the proof of the direct part of Theroem 2.10, $\overline{\sigma^*} \cap \mathcal{H}^* = \overline{\sigma^*} \cap \mu$. On the other hand, by Lemma 2.6, $\sigma \vee \delta$ is a good congruence on S, and so $(\sigma \vee \delta) \cap \mu$ is also good. Notice that $[(\sigma \vee \delta) \cap \mu] \subseteq \mathcal{H}^*$, $(\sigma \vee \delta) \cap \mu$ is, by Theorem 2.1, an idempotent-separated good congruence on S. But by Definition 2.5 and the meaning of σ^*, now $\sigma \vee \delta = \overline{\sigma^*}$ so that

$$\overline{\sigma^*} \cap \mu = (\sigma \vee \delta) \cap \mu \qquad \ldots (5)$$

We next prove that $\sigma = \overline{\sigma^*} \cap \mu$ $(= \sigma \cap \mathcal{H}^*)$. To see this, let $(a, b) \in (\sigma \vee \delta) \cap \mu$, that is, $(a, b) \in \sigma \vee \delta$ and $(a, b) \in \mu$. Then $(a\delta, b\delta) \in \sigma^*$ and, since $\mu \subseteq \mathcal{H}^*$, $a\mathcal{H}^*b$. It follows from Lemma 2.7 that $(a, b) \in \sigma$ so that $(\sigma \vee \delta) \cap \mu \subseteq \sigma$, and so by (5), $(\overline{\sigma^*} \cap \mu) \subseteq \sigma$. But, since $\sigma \subseteq \mu$, $\sigma \subseteq (\sigma \vee \delta) \cap \mu$ so that $\sigma \subseteq \overline{\sigma^*} \cap \mu$. Thus $\sigma = \overline{\sigma^*} \cap \mu$ $(= \overline{\sigma^*} \cap \mathcal{H}^*)$. Therefore $\overline{\sigma^*}$ is the idempotent-separated good congruence τ which is required in the theorem.

Finally, we prove the uniqueness of τ. To show this, we only prove that $\overline{\eta} \cap \mathcal{H}^* \subseteq \overline{\pi} \cap \mathcal{H}^*$ where η, π are idempotent-separated good congruences on T, contained in μ^*, may imply that $\eta \subseteq \pi$. For all $x, y \in S$, if $(x\delta, y\delta) \in \eta$, then $(x, y) \in \overline{\eta}$ and, since $\eta \subseteq \mathcal{H}^*$, $(x\delta, y\delta) \in \mathcal{H}^*$. By Lemma 2.8, there exists $z \in \delta_y$ such that $(x, z) \in \mathcal{H}^*$. Obviously, $(x\delta, z\delta) \in \eta$ so that $(x, z) \in \overline{\eta}$. Hence $(x, z) \in \overline{\eta} \cap \mathcal{H}^* \subseteq \overline{\pi} \cap \mathcal{H}^*$. It follows that $(x, z) \in \overline{\pi}$ so that $(x\delta, y\delta) = (x\delta, z\delta) \in \pi$. Thus $\eta \subseteq \pi$, as required.

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REFERENCES

1. A El-Qallali and J. B. Fountain, Idempotent-connected abundant semigroups, Proc. Roy. Soc. Edinburgh 91A (1981), 79-90.

2. A El-Quallai and J. B. Fountain, Quasi-adequate semigroups. Proc. Roy. Soc. Edinburgh 91A (1981), 91-99.

3. J. B. Fountain, Adequate semigroups. Proc. Edinburgh Math. Soc. 22 (1979), 113-125.

4. J. B. Fountain, Abundant semigroups Proc. London Math. Soc. (3) 44 (1982), 103-129.

5. Xiaojiang. Guo, F-abundant semigroups (submitted).

6. J. M. Howie. An introduction to semigroup theory, Academic Press. London. 1976.

7. M. V. Lawson. The natural partial orders on an abundant semigroup. Proc. Edinburgh Math. Soc. 30 (1987), 169-186.

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ON A SEMI SYMMETRIC METRIC CONNECTION IN A KENMOTSU MANIFOLD

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ABSTRACT : It is proved that the semi-symmetric metric connection in a Kenmotsu manifold is a ϕ-connection and Kenmotsu manifold is of constant curvature if and only if it is of constant curvature for the semi-symmetric metric connection in it. Moreover, if M is either a conformally flat Kenmotsu manifold of dimension > 3 or a Kenmotsu manifold of constant ϕ-sectional curvature, then M becomes flat with respect to the semi-symmetric connection.

Mathematics Subject Classification : 53B15

1. INTRODUCTION

Let M be an almost contact metric manifold [1] with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is a (1,1) tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M such that

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi(\xi) = 0, \ \eta \circ \phi = 0 \qquad \dots (1)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\, \eta(Y), \qquad \dots (2)$$

$$\Phi(X, Y) \equiv g(\phi\, X, Y) = -\Phi(Y, X), \ g(\xi, X) = \eta(X) \qquad \dots (3)$$

for all $X, Y \in TM$.

An almost contact metric manifold is called a Kenmotsu manifold if

$$(\nabla X\ \phi)\ Y = \eta(Y)\ \phi\ X - g(\phi\ X, Y)\ \xi \qquad \dots (4)$$

From (4) it follows that

$$\nabla X \xi = \phi^2 X. \qquad \dots (5)$$

A linear connection $\tilde{\nabla}$ in an almost contact metric manifold M is said to be a *semi-symmetric connection* [3, 4] if its torsion tensor

$$T(X, Y) = \tilde{\nabla}_{x} Y - \tilde{\nabla}_{y} X - [X, Y]$$

satisfies

$$T(X, Y) = \eta(Y)\, X - \eta(X)\, Y \qquad \ldots (6)$$

a *metric connection* [3, 4] if

$$\tilde{\nabla} g = 0 \qquad \ldots (7)$$

a *ϕ-connection* [5] if

$$\tilde{\nabla} \phi = 0 \qquad \ldots (8)$$

If ∇ is Levi-Civita connection of g in an almost contact metric manifold, then putting

$$\tilde{\nabla}_{x} Y = \nabla_{x} Y + \eta(Y)\, X - g(X, Y)\, \xi, \qquad \ldots (9)$$

it is known that $\tilde{\nabla}$ is a semi-symmetic metric connection in M (See [3, 4]).

Let R be the curvature tensor of ∇ and $\bar{R}$ be the curvature tensor of ∇. Then $\tilde{R}$ and R are related by [3, 4]

$$\tilde{R}(X, Y)Z = R(X, Y)Z - \alpha(Y, Z)X + \alpha(X, Z)Y - g(Y, Z)\beta X + g(X, Z)\beta Y \ldots (10)$$

where

$$\alpha(X, Y) = (\tilde{\nabla}_{x}\eta)Y - \frac{1}{2}g(X, Y), \qquad \ldots (11)$$

$$\beta(X), Y = \tilde{\nabla}_{x}\xi - \frac{1}{2}X. \qquad \ldots (12)$$

In this paper we prove the following theorems :

Theorem 1. *The semi-symmetric metric connection $\tilde{\nabla}$ in a Kenmotsu manifold is a ϕ-connection.*

Theorem 2. *The Kenmotsu manifold is of constant curvature if and only if it is of constant curvature for the connection $\tilde{\nabla}$.*

Theorem 3. *If M is a conformally flat Kenmotsu manifold of dimension > 3 then M becomes flat with respect to the semi-symmetric metric ϕ-connection $\tilde{\nabla}$.*

Theorem 4. *If M is a Kenmotsu manifold of constant ϕ-sectional curvature then M becomes flat with respect to the semi-symmetric metric ϕ-connection $\tilde{\nabla}$.*

2. PROOF OF THE THEOREMS

Proof of the Theorem 1.

We have

$$\begin{aligned}
(\tilde{\nabla} x\phi)Y &= \tilde{\nabla} x\phi Y - \phi\tilde{\nabla} xY \\
&= \nabla x\phi Y + \eta(\phi Y)X - g(X, \phi Y)\xi \\
&\quad - \phi(\nabla xY + \eta(Y)X - g(X, Y)\xi \\
&= (\nabla x\phi)(Y) - (\eta(Y)\phi X - g(\phi X, Y)\xi) \\
&= 0,
\end{aligned}$$

where (9), (1) and (4) have been used.

Proof of the Theorem 2.

We have

$$\begin{aligned}
\tilde{\nabla} x\xi &= \nabla x\xi + \eta(\xi)X - g(X, \xi)\xi \\
&= \nabla x\xi + (X - \eta(X)\xi) \\
&= \phi^2 X + (X - \eta(X)\xi) \\
&= 0,
\end{aligned}$$

where (9), (1) and (5) have been used. We also have

$$\begin{aligned}
(\tilde{\nabla} x\eta)Y &= \tilde{\nabla} x\eta(Y) - \eta(\tilde{\nabla} xY) \\
&= X\eta(Y) - \eta(\nabla xY + \eta(Y)X - g(X, Y)\xi) \\
&= (\nabla x\eta)Y + g(X, Y) - \eta(X)\,\eta(Y) \\
&= g(\nabla x\xi, Y) + g(\phi X, \phi Y) \\
&= g(\phi^2 X, Y) + g(\phi X, \phi Y) \\
&= 0,
\end{aligned}$$

where (9), (1) and (5) have been used. Using $\tilde{\nabla}_X\xi = 0 = (\tilde{\nabla}_X\eta)Y$ in (11) and (12) we get

$$\alpha(X, Y) = -\frac{1}{2}g(X, Y),\ \beta(X) = -\frac{1}{2}X,$$

and consequently (10) becomes

$$\tilde{R}(X, Y)Z = R(X, Y)Z + (g(Y, Z)X - g(X, Z)Y). \qquad \ldots (13)$$

It is known that if a Riemannian manifold is of constant curvature then

$$R(X, Y)Z = c\ (g(Y, Z)X - g(X, Z)Y). \qquad \ldots (14)$$

Consequently, in (13), $\tilde{R}(X, Y)Z$ is of the form (14) if and only if $R(X, Y)Z$ is of the form (14). This proves Theorem 2.

Proof of the Theorem 3.

Let M be a Kenmotsu manifold of dimension > 3.If M is conformally flat, then M is a space of constant negative curvature −1 [2]. Then from (13) we get

$$\tilde{R}(X, Y)Z = R(X, Y)Z + (g(Y, Z)X - g(X, Z)Y) = 0$$

for $\tilde{\nabla}$. Thus M becomes flat with respect to the semi-symmetric metric connection $\tilde{\nabla}$.

Proof of the Theorem 4.

Let M be a Kenmotsu manifold. If it is of constant ϕ-sectional curvature c, then M is a space of constant curvature and $c = -1$ [2]. Therefore, as in Theorem 3, M becomes flat with respect to the semi-symmetric metric connection $\tilde{\nabla}$.

REFERENCES

1. D. E. Blair, *Contact manifolds in Riemannian geometry,* Lecture Notes in Math. **509,** Springer Verlag, 1976.

2. K. Kenmotsu, *A class of almost contact Riemannian manifolds,* Tohoku Math. J. **24**(1972), 93-103.

3. A. Sharfuddin and S. I. Husain, *Semi symmetric metric connexions in almost contact manifolds,* Tensor, **30**(1976), 133-139.

4. K. Yano, *On semi-symmetric metric connexions,* Rev. Roumaine Math. Pures Appl. **15**(1970), 1579-1586.

5. K. Yano and T. Imai, *On semi-symmetric metric ϕ-connections in a Sasakian manifold,* Kodai Math. Sem. Rep. **28**(1977), 150-158.

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THE INFLATIONS OF THE TOTALLY ORDERED SEMILATTICES OF $M_{F,n}$*

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ABSTRACT : On the base inflation of a zero matrix, we give the structure of the totally ordered semilattice of $M_{F,n}$. Considering the nonnegative inflation of zero matirx, we give a method how to characterize it.

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INTRODUCTION

Every element s in a semigroup S associates a set Z_s such that Z_s are pairwise disjoint and $Z_s \cap S = \{s\}$. The set $V = \cup_{s \in S} Z_s$ together with the multiplication $x * y = ab$ if $x \in Z_a$, $y \in Z_b$, is called *an inflation* of S[2] [4]. If S is a semilattice, in such case, V is *an inflation of the semilattice.*

In this paper, we have solved the inflation of such semilattice of $M_{F,n}$ which contains only one idempotent matrix and the inflations of totally ordered semilattices of $M_{F,n}$. Firstly we have discussed such semilattice which contains only one idempotent matrix. For those which have discussed such semilattice which contains only one idempotent matrix. For those which have proper inflations, we find their inflations have a close relation to the inflation of $\{0\}$ where 0 is the zero matrix. So we give some properties of the inflation of $\{0\}$ in general and limited in nonnegative matirces. Finally we have solved the problem of the form of the totally ordered semilattice of $M_{F,n}$ and prove that Y subsemilattice of $M_{F,n}$ if and only if, under the isomorphism, Y is one of the subset of $\{ \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} | \; r = 0, 1, 2, \ldots, n\}$. On the base of this, we give the inflations of the totally ordered semilattices of $M_{F,n}$.

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By a *groupoid* (S, μ) we shall mean a nonempty set S *on which a binary operation* μ is defined. That is to say, we have a mapping $\mu : S \times S \to S$. We shall say that (S, μ) is a *semigroup* if μ is associative, i.e., if

$$(\forall x, y, z \in S)\ ((x, y)\ \mu, z)\ \mu = (x, (y, z)\mu)\mu^{[3]}.$$

We shall not in fact use this rather cumbersome notation. Following the usual practice in algebra we shall write $(x, y)\ \mu$ simply as xy and usually refer to the semigroup operation as *multiplication*. The above formula then becomes

$$(xy)z = x(yz).$$

$M_{F,n}$ is the set of all order n square matrices over a field F. Certainly, it is well known that on the set $M_{F,n}$ there are two binary operation "+" and "×". For "+", $(M_{F,n}, +)$ is an Abel group. For "×"—the usual multiplication on matrices, $(M_{F,n}, \times)$ is a regular semigroup [2]. Throughout this paper, we say matrices multiplication semigroup $M_{F,n}$ is $(M_{F,n}, \times)$. Also we write $A \times B$ simply as AB for all $A, B \in M_{F,n}$.

An element a of a semigroup S is *idempotent* if $a^2 = a$. A *band* is a semigroup in which every element is idempotent. If $(S, \bullet)$ has the additional property that

$$(\forall x, y \in S)xy = xy,$$

we shall say that it is a *commutative semigroup*. A commutative band is a *semilattice*. A binary relation ω on a set X (i.e. a subset of $X \times X$) will be called a *partial order* if (i) $(x, x) \in \omega$ for all x in X ; (ii) $(\forall x, y \in X)\ (x, y) \in \omega$ and $(y, x) \in \omega \Rightarrow x = y$; (iii) $(\forall x, y, z \in X)\ (x, y) \in \omega$ and $(y, z) \in \omega \Rightarrow (x, z) \in \omega$.

Traditionally one writes $x \le y$ rather than $(x, y) \in \omega$ and write $x < y$ to mean $x \le y$ and $x \ne y$. A partial order having the extra property (iv) $(\forall x, y \in X)\ x \le y$ or $y \le x$ will be called a total order. We shall refer to $(X, \le)$, or just to X, as a totally ordered set. If e, f are idempotents of a semigroup S, we shall write $e \le f$ if $ef = fe = e$. It is a routine matter to check that $\le$ is a partial order relation on the set E of idempotents of S[3]. Throughout this paper, we say a totally ordered semilattice Y of $M_{F,n}$ refers to a semilattice Y of $M_{F,n}$ in which $A \le B$ (if and only if $AB = BA = A$) is a total order. From [2] or [4], we easily deduce the following lemma.

Lemma 1. *A semigroup S is an inflation of a semilattice if and only if it satisfies the identity*

$$(\forall x, y \in S)xy = yx = x^2y.$$

According to the lemma 1, we easily deduce the lemma 2

Lemma 2. *For every $E \in E(M_{F,n})$ (i.e. the set of idempotents matrices in $M_{F,n}$.), $L = \{E\}$ has proper inflations if and only if there exists $E \neq A \in M_{F,n}$ such that $AE = EA = E$ and $A^2 = E$.*

As an important base and the end of this section, we finally give three lemmas about the idempotent matrix in $M_{F,n}$.

Lemma 3. *Let $A \in M_{F,n}$. Then A is an idempotant matrix ($A^2 = A$) if and only if there exists an invertible matrix R such that*

$$RAR^{-1} = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}$$

(r = rank A, $0 \leq r \leq n$ and E_r is the identity matrix in $M_{F,r}$.

Then the following lemmas are obvious.

Lemma 4.

$$\{\textit{rank } r \textit{ idempotent matrices in } M_{F,n}\}$$

$$= \{R \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} R^{-1} \mid R \textit{ is any invertible matrix in } M_{F,n}\} \ (0 \leq r \leq n).$$

Lemma 5.

$$E(M_{F,n}) = \{R \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} R^{-1} \mid r = 0, 1, 2, \ldots, n, R \textit{ is any invertible matrix in } M_{F,n}\}.$$

Throughout this paper, Z_A is representing an inflation of the semilattice $\{A\}$ for any $A \in E(M_{F,n})$. E_n is the identity matrix in $M_{F,n}$. $E_{r,n} = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}_{n \times n}$ and $E_{r,n,R} = R \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} R^{-1}$ in which R is any invertible matrix in $M_{F,n}$.

1 THE INFLATION OF AN IDEMPOTENT MATRIX IN $M_{F,N}$

Theorem 1.1. *The semilattice $\{0\}$ of $M_{F,n}$ ($n > 1$) has a proper inflation.*

Proof. By the lemma 2 and multiplication between matirces, we find that

$$\left\{ A_a = \begin{pmatrix} 0 & a & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & & & & \dots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \middle| a \in F \right\} = Z_{0_{n\times n}}.$$

So the semilattice $\{0\}$ of $M_{F,n}$ has proper inflation.

Theorem 1.2. *In* $M_{F,n}$ $(n > 1)$, $\{E_{n-1,n}\}$ *and* $\{E_n\}$ *have no proper inflations.*

Proof. Clearly $\{E_n\}$ has no proper inflation. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ (puting into same partitioned matrix like $E_{r,n} \in Z_{E_{r,n}}$ By solving $AE_{r,n} = E_{r,n} = E_{r,n} A$ and $A^2 = E_{r,n,}$ we have $A_{11} = E_r$ $A_{12} = 0$, $A_{21} = 0$ and $A_{22}^2 = 0$. When $n - r = 1$, $A_{22} \in F$ and there is only zero in F satisfying whose square is equal to zero. Now $r = n - 1$ and $A = \begin{pmatrix} E_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$ $= E_{n-1,n}$. So $E_{n-1,n}$ has no proper inflation.

From the proof of theorem 1.2 we know that $n - r \geq 2$, $A = \begin{pmatrix} E_r & 0 \\ 0 & A_{22} \end{pmatrix} \in Z_{E_{r,n}}$ if and only if $A_{22} \in Z_{0_{(n-r)\times(n-r)}}$. Since $Z_{0_{(n-r)\times(n-r)}}$, when $n - r \geq 2$, has proper inflations, every semilattice $\{E_{r,n}\}$, when $n - r \geq 2$, has proper inflations.

Clearly $Z_{E_{r,n,R}} = \{RAR^{-1} \mid A \in Z_{E_{r,n}}\}$ where R is any invertible matrix in $M_{F,n}$. So $\{E_{r,n}\}$ has proper inflations, so does $\{E_{r,n,R}\}$. And when $\{E_{r,n}\}$ has no proper inflations, neither does $\{E_{r,n,R}\}$.

Up to now, we have well known every semilattice of $M_{F,n}$ which contains only one idempotent matrix has proper inflations except $\{RE_{n-1,n}R^{-1} \mid$ where R is any invertible matirx in $M_{F,n}\}$ and $\{E_n\}$.

From lemma 2, we have the following two corollarys.

Corollary 1.3. $Z_0 = \{A, B\}$ *if and only if* $A^2 = 0 = B^2$ *and* $AB = BA = 0$.

Let the multiplication between F and $M_{F,n}$ be

$$aA = \begin{pmatrix} aa_{11} & aa_{12} & \dots & aa_{1n} \\ aa_{21} & aa_{22} & \dots & aa_{2n} \\ \dots & & & \dots \\ aa_{n1} & aa_{n2} & \dots & aa_{nn} \end{pmatrix}$$

if

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

and $a \in F$. Then $(M_{F,n}, +, \cdot, F)$ is a linear space.

Corollary 1.4. *Z_0 is a subspace of $M_{F,n}$.*

Theorem 1.5. *Let F is the real number field. If Z_0 contain only nonnegative matrices and satisfies that for any A, $B \in Z_0$, $A + B \in Z_0$, then there exists a permutation matirx P such that any matrix A in PZ_0P^T can be written as*

$$A = \begin{pmatrix} A_{11} & 0 & \dots & 0 & 0 \\ A_{21} & A_{22} & \dots & 0 & 0 \\ \dots & & & & \dots \\ A_{t1} & A_{t2} & \dots & A_{tt} & 0 \\ A_{t+1\ 1} & A_{t+1\ 2} & \dots & A_{t+1\ t} & 0 \end{pmatrix} \qquad \dots (1)$$

where for any $1 \leq i, j \leq t$, A_{ij} satisfies that the first column is zero and the first row may be nozero but other rows are all zeros ; for any i, A_{ii} is a square matirx ; any j, $A_{t+1\ j}$ satisfies that the first column is zero.

Conversely, the set of all matrices satisfying (1) constructs a inflation of a zero matrix.

Proof. Suppose that $Z_0 \neq 0$, and for any $A \in Z_0$, define a mapping ϕ from Z_0 into all nonnegative natural numbers

$$\phi : A \rightarrow \phi_A,$$

where ϕ_A = max {the number of nonzero entries in any row of A}. Since Z_0 is an inflation of a zero matrix, for any $A = (a_{ij})_{n\times n} \in Z_0$, $a_{ii} = 0 (i = 1, 2, \dots, n)$. Let r_1 = max

$\{\phi_A \mid A \in Z_0\}$. Then $0 < r < n$ and there exists $A \in Z_0$ such that $\phi_A = r_1$. So there exists a permutation matrix P such that

$$PAP^T = \begin{pmatrix} 0 & a_{12} & \ldots & a_{1r_1+1} & 0 & \ldots & 0 \\ \ldots & & & & & \ldots & \\ \ldots & & & & & \ldots & \end{pmatrix}. \qquad \ldots (2)$$

Without the loss of generality, suppose that $A = PAP^T$.

For any $B \in Z_0$ and $AB = BA = 0$, we deduce that $a_{ij} = 0$, $i = 2,3, \ldots , r_1 + 1$ and A is written as

$$A = \begin{pmatrix} a_{11} & 0 \\ A'_{21} & A'_{22} \end{pmatrix}$$

where A_{11} is a $(r_1 + 1) \times (r_1 + 1)$ matrix such that any entries in the first row are all nonzero except the first entry 0 and other rows are all zero. The first column of A'_{22} is zero.

Let $B \in Z_0$. Then B can be written as

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

where B_{11} is a $(r_1 + 1) \times (r_1 + 1)$ matrix. By $AB = BA = 0$ and the property of r_1, we have $B_{12} = 0$ and the rows in B_{11} are all zero except the first row. Let

$$Z'_0 = \{B_{22} \mid (\exists B \in Z_0)\ B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix} \text{ where } B_{11} \text{ is a } (r_1 + 1) \times (r_1 + 1) \text{ matrix}\}.$$

Then Z'_0 is an inflation of a $(n - r_1 - 1) \times (n - r_1 - 1)$ zero matrix 0. Similar to Z_0, if $Z'_0 \neq \{0\}$, any B_{22} in Z'_0 can be written as

$$\begin{pmatrix} B_{33} & 0 \\ B_{43} & B_{44} \end{pmatrix}$$

where the first row of B_{33} may be nonzero, other rows and the first column are all zero and the first column of B_{43} is zero. By $AB = BA = 0$, for any $B \in Z_0$, it can be written as

$$\begin{pmatrix} B_{11} & 0 & 0 \\ B_{21} & B_{22} & 0 \\ B'_{31} & B'_{32} & B'_{33} \end{pmatrix}$$

where the first rows of B_{11}, B_{21} and B_{22} may be nonzeros, other rows are zero and the first columns of B_{11}, B_{21}, B_{22}, B'_{31} and B'_{32} are zero. If we continue to do it, we prove the direct part of theorem.

Conversely, it is clear.

2 THE INFLATIONS OF THE TOTALLY ORDERED SEMILATTICES OF $M_{F,n}$

Let $TST^{-1} = \{TAT^{-1} \mid A \in S\}$ if $S \subseteq M_{F,n}$.

Lemma 2.1. *If* $S \subseteq M_{F,n}$ *is a semigroup, then* $S \simeq TST^{-1}$ *where* T *is any invertible matrix in* $M_{F,n}$.

Lemma 2.2. *If* Y *is a totally ordered semilattice of* $M_{F,n}$, *there is at most one rank* $r(0 \leq r \leq n)$ *idempotent matrix in* Y.

Proof. Since Y is a totally ordered semilattice of $M_{F,n}$, there are two cases : $A \leq B$ or $B \leq A$ for all $A, B \in Y$. Let rank A = rank $B = r$.

Suppose $A \leq B$. Since $A \in Y \subseteq E(M_{F,n})$ and rank $A = r$, there exists an invertible matrix $T \in M_{F,n}$ such that $TAT^{-1} = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}$. Then we can prove the lemma in TYT^{-1}. But because of the lemma 2.1, we will view Y and TYT^{-1} as the same. So $A = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} \in Y$. From $A \leq B$, we have $AB = BA = A$. Let $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$. Then

$$\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, $B = \begin{pmatrix} E_r & 0 \\ 0 & B_{22} \end{pmatrix}$. Assume $B_{22} \neq 0$. Then $r_{B_{22}} \geq 1$. So $r_B > r$. This contradicts with $r_B = r$. Thus, $B_{22} = 0$. This is to say,

$$B = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} = A.$$

Corollary 2.3. *In $M_{F,n}$, idempotent matrices which have the same rank can't compare.*

Lemma 2.4. *If A, $B \in Y$ and Y is a totally ordered semilattice of $M_{F,n}$, and $r_A \le r_B$, then $A \le B$.*

Proof. Since $B \in Y \subseteq E(M_{F,n})$, there exists an invertible matrix $T \in M_{F,n}$ such that

$$TBT^{-1} = \begin{pmatrix} E_{r_B} & 0 \\ 0 & 0 \end{pmatrix}.$$

Then we can regard $B = \begin{pmatrix} E_{r_B} & 0 \\ 0 & 0 \end{pmatrix} \in Y$ because of the lemma 2.1. Assume $A > B$, then $AB = B = BA$. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$. So

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} E_{r_B} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} E_{r_B} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

By solving the matrix equation, we have

$$A = \begin{pmatrix} E_{r_B} & 0 \\ 0 & A_{22} \end{pmatrix} \ne \begin{pmatrix} E_{r_B} & 0 \\ 0 & 0 \end{pmatrix}.$$

So $A_{22} \ne 0$. Thus $r_A > r_B$. This is a contradiction. Therefore if $r_A \le r_B$, then $A \le B$.

Theorem 2.5. *Every totally ordered semilattice of $M_{F,n}$ is isomorphic to one of the subset of*

$$\left\{ \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} \Big| \; r = 0, 1, 2, \dots, n \right\}.$$

Proof. When $0 \le r_i \le r_j \le n$,

$$\begin{pmatrix} E_{r_i} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} E_{r_j} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} E_{r_i} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} E_{r_j} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} E_{r_i} & 0 \\ 0 & 0 \end{pmatrix}.$$

So $\left\{\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} \middle| \, r = 0, 1, 2, \dots, n \right\}$ and every subset of it is a totally ordered semilattice of $M_{F,n}$.

For any totally ordered semilattice Y of $M_{F,n}$, we have reason to suppose the form of the matrix A_{r_t} which has the biggest rank r_t $(0 \le r_t \le n)$ in Y is

$$A_{r_t} = \begin{pmatrix} E_{r_t} & 0 \\ 0 & 0 \end{pmatrix}.$$

And the second biggest is $A_{r_{t-1}}$, the third is $A_{r_{t-2}}$ and so on. That is to say. $r_t > r_{t-1} > \dots > r_2 > r$. Then $A_{r_t} > A_{r_{t-1}} > \dots > A_{r_1}$. Let

$$A_{r_{t-1}} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where A_{11} and A_{22} are order r_t and $n - r_t$ square matrix, respectively. We obtain $A_{r_{t-1}} = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix}$ from $A_{r_t} > A_{r_{t-1}}$. Since $A_{r_{t-1}}^2 = A_{r_{t-1}}$, $A_{11}^2 = A_{11}$. Then there exists an invertible order r_t matrix T_{r_t} such that

$$T_{r_t} A_{11} T_{r_t}^{-1} = \begin{pmatrix} E_{r_{t-1}} & 0 \\ 0 & 0 \end{pmatrix}$$

Let

$$T_1 = \begin{pmatrix} T_{r_t} & 0 \\ 0 & E_{n-r_t} \end{pmatrix}.$$

T_1 is an invertible matrix in $M_{F,n}$. And

$$T_1 A_{r_{t-1}} T_1^{-1} = \begin{pmatrix} T_{r_t} & 0 \\ 0 & E_{n-r_t} \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_{r_t}^{-1} & 0 \\ 0 & E_{n-r_t} \end{pmatrix}$$

$$= \begin{pmatrix} T_{r_t} A_{11} T_{r_t}^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} E_{r_{t-1}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T_1 A_{r_t} T_1^{-1} = \begin{pmatrix} T_{r_t} & 0 \\ 0 & E_{n-r_t} \end{pmatrix} \begin{pmatrix} E_{r_t} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_{r_t}^{-1} & 0 \\ 0 & E_{n-r_t} \end{pmatrix}$$

$$= \begin{pmatrix} T_{r_t} E_{r_t} T_{r_t}^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} E_{r_t} & 0 \\ 0 & 0 \end{pmatrix}.$$

We find, in $T_1 Y T_1^{-1}$, A_{r_t} and $A_{r_{t-1}}$ are simultaneously the forms of $\begin{pmatrix} E_{r_t} & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} E_{r_{t-1}} & 0 \\ 0 & 0 \end{pmatrix}$ respectively. Since $Y \simeq T_1 Y T_1^{-1}$, we might as well to regard

$$A_{r_t} = \begin{pmatrix} E_{r_t} & 0 \\ 0 & 0 \end{pmatrix}, \ A_{r_{t-1}} = \begin{pmatrix} E_{r_{t-1}} & 0 \\ 0 & 0 \end{pmatrix} \in Y.$$

Since $A_{r_{t-1}} > A_{r_{t-2}}$, if we use the same method like before, we can also construct an invertible order n matrix

$$T_2 = \begin{pmatrix} E_{r_{t-1}} & 0 \\ 0 & E_{n-r_{t-1}} \end{pmatrix}$$

where $T_{r_{t-1}}$ is some invertible matrix whose function is like T_{r_t} and in $T_2 Y T_2^{-1}$, $A_{r_{t-1}}$ and $A_{r_{t-2}}$ are the form of $\begin{pmatrix} E_{r_{t-1}} & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} E_{r_{t-2}} & 0 \\ 0 & 0 \end{pmatrix}$ respectively. Let's look the following,

$$
\begin{aligned}
T_2 A_{r_t} T_2^{-1} &= \begin{pmatrix} T_{r_{t-1}} & 0 \\ 0 & E_{n-r_{t-1}} \end{pmatrix} \begin{pmatrix} E_{r_t} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T^{-1}_{r_{t-1}} & 0 \\ 0 & E_{n-r_{t-1}} \end{pmatrix} \\
&= \begin{pmatrix} T_{r_{t-1}} & 0 & 0 \\ 0 & E_{r_t-r_{t-1}} & 0 \\ 0 & 0 & E_{n-r_t} \end{pmatrix} \begin{pmatrix} E_{r_{t-1}} & 0 & 0 \\ 0 & E_{r_t-r_{t-1}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T^{-1}_{r_{t-1}} & 0 & 0 \\ 0 & E_{r_t-r_{t-1}} & 0 \\ 0 & 0 & E_{n-r_t} \end{pmatrix} \\
&= \begin{pmatrix} T_{r_{t-1}} E_{r_{t-1}} T^{-1}_{r_{t-1}} & 0 & 0 \\ 0 & E_{r_t-r_{t-1}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} E_{r_{t-1}} & 0 & 0 \\ 0 & E_{r_t-r_{t-1}} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} E_{r_t} & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}
$$

That is to say, in $T_2 Y T_2^{-1}$, not only $A_{r_{t-1}} = \begin{pmatrix} E_{r_{t-1}} & 0 \\ 0 & 0 \end{pmatrix}$ and $A_{r_{t-2}} = \begin{pmatrix} E_{r_{t-2}} & 0 \\ 0 & 0 \end{pmatrix}$, but also $A_{r_t} = \begin{pmatrix} E_{r_t} & 0 \\ 0 & 0 \end{pmatrix}$. We might as well to regard they are these forms and in Y.

Similarly, we can construct T_3, T_4, … , T_{t-1} such that in $T_{t-1} Y T_{t-1}^{-1}$, $A_t = \begin{pmatrix} E_{r_t} & 0 \\ 0 & 0 \end{pmatrix}$, $A_{r_{t-1}} = \begin{pmatrix} E_{r_{t-1}} & 0 \\ 0 & 0 \end{pmatrix}$, $A_{r_{t-2}} = \begin{pmatrix} E_{r_{t-2}} & 0 \\ 0 & 0 \end{pmatrix}$, … , $A_{r_1} = \begin{pmatrix} E_{r_1} & 0 \\ 0 & 0 \end{pmatrix}$. Since $T_{t-1} Y T_{t-1}^{-1} \simeq Y$, they are these forms and in Y.

On one hand,

$$
\left\{ \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} \,\middle|\, r = 0, 1, 2, \dots , n \right\}
$$

is the largest totally ordered semilattice of $M_{F,n}$, and any subset of it is a totally ordered semilattice of $M_{F,n}$. On the other hand, if Y is a totally ordered semilattice of $M_{F,n}$, then under one isomorphism mapping, Y is one of subset of

$$\left\{\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} \middle| \ r = 0, 1, 2, \dots, n \right\}.$$

So we have the following corollary.

Corollary 2.6. *$M_{F,n}$ has only $2^{n+1} - 1$ totally ordered semilattices, that is all subsets of*

$$\left\{\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} \middle| \ r = 0, 1, 2, \dots, n \right\}$$

except empty set.

Theorem 2.7. *If Y is a totally ordered semilattice of $M_{F,n}$, then there exists an invertible matrix $T \in M_{F,n}$ such that $Y = T \left\{\begin{pmatrix} E_{r_i} & 0 \\ 0 & 0 \end{pmatrix} \middle| \ i = 1, 2, \dots, t \ ; \ r_1 < r_2 \dots < r_t \right\} T^{-1}$ and the inflation S_Y* of *Y is constructed by*

$$S_Y = T \left(\bigcup_{i = 1, 2, \dots, t} \begin{pmatrix} E_{r_i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A \end{pmatrix} \right) T^{-1}.$$

$$A \in Z_{0_{(n-r_i)\times(n-r_i)}}$$

Proof. By theorem 2.5, we can suppose that

$$Y = \left\{\begin{pmatrix} E_{r_i} & 0 \\ 0 & 0 \end{pmatrix} \middle| \ i = 1, 2, \dots, t \ ; \ r_1 < r_2 \dots < r_t \right\}.$$

Since

$$(\forall B \in S_Y) \ (\exists \ r_i \in \{r_1, r_2 \dots, r_t\}) \ B \in Z_{E_{r_i,n}},$$

$$B = \begin{pmatrix} E_{r_i} & 0 \\ 0 & B_{22} \end{pmatrix}$$

where $B_{22} \in Z_{0_{(n-r_i)\times(n-r_i)}}$. By the lemma 2, we have

$$BE_{r_i,n} = E_{r_i,n} = E_{r_i,n}B.$$

Thus
$$B = \begin{pmatrix} E_{r_i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_{33} \end{pmatrix}$$

and $B_{22} = \begin{pmatrix} 0 & 0 \\ 0 & B_{33} \end{pmatrix}$ where $B_{33} \in Z_{0_{(n-r_i)\times(n-r_i)}}$ is an order $n - r_i$ square matrix. Let $i < j$. Then for all

$$\begin{pmatrix} E_{r_i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B \end{pmatrix}\begin{pmatrix} E_{r_j} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C \end{pmatrix}$$

$$= \begin{pmatrix} E_{r_i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B \end{pmatrix}\begin{pmatrix} E_{r_i} & 0 & 0 \\ 0 & E_{r_i} & 0 \\ 0 & 0 & C \end{pmatrix}$$

$$= \begin{pmatrix} E_{r_i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & BC \end{pmatrix} = \begin{pmatrix} E_{r_i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ (since } B,\ C \in Z_{0_{(n-r_i)\times(n-r_i)}}\text{)}.$$

Similary, we can find

$$\begin{pmatrix} E_{r_j} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C \end{pmatrix}\begin{pmatrix} E_{r_i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B \end{pmatrix} = \begin{pmatrix} E_{r_i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} E_{r_i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B \end{pmatrix}^2\begin{pmatrix} E_{r_j} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C \end{pmatrix} = \begin{pmatrix} E_{r_i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So by lemma 1, we know that S_Y is an inflation of Y.

Corollary 2.8. *Let Y be a totally ordered semilattice of $M_{F,n}$. If E_n or $TE_{n-1,n}T^{-1} \in Y$ where T is an invertible matrix in $M_{F,n}$, then Y has no proper inflation.*

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REFERENCES

1. Zhang Wajin, Generalized inverses of matrices over a skew field, *J. of Math. (PRC),* Vol, No. 1, 1986.
2. Mario petrich, *Introduction to semigroups,* Merrill, Columbus, 1973.
3. J. M. Howie, *An introduction to semigroup theory,* London New York San Francisco, 1976.
4. Stojan Bogdanovic, *Semigroups with a system of subsemigroups,* Novi Sad, 1985.

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SEMI-BOUNDEDNESS OF A SECOND-ORDER MATRIX DIFFERENTIAL OPERATOR

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ABSTRACT : Considering the semi-boundedness of the matrix differential operator

$$T = \begin{pmatrix} -D^2 + p_1(x) & q(x) \\ q(x) & -D^2 + p_2(x) \end{pmatrix}, \ D = d/dx$$

it has been proved that there is no spectrum below $C\text{-}K^2$, where $C = \text{Min}\ (C_1, C_2)$ and $K = \text{Max}\ (K_1, K_2)$ with

$$C_j = \underset{0 \le x < \infty}{Inf} \{ \frac{1}{W} \int_x^{x+w} | p_j - q | \, dt \}, \ k_j = \underset{0 \le x < \infty}{Sup} \{ \int_x^{x+w} | p_j - q | \, dt \},$$

$j = 1, 2$ and w is any constant greater than zero, $p_1(x)$, $p_2(x)$, $q(x)$ are real-valued and absolutely continuous in $[0, \infty)$ and $p_1, q \ge 0$.

Key words : Hilbert space, Limit-n at infinity, Variational principle, Eigenvalue, Spectrum, Holder's inequality.

AMS-MOS Mathematics Subject Classifications : 34 B 25, 47 A 10

1. INTRODUCTION

We consider the matrix differential expression,

$$Ly(x) = \begin{pmatrix} -D^2 + p_1(x) & q(x) \\ q(x) & -D^2 + p_2(x) \end{pmatrix} y(x), \qquad \dots (1.1)$$

$y = (y_1, y_2)^t$, $p_1, q \ge 0$, p_2 be real-valued and absolutely continuous in $[0, \infty)$ and introduce the differential operator T defined in the Hilbert space $L^2\ (0, \infty)$ as follows :

The domain D(T) consists of those functions $f = (f_1, f_2)^t$ in $L^2(0, \infty)$ such that

(i) $f_1'(x)$, $f_2'(x)$ are absolutely continuous in $[0, X]$, $X > 0$

(ii) $Lf(x)$ is an $L^2(0, \infty)$

(iii) $a_{j1} f_1(0) + a_{j2} f_2(0) + a_{j3} f_1'(0) + a_{j4} f_2'(0) = 0, \; j = 1, 2$ ... (1.2)

Then, for f in D(T)

$$(Tf)(x) = \begin{pmatrix} -D^2 + p_1(x) & q(x) \\ q(x) & -D^2 + p_2(x) \end{pmatrix} f(x),$$

a_{jk} ($j = 1, 2$; $k = 1, 2, 3, 4$) are real-valued constants and satisfy

$$a_{12} a_{24} - a_{14} a_{22} + a_{11} a_{23} - a_{13} a_{21} = 0 \quad \text{... (1.3)}$$

$$a_{13} a_{22} - a_{12} a_{23} + a_{14} a_{21} - a_{11} a_{24} = 1 \quad \text{... (1.4)}$$

$$a_{13} a_{24} - a_{14} a_{23} \text{ and } a_{11} a_{22} - a_{12} a_{21} \neq 0 \quad \text{... (1.5)}$$

The operator T will be marked as T_D or T_N when the boundary condition (1.2) is replaced either by Dirichlet boundary condition

$$f_1(0) = f_2(0) = 0 \quad \text{... (1.6)}$$

or, by Neumann boundary condition

$$f_1'(0) = f_2'(0) = 0 \quad \text{... (1.7)}$$

and accordingly the problem will be termed as Dirichlet problem (DP) or Neumann problem (NP).

In this paper we shall establish a lower bound of the spectrum utilising the semi-boundedness of the operator T_D and in a similar way for T_N, then considering the operator T in limit-2 case at infinity it can be seen that the same result holds in this case also. Previously, in [(6), (7)] the authors considered the differential expression defined by

$$M\, y(x) = - y''(x) + q(x)\, y(x),$$

q(x) is real-valued and continuous in $[0, \infty)$, and introduce the differential operator T_α $(0 \leq \alpha < \pi)$ in the Hilbert space $L^2(0, \infty)$ such that the domain $D(T_\alpha)$ consists of those functions which satisfy

(i) $f'(x)$ is absolutely continuous in $[0, X]$, for all $X > 0$

(ii) $Mf(x)$ is $L^2(0, \infty)$

(iii) $f(O) \cos \alpha + f'(0) \sin\alpha = 0$

Then, for f in $D(T_\alpha)$

$$T_\alpha f = Mf$$

The object of study of these papers was to achieve the operator T_α to be bounded below. For which the necessary condition is that T_α is to be self-sadjoint. As we know the limit-point case ensures T_α to be self-adjoint ; to make $M[\cdot]$ in the limit point case at ∞, the restrictions on q were taken as

(i) $q(x)$ is $L^p(0, \infty)$ for some $p \geq 1$ in [7] and

(ii) $\int_x^{x+w} | q(x) | \, dx$ is bounded for all x in $[0, \infty)$, for fixed $\omega > 0$, in [6]

In [7] it was shown that the lower bound for T_0 is $- k(p)$, where,

$$k(p) = (2p - 1)\, 2^{-2(p-1)/(2p-1)} p^{-2p/(2p-1)} \left(\int_0^\infty | q(x) |^p dx \right)^{1/p}$$

while for $\alpha \neq 0$, the spectra of T_α has atmost one point below $- k(p)$.

In [6] the operators were shown to be bounded below and obtained the lower bound $- k = C - K^2$, for T_0 or $T_{\pi/2}$, where

$$K = \sup_{0 \leq x < \infty} \left(\int_x^{x+w} | q (t) | \, dt \right), \quad C = \inf_{0 \leq x < \infty} \left(w^{-1} \int_x^{x+w} q (t) \, dt \right)$$

while for α in the open ranges $(0, \pi/2)$ or $(\pi/2, \pi)$ there is atmost one point in the spectrum of T_α below $- k$.

To make our problem limit-2 case at infinity, we assume the restrictions on $p_1(x)$, $p_2(x)$, $q(x)$ as in [12], which states that :

For $N(x)$ be a positive, absolutely continuous function of x satisfying

(i) $\int_0^\infty [N(x)]^{-1/2} \, dx$ diverges ... (1.8)

(ii) $\lim \sup_{x \to \infty} N'(x) [N(x)]^{-3/2}$ exist finitely, ... (1.9)

then if $p_j(x) \geq -k_j N(x)$, $j = 1, 2$; $| q(x) | \leq k_3 N(x)$ hold for all positive k_j,... (1.10)

$M[\cdot]$ is in the limit-2 case at infinity.

Clearly, this result holds for $N(x) = 0(x^k)$, $2 \geq k > -2$ and (1.10) implies $| p_j(x) - q(x) | = 0(x^k)$, $2 \geq k > -2$. For, k = positive, there are constants w and k, where $w > 0$, such that

$$\int_{x}^{x+w} | p_j(t) - q(t) | \, dt \leq k, \; j = 1,2 \qquad \ldots (1.11)$$

for all x in $[0, \infty)$. For k = negative,

$$| p_j(x) - q(x) | \in L^p (0, \infty), \; j = 1,2 \qquad \ldots (1.12)$$

for some $p \geq 1$.

2. SPECTRAL CLASSIFICATION

Let T be a self-adjoint operator defined on a manifold $\mathcal{D}_T$ which is dense in $\mathcal{H}$. We consider the equation

$$Tf - \lambda f = g$$

where λ is a complex parameter. We denote the range of the operator $T - \lambda I$ by $\Delta_T(\lambda)$ and is defined by $\Delta_T(\lambda) = (T - \lambda I)\mathcal{D}_T$. Thus the operator $T - \lambda I$ gives rise to a correspondence between $\mathcal{D}_T$ and $\Delta_T(\lambda)$. The correspondence is one to one when the operator $(T - \lambda I)$ has an inverse operator $(T - \lambda I)^{-1}$ with domain $\Delta_T(\lambda)$ and range $\mathcal{D}_T$.

We say that λ is a regular point of the operator T if $(T - \lambda I)^{-1}$ exists and is bounded operator defined on the whole spece $\mathcal{H}$. The collection of all regular points of T is called the resolvent set of T and denoted by $\rho(T)$ and since T is self adjoint, $\lambda \in \rho(T)$ if im $\lambda \neq 0$. The complement of $\rho(T)$ in the complex plane is called the spectrum of T and denoted by $\sigma(T)$. Thus all points in $\sigma(T)$ are real and also it is known that $\sigma(T)$ is a closed set, both on the real line and in the complex plane, so $\rho(T)$ is an open set on the plane.

The operator $R_\lambda = (T - \lambda I)^{-1}$, which depends on the parameter λ, is called the resolvent of the operator T, and is defined for all values of λ for which it exists and for which its domain of definition $\Delta_T(\lambda)$ is dense in $\mathcal{H}$.

With the resolvent operator defined for all λ in the complex plane we have the following classification of points in $\rho(T)$ and $\sigma(T)$ in terms of the resolvent operator R_λ :

(A) $\lambda \in \rho(T)$ if and only if R_λ is a bounded operator defined on the whole of $\mathcal{H}$.

(B) $\lambda \in \rho\sigma(T)$ if and only if R_λ is a bounded operator defined on a set which is not dense in $\mathcal{H}$.

(C) $\lambda \in C\sigma(T)$ if and only if R_λ is an unbounded operator defined on a set which is dense in $\mathcal{H}$.

(D) $\lambda \in PC\sigma(T)$ if and only if R_λ is an unbounded operator defined on a set which is dense in $\mathcal{H}$.

[This refers Akhiezer & Glazman[1] Chapter IV, §43 and Chaudhuri & Everitt[4], §4].

3. PRELIMINARIES

We state below some useful results in order to use them in the limit-2 case of the operator T and T_D (or T_N).

Let $\phi_r\,(x, \lambda) = (u_r\,(x, \lambda), v_r(x, \lambda))^t$, r = 1, 2 be the boundary condition vectors of (1.2), solutions of

$$Ly(x) = \lambda y(x) \qquad \dots (3.1)$$

and satisfy

$$\left.\begin{aligned} u_r\,(0, \lambda) &= -a_{r3}, \quad u_r'(0, \lambda) = a_{r1} \\ v_r\,(0, \lambda) &= -a_{r4}, \quad v_r'(0, \lambda) = a_{r2} \end{aligned}\right\} (r = 1, 2) \qquad \dots (3.2)$$

There are two more solutions of (3.1), $\theta_r(x, \lambda) = (x_r(x, \lambda), y_r\,(x, \lambda))^t$, r = 1, 2, which are not unique, determined by :

$$\left.\begin{aligned} x_k\,(0, \lambda) &= (-1)^{k+1}\, a_{l4}, \quad x_k'\,(0, \lambda) = (-1)^k\, a_{l2} \\ y_k\,(0, \lambda) &= (-1)^{k+1}\, a_{l3}, \quad y_k'\,(0, \lambda) = (-1)^k\, a_{l1} \end{aligned}\right\} \qquad \dots (3.3)$$

[when k = 1, l = 2 and when k = 2 l = 1 ; see (11), §2]

It has been proved by Chakravarty[2] that (3.1) has atleast two linearly independent solutions $\psi_1(x, \lambda)$ and $\psi_2(x, \lambda)$ belonging to $L^2\,(0, \infty)$ given by

$$\psi_r(x, \lambda) = \theta_r\,(x, \lambda) + \sum_{s=1}^{2} M_{rs}\,\phi_s\,(x, \lambda), \; (r = 1, 2) \qquad \dots (3.4)$$

for $\lambda \in \rho(T)$, the resolvent set of the operator T.

In the case of the operator T_D, the two square-integrable linearly independent solutions are

$$\tilde{\psi}_r(x, \lambda) = \tilde{\theta}_r\ (x, \lambda) + \sum_{s=1}^{2} m_{rs}\ \tilde{\phi}_s\ (x, \lambda),\ (r = 1, 2) \qquad \ldots (3.5)$$

$\lambda \in \rho(T_D)$, the resolvent set of the operator T_D, where $\tilde{\phi}_r$ (the boundary condition vector of (1.6)), $\tilde{\theta}_r$, r = 1, 2 satisfy initial conditions at x = 0

$$\left.\begin{aligned} \tilde{\phi}_1(0, \lambda) &= \begin{pmatrix}1\\0\end{pmatrix},\ \tilde{\phi}_1'(0, \lambda) = \begin{pmatrix}0\\0\end{pmatrix} \\ \tilde{\phi}_2(0, \lambda) &= \begin{pmatrix}0\\1\end{pmatrix},\ \tilde{\phi}_2'(0, \lambda) = \begin{pmatrix}0\\0\end{pmatrix} \end{aligned}\right\} \qquad \ldots (3.6)$$

$$\tilde{\theta}_1(0, \lambda) = \tilde{\theta}_2(0, \lambda) = \begin{pmatrix}0\\0\end{pmatrix},\ \tilde{\theta}_1'(0, \lambda) = \begin{pmatrix}1\\0\end{pmatrix}\ \tilde{\theta}_2'(0, \lambda) = \begin{pmatrix}0\\1\end{pmatrix} \qquad \ldots (3.7)$$

4. AN USEFUL LEMMA

Here we establish a result regarding the lowest eigenvalue of the Neumann problem

$$Ly(x) = \lambda\ y(x) \qquad \ldots (4.1)$$

in an interval [a, b] and the boundary conditions are

$$\left.\begin{aligned} y_1'(a) &= y_2'(a) = 0 \\ y_1'(b) &= y_2'(b) = 0 \end{aligned}\right\} \qquad \ldots (4.2)$$

using a method of Kato[8], modified by Eastham[6].

Lemma. For the DP or NP the lowest eigen value λ_0 satisfies an inequality

$$\text{Min}\ [C_j - \frac{1}{4}(\int_a^b |\ p_j - q - C_j\ |\ dx)^2] \le \lambda_0 \le \frac{C_1 + C_2}{2},\ j = 1, 2 \qquad \ldots (4.3)$$

where $C_j = \frac{1}{b-a} \int_a^b (p_j - q)\ dx,\ j = 1, 2.$

Proof. Using Ritz variational principle, we have for the Neumann problem

$$\lambda_0 = \text{Min} \left[\frac{\int_a^b (f_1'^2 + f_2'^2 + p_1 f_1^2 + 2q f_1 f_2 + p_2 f_2^2) dx}{\int_a^b (f_1^2 + f_2^2) dx} \right] \qquad \ldots (4.4)$$

The minimum being taken over all real-valued functions $f(x) = (f_1(x), f_2(x))^t$ where $f_1(x)$, $f_2(x)$, are continuous and piece-wise smooth in [a, b]. The right hand inequality in (4.3) follows on choosing $f(x) = k\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or $k\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, where k be any constant.

Let $\psi(x) = (\psi_1(x), (\psi_2(x))^t$ denote a real-valued normalized eigenvector corresponding to λ_0 for the NP and let $h(x) = (h_1(x), h_2(x))^t$ be any real-valued, continuous and piece-wise smooth function in [a, b] such that

$$h(a) = h(b) = 0. \qquad \ldots (4.5)$$

Then

$$\begin{aligned} \lambda_0 &= \int_a^b \{\psi_1'^2 + \psi_2'^2 + p_1 \psi_1^2 + 2q\psi_1\psi_2 + p_2 \psi_2^2\} dx \\ &= \int_a^b [(\psi_1' + h_1\psi_1)^2 + (\psi_2' + h_2\psi_2)^2 + (p_1 - h_1^2)\, \psi_1^2 + (p_2 - h_2^2)\psi_2^2] dx \\ &\quad -2 \int_a^b (h_1\psi_1\psi_1' + h_2\psi_2\psi_2')\, dx + 2\int_a^b q\psi_1\psi_2 dx \\ &\geq \int_a^b [(p_1 - q - h_1^2 + h_1')\, \psi_1^2 + (p_2 - q - h_2^2 + h_2')\psi_2^2] dx \end{aligned}$$

using (4.5) and the fact that $q \geq 0$.

So that

$$\lambda_0 \geq \text{Min}\ [\inf_{a \leq x \leq b} (p_j - q - h_j^2 + h_j')],\ j = 1, 2 \qquad \ldots (4.6)$$

we choose $h_j(x) = -\int_a^x \{p_j(x) - q(x) - C_j\} dx \qquad \ldots (4.7)$

Clearly h_1, h_2 satisfy (4.5). Then (4.6) can be expressed as

$$\lambda_0 \geq \text{Min}\ [C_j - \sup_{a \leq x \leq b} (\int_a^x (p_j - q - C_j) dx)^2],\ j = 1, 2 \qquad \ldots (4.8)$$

Now let $Q_j(x) = p_j(x) - q(x) - C_j$, $j = 1, 2$ then by (4.7),

$$\int_a^b Q_j(x)dx = 0, \; j = 1, 2.$$

Writing these as

$$\int_a^\lambda Q_j(x)dx + \int_\lambda^b Q_j(x)dx = 0$$

we have,

$$2\int_a^\lambda Q_j(x)dx = \int_a^\lambda Q_j(x)dx - \int_x^b Q_j(x)dx$$

so that

$$-\int_a^\lambda | Q_j(x) | dx - \int_x^b | Q_j(x) | dx \le 2\int_a^x Q_j(x)dx \le \int_a^x | Q_j(x) | dx + \int_x^b | Q_j(x) | dx$$

$$\text{i.e.} \; -\frac{1}{2}\int_a^b | Q_j(x) | dx \le \int_a^\lambda Q_j(x)dx \le \frac{1}{2}\int_a^b | Q_j(x) | dx \qquad \ldots (4.9)$$

Utilising (4.9) in (4.8) we obtain the left hand inequality of (4.3). Again since

$$\int_a^b | p_j - q - C_j | dx \le 2\int_a^b | p_j - q | dx, \; j = 1, 2$$

we can write from (4.3)

$$\lambda_0 \ge \text{Min} \left[C_j - \int_a^b | p_j - q | dx\right], \; j = 1, 2 \qquad \ldots (4.10)$$

5. PROOF OF THE MAIN THEOREM

Theorem. For the case (1.11), let

(i) $\underset{0 \le x < \infty}{Sup} \int_x^{x+w} | p_j(x) - q(x) | dx = K_j$, $j = 1, 2$

(ii) $K = \text{Max} (K_1, K_2)$

(iii) $C_j = \underset{0 \le x < \infty}{Inf} \{ \frac{1}{w} \int_x^{x+w} | p_j(x) - q(x) | \, dx \}, \ j = 1, 2$

(iv) $C = \text{Min}\,(C_1, C_2)$

Then DP or NP, there is no spectrum below $-k$, where $k = K^2 - C$, the same result holds for the operator T problem also. [Compare Naimark[10] corollary 4, § 14.11].

Proof. We divide up $[0, \infty)$ by the points

$$nw, \ (n = 0, 1, 2, \ldots,)$$

Let $\lambda_0(N)$ denote the least eigenvalue associated with DP in [0 Nw], N is an integer which we shall let tend to infinity and $\mu_0'(n)$ denote the least eigenvalue associated with NP in [nw, (n + 1)w]. Now following as in Th. 7.2(i), Chakravarty and Sengupta[3] we obtain

$$\lambda_0(N) \ge \underset{0 \le n \le N-1}{Min} \mu_0'(n) \qquad \ldots (5.1)$$

and applying (4.10) to $\mu_0'(n)$ we find

$$\lambda_0(N) \ge \text{Min}\,(C_1 - K_1^2, \ C_2 - K_2^2) = -k \qquad \ldots (5.2)$$

holds for all N.

It is known from Chakravarty[2] that $m_{rs}(\lambda)$, (r, s = 1, 2) as defined there, have poles on the real axis and that they are all simple ; from these and (5.2) it follows that all $m_{rs}(\lambda)$ for T_D or T_N are regular for Re $\lambda < -k$. It can be proved in a similar way as in Das and Banerjee[5], Theorem I that all $m_{rs}(\lambda)$ are regular only when λ belongs to the resolvent set of T_D (or T_N). The set of spectrum is the complement of the resolvent set and hence the spectrum of T_D or T_N is bounded below by $-k$.

To prove the last part of the theorem, first we consider the operator T in the limit -2 case at infinity and show that all $M_{rs}(\lambda)$ exist finitely for Re $\lambda < -k$.

From (3.4) and (3.5) we see $\psi_r(x, \lambda)$ and $\tilde{\psi}_r(x, \lambda)$, r = 1, 2 are the two sets of square-integrable solutions in $(0, \infty)$ for the T problem and T_D (or T_N) problem respectively and since both the problems belong to limit-2 case at infinity there exist constants α, β, γ, δ independent of x such that

$$\left. \begin{aligned} \psi_1(x, \lambda) &= \alpha \tilde{\psi}_1(x, \lambda) + \beta \tilde{\psi}_2(x, \lambda) \\ \psi_2(x, \lambda) &= \gamma \tilde{\psi}_1(x, \lambda) + \delta \tilde{\psi}_2(x, \lambda) \end{aligned} \right\} \qquad \ldots (5.3)$$

Applying initial conditions (3.2), (3.3), (3.6) and (3.7) to these equations we obtain following eight equations (compare Majumder and Das[9], § 4).

$$\begin{aligned}
a_{13}M_{11} + a_{23}M_{12} + m_{11}\alpha + m_{21}\beta &= a_{24}\\
a_{14}M_{11} + a_{24}M_{12} + m_{12}\alpha + m_{22}\beta &= a_{23}\\
a_{11}M_{11} + a_{21}M_{12} - \alpha &= a_{22}\\
a_{12}M_{11} + a_{22}M_{12} - \beta &= a_{21}\\
a_{13}M_{21} + a_{23}M_{22} + m_{11}\gamma + m_{21}\delta &= -a_{14}\\
a_{14}M_{21} + a_{24}M_{22} + m_{12}\gamma + m_{22}\delta &= -a_{13}\\
a_{11}M_{21} + a_{21}M_{22} - \gamma &= -a_{12}\\
a_{12}M_{21} + a_{22}M_{22} - \delta &= -a_{11} \qquad \dots (5.4)
\end{aligned}$$

Since these eight functions $M_{rs}(\cdot)$, $(r, s = 1, 2)$ and $\alpha, \beta, \gamma, \delta$ can be determined uniquely, the rank of the co-efficient matrix in the equations (5.4) should be non-zero. Whence it follows that Δ not equal to zero. Therefore, as the poles of $m_{rs}(\lambda)$ occur only when $\operatorname{Re}\lambda \geq -k$, the poles of $M_{rs}(\lambda)$ cannot occur for $\operatorname{Re}\lambda < -k$.

Corollary : For the case (1.12), the spectrum associated with the problem T_D (or T_N) are bounded below by $-l_s$ where

$$l_s = (2s-1)\,\{2(s-1)\}^{-2(s-1)/2s-1}\operatorname{Max}\,[\| p_j - q \|_s^{2s/2s-1}],\ j = 1, 2 \qquad \dots (5.5)$$

with $\| f \|_s = \left(\int_0^\infty | f |^s\right)^{1/s}$.

Proof. We first prove (5.5) for $s = 1$. From the expressions of K_1, K_2 as given in the Theorem, we find

$$K_j^2 - C_j \leq \| p_j - q \|_1^2 + w^{-1} \| p_j - q \|_1,\ j = 1, 2$$

so that the spectrum of the problem T_D (or T_N) are bounded below by

$$-\operatorname{Max}\,\{ \| p_j - q \|_1^2 + w^{-1} \| p_j - q \|_1 \},\ j = 1, 2$$

and as $w \to \infty$ the lower bound turns out to be

$$-\operatorname{Max}\,\{ \| p_j - q \|_1^2 \},\ j = 1, 2$$

Now for $s > 1$, we introduce h such that $\frac{1}{s} + \frac{1}{h} = 1$, to utilise the Holder's inequality to obtain

$$\begin{aligned} K_j^2 - C_j &\le \| p_j - q \|_1 (\| p_j - q \|_1 + w^{-1}) \\ &\le \| p_j - q \|_s w^{1/h} (w^{-1} + \| p_j - q \|_s + w^{1/h}) \\ &= w^{2/h} \| p_j - q \|_s^2 + w^{-1/s} \| p_j - q \|_s \end{aligned}$$

for $j = 1, 2$

Thus for the problem T_D (or T_N) the spectrum are bounded below by

$$- \text{Max} \, [\| p_j - q \|_s (w^{2/h} \| p_j - q \|_s + w^{-1/s})], \; j = 1, 2 \qquad \ldots (5.6)$$

Now the expressions $\{2(s - 1)\}^{s/1 - 2a} \| p_j - q \|_s^{s/1 - 2s}$, $j = 1, 2$ for w minimizes $\| p_1 - q \|_s (w^{2/h} \| p - q \|_s + w^{-1/s})$, $j = 1, 2$ respectively. Substituting these expressions for w in (5.6) we find that

$$l_s = (2s - 1) \{2(s - 1)\}^{-2(s-1)/2s-1} \, \text{Max} \, [\| p_j - q \|_s^{2s/2s-1}], \; j = 1, 2$$

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REFERENCES

1. N. I. Akhiezer and I. M Glazman ; *Theory of linear operators in Hilbert space,* Vol. I, New York, Ungar.
2. N. K. Chakravarty, Some problems in eigen-function expansions (III), *Q, J. Math. Oxford* (2) 19 (1968), 397-415.
3. N. K. Chakravarty and P. K. Sengupta, On the distribution of the eigenvalues of a matrix differential operator, *Jour. Ind. Inst. Sci.,* 61(B), 1979, 19-42.
4. J. Chaudhuri and W. N. Everitt, The spectrum of a fourth-order differential operator, *Proc. Roy. Soc. Edinb.,* 68A, (1968) 95-119.
5. J. Das and S K. Banerjee, An alternative characterisation of the spectrum of a self-adjoint operator associated with certain fourth-order formally self-adjoint ordinary differential expression, *Indian. Jour. Mathematics,* 30(1), 1988, 1-8.
6. M. S. P. Eastham, Semi-bounded second-order differential operators, *Proc. Roy. Soc. Edinb.* 72A (1972/73), 9-16.
7. W. N. Everitt, On spectrum of a second order linear differential equation with p-integrable coefficient. *Applicable Analysis* (2) (1972), 143-160.

8. T. Kato, Note on the least eigenvalues of Hille's equation *Q, Appl. Math.* 10 (1952), 292-294.

9. M. Majumder and J. Das, On the invariance of the nature of singularities of the m-coefficients associated with an ordinary linear fourth-order differential equation. *Jour. Pure. Math* 3 (1983), 25-30

10. M. A. Naimark, *Linear Differential operators,* Part II, Frederick Ungar Publishing Co., New York, 1968.

11. P. K. Sengupta, On the convergence of eigenfunction expansions associated with a vector matrix differential equation, *Jour. Ind. Inst. Sci.,* 60(A), 1978, 89-110.

12. _____, The limit-2 case of a second-order differential system. *J. Indian Inst. Sci.,* 65(B), 1984, 163-172.

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A GENERALIZED FK-SPACE OF ENTIRE FUNCTIONS OF *M*-BOUNDED INDEX

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ABSTRACT : In this paper we have shown that the set of all entire functions of M-bounded index B_M is a generalized FK-space. We have also characterized the form of continuous linear functionals on B_M. Finally we have constructed a total set on B_M.

Key words : M-bounded index, Locally convex linear topological vector space. Total set

AMS Subject Classification : 32A

1. INTRODUCTION

Definition 1. [1] An entire function f is said to be of M-bounded index if and only if there exists an integer N such that

$$\frac{M_n(r)}{n!} \leq \max_{0\leq j\leq N} \left\{\frac{M_j(r)}{j!}\right\} \text{ for all } r > 0 \text{ and for all integres } n \geq 0, \text{ where}$$

$$M_n(r) = \max_{|z|=} \{|f^{(n)}(z)|\} \text{ for all } n \geq 0,\ f^{(0)}(z) = f(z) \text{ and } M_0(r) = M(r).$$

The least of such integers N is called the M-index of f. It was shown [1] that an entire function f is of M-bound index if and only if f is of exponential type, i.e. f is of order 1 and finite type or f is of order less than 1.

Definition 2. Let X be a vector space over the field K. A paranorm on X is a function $g : X \to R$ which satisfies the following axioms : for $x, y \in X$, (i) $g(0) = 0$, (ii) $g(x) = g(-x)$, (iii) $g(x + y) \leq g(x) + g(y)$, and (iv) $\lambda \to \lambda_0$, $x \to x_0$ imply $\lambda x \to \lambda_0 x_0$ where $\lambda, \lambda_0 \in K$ and $x_0 \in X$.

Definition 3. We define a generalized FK-space X as one which satisfies the following conditions :

(a) X is a linear space over the field of real or complex numbers and that elements of X can be expressed as real or complex sequences.

(b) X is a locally convex topological linear space in which the topology is given by a countable family of paranorms $\{p_n\}$.

(c) X is metrizable and as a metric space (the metric which induces the locally convex topology) it is complete.

(d) If $\{\alpha_n\}$ is a base for X so that for $x \in X$, $x = \sum_{n=1}^{\infty} \phi_n(x)\alpha_n$

then $\phi_n(x)$ $(n \geq 1)$ are continuous linear functionals on X.

§2. Throughout this paper we shall adopt the following notations :

$B_M = \{f : f$ is an entire function of M-bounded index$\}$

Throughout we shall take $f(z) = \sum_{n=0}^{\infty} a_n z^n$, unless otherwise stated.

X = A fixed monotonic increasing sequence of positive real numnbers x_i such that $x_i \to \infty$ as $i \to \infty$.

Theorem 1. B_M is a generalized FK-space.

Proof. Addition and scalar multiplication in B_M are defined as follows :

$$(f+g)(z) = f(z) + g(z) \text{ and } (\lambda f)(z) = \lambda f(z),\ \lambda \in ¢.$$

Since $(f+g)$, $\lambda f \in B_M$ [4], it is easy to see that B_M is a complex linear space. The zero element of B_M i.e. $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $a_n = 0$ for all n, will be denoted by 0^*. We consider the sequence $\{\alpha_n\}$ in B_M, where $\alpha_n = z^n$, $n = 0, 1, 2\ldots$ Then the sequence $\{\alpha_n\}$is a basis for B_M since for any $f \in B_M$ we can write $f(z) = \sum_{n=0}^{\infty} a_n(f) z^n$ where

$$a_n(f) = \left.\frac{d^n f(z)}{dz^n}\right|_{z=0}.$$

Again if $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers satisfying the conditions :

(1) $\lim\limits_{n\to\infty} |a_n|^{1/n} = 0$

(2) $\limsup\limits_{n\to\infty} \dfrac{n\log n}{-\log|a_n|} \leq 1$

(3) If $\limsup\limits_{n\to\infty} \dfrac{n\log n}{-\log|a_n|} = 1$, and $\dfrac{1}{e} \limsup\limits_{n\to\infty} n|a_n|^{1/n} < \infty$, then $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n \equiv B_M$.

Here the condition (1) ensures that f is an entire function whereas condition (2) tells that order of f is ≤ 1 and condition (3) states that if the order of f is 1, then its type is finite. Conversely any function $f \in B_M$ will satisfy the conditions (1), (2) and (3). So, $f \in B_M$ can be expressed as a sequence of complex numbers : $\{a_0(f), \ldots, (a_r(f), \ldots\}$, satisfying the conditions (1), (2) and (3).

Let us now define a countable family of paranorms in B_M as follows : For $x_i \in X$, $i = 1, 2, \ldots$

$$\|f;x_i\| = x_i \sup\left\{|a_0|, |a_n|^{1/n} : n = 1, 2, \ldots\right\}$$

Consequently B_M becomes a locally convex linear topological space in which the locally convex topology is induced by the family of paranorms $\|f;x_i\|$, $i = 1, 2, \ldots$ [3].

Let us now define

$$\|f\| = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|f;x_i\|}{1+\|f;x_i\|} \text{ and } d(f, g) = \|f-g\|, f, g \in B_M .$$

The topology induced by the metric d is the same as induced by the family of paranorms $\left\{\|\ldots,x_i\| : i = 1, 2, \ldots\right\}$. Consequently, B_M is metrizable.

We now show that the metric space (B_M, d) is complete. Let $\{f_m\}_{m=1}^{\infty}$ be a Cauchy sequence in (B_M, d) where $f_m(z) = \sum\limits_{n=0}^{\infty} a_n^{(m)} z^n$ for $m = 1, 2, \ldots$ Then for any given $\varepsilon > 0$, there exists a positive integer N such that

$d(f_m, f_n) < \varepsilon$ for all $m, n \geq N$.

i.e. $\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|f_m - f_n ; x_i\|}{1+\|f_m - f_n ; x_i\|} < \varepsilon$ for all m, $n \geq N$.

i.e.

(4) $(1 - 2^i\varepsilon)\, x_i \sup \left\{ \left|a_0^{(m)} - a_0^{(n)}\right|, \left|a_j^{(m)} - a_j^{(n)}\right|^{1/j} : j = 1, 2 \ldots \right\} < 2^i \varepsilon$ for all m, $n \geq N$ and $i \geq 1$.

Hence

$$(5) \begin{cases} (1 - 2^i\varepsilon)\, x_i \left|a_0^{(m)} - a_0^{(n)}\right| < 2^i\varepsilon \\ \text{and for all } m,\ n \geq N \text{ and } i,\ j \geq 1 \\ (1 - 2^i\varepsilon)\, x_i \left|a_j^{(m)} - a_j^{(n)}\right|^{1/j} < 2^i\varepsilon \end{cases}$$

We fix i and choose ε such that $2^i\varepsilon < 1$. Now we fix j and consider the sequence $a_j^{(1)},\ a_j^{(2)}, \ldots$

Due to (5) this sequence will converge to a limit, say, a_j i.e.

(6) $\lim_{m \to \infty} a_j^{(m)} = a_j,\ j = 0,\ 1,\ 2,\ \ldots$

Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$. We now show that $f \in B_M$ and $f_m \to f$ as $m \to \infty$.

From (6) we have for any $\varepsilon > 0$,

$\left|a_j^{(m)} - a_j\right| < \varepsilon$ for all $m \geq N$ and $j \geq 0$

which implies that

(7) $|a_j|^{1/j} \leq \left|a_j^{(N)}\right|^{1/j} + \varepsilon$ for $j \geq 0$.

Since $\lim_{j \to \infty} \left|a_j^{(N)}\right|^{1/j} = 0$, we have from (7) $\lim_{j \to \infty} |a_j|^{1/j} = 0$. Hence f is an entire function.

We next show that f is of exponential type. Now, from (7), it follows that

$$\limsup_{j\to\infty} \frac{j\ln j}{-\ln|a_j|} \leq \limsup_{j\to\infty} \frac{j\ln j}{-\ln\left|a_j^{(N)}\right|}$$

$$= \text{order of } f_N \leq 1 \ [\ \because f_N \in B_M\,]$$

Hence the order of $f \leq 1$. Now, if the order of $f = 1$, then the order of $f_N = 1$. Hence $\frac{1}{e}\limsup_{j\to\infty} j\left|a_j^{(N)}\right|^{1/j}$ = type of $f_N < \infty$. Therefore from (7) it follows that f is of finite type when its order is 1 and consequently $f \in B_M$. Now, we show that $f_m \to f$ as $m \to \infty$ in (B_M, d). In (4) we fix i and choose ε such that $2^i\varepsilon < 1$ and making $n \to \infty$, we obtain

$$x_i \sup\left\{\left|a_0^{(m)} - a_0\right|, \left|a_j^{(m)} - a_j\right|^{1/j} : j = 1, 2, \ldots\right\} < \frac{2^i\varepsilon}{1-2^i\varepsilon} = \epsilon \text{ (say), for all } m \geq N$$

and for each $i \geq 1$ i.e. $\|f_m - f; x_i\| < \epsilon$ for all $m \geq N$ and for each i,which implies,

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|f_m - f; x_i\|}{1+\|f_m - f; x_i\|} \to 0 \text{ as } m \to \infty$$

i.e. $d(f_m, f) \to 0$ as $m \to \infty$. Hence $f_m \to f$ as $m \to \infty$ in (B_M, d).

Thus B_M is metrizable and as a metric space it is complete.

Let $f \in B_M$ and $f(z) = \sum_{n=0}^{\infty} a_n(f)z^n$. We now show that the mappings ϕ_n : $B_M \to$ ¢ defined by $\phi_n(f) = a_n(f)$, $n = 0, 1, 2, \ldots$, are continuous linear functionals. ϕ_n are obviously linear functionals on B_M. Since B_M is paranormed linear space, it is sufficient to prove the continuity of ϕ_n at zero element of B_M. Let $\{f_p\}_{p=1}^{\infty}$ be a sequence in B_M such that $f_p \to 0^*$ as $p \to \infty$, where $f_p(z) = \sum_{n=0}^{\infty} \phi_n(f_p)z^n$.

Then, $d(f_p, 0^*) < \varepsilon$ for all $p \geq p_0$ i.e., $\sum_{i=1}^{\infty} \frac{1}{2^i}\frac{\|f_p; x_i\|}{1+\|f_p; x_i\|} < \varepsilon$ for all $p \geq p_0$

which implies $\frac{1}{2^i}\frac{\|f_p; x_i\|}{1+\|f_p; x_i\|} < \varepsilon$ for all $p \geq p_0$ and $i = 1, 2, \ldots$

i.e., $(1 - 2^i\varepsilon) \parallel f_p \; ; \; x_i \parallel < 2^i\varepsilon$ for all $p \geq p_0$ and $i = 1, 2,\ldots$

Fixing i and choosing ε such that $2^i \varepsilon < 1$, we see that

$x_i \sup \{\phi_0 \, (f_p), \mid \phi_n \, (f_p) \mid^{1/n} : n = 1, 2,\ldots\} < \in$ for all $p \geq p_0$ and for each $i \geq 1$,

which implies that ϕ_n is continuous at 0^* for each $n \geq 0$ and consequently ϕ_n is continuous on B_M for each $n \geq 0$. Thus B_M is a generalized FK-space.

Theorem 2. Every continuous linear functional ϕ on the paranormed linear space $(B_M ,\|\ldots; x_i\| : i = 1, 2, \ldots)$ is precisely of the form $\phi\,(f) = \sum_{k=0}^{\infty} a_k\mu_k$ where $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $\{\mu_k\}_{k=0}^{\infty}$ is a sequence of complex numbers such that $\left\{\frac{\mu_k}{x_i}\right\}_{k=0}^{\infty}$ is bounded for each i.

Proof. Let ϕ be a continuous linear functional on the space $(B_M \,, \parallel \ldots \; ; \, x_i \parallel)$ and so there exists a positive constant M such that

$$\mid \phi \; (f) \mid \leq M \parallel f \; ; \; x_i \parallel \text{ for all } f \in B_M \text{ and } i = 1, 2,\ldots$$

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k \in (B_M ,\|\ldots; x_i\|)$ and $f_N(z) = \sum_{k=0}^{N} a_k z^k$, $N = 1, 2, \ldots$ Evidently, $f_N \in (B_M \,, \parallel \ldots \; ; \, x_i \parallel)$ for each $N = 1, 2, \ldots$ Let $\phi(z^k) = \mu_k$. Then the numbers μ_k are uniquely determined ϕ.

Now, $\parallel f - f_N \; ; \; x_i \parallel = x_i \sup \left\{|a_{N+1}|^{1/N+1}, |a_{N+2}|^{1/N+2}, \ldots\right\}$

Since $\lim\limits_{N\to\infty} |a_{N+1}|^{1/N+1} = 0$, we have $\parallel f - f_N \; ; \; x_i \parallel \to 0$ as $N \to \infty$ for each $i =$ 1, 2, ... Hence $\{f_N\}$ converges to f in $(B_M, \|\ldots; x_i\|)$ and so by the continuity of ϕ,

$$\phi(f) = \phi\left(\lim_{N\to\infty} f_N\right) = \lim_{N\to\infty} \phi(f_N) = \lim_{N\to\infty} \phi\left(\sum_{k=0}^{N} a_k z^k\right) = \lim_{N\to\infty} \sum_{k=0}^{N} a_k \phi(z^k)$$

$$= \lim_{N\to\infty} \sum_{k=0}^{N} a_k\mu_k = \sum_{k=0}^{\infty} a_k\mu_k$$

Now $\mid \mu_k \mid = \mid \phi(z_k) \mid \leq M \parallel z^k \, ; x_i \parallel = M x_i$

Hence $\left\{\frac{\mu_k}{x_i}\right\}_{k=0}^{\infty}$ is bounded for each x_i. Conversely, let the mapping $\phi : (B_M, \|\ldots; x_i\|) \to$ ¢ be given by $\phi(f) = \sum_{k=0}^{\infty} a_k \mu_k$ where the sequence $\left\{\frac{\mu_k}{x_i}\right\}_{k=0}^{\infty}$ is bounded for each i.

Since $\sum_{k=0}^{\infty} | a_k |$ is convergent and $\left\{\frac{\mu_k}{x_i}\right\}_{k=0}^{\infty}$ is a bounded sequence of complex numbers, it therefore implies that $\sum_{k=0}^{\infty} a_k \mu_k$ is convergent and ϕ is well-defined. Evidently, ϕ is a linear functional.

Now, for every $f \in (B_M, \|\ldots; x_i\|)$, $| \phi (f) | \le \sum_{k=0}^{\infty} | a_k \mu_k | \le M x_i \sum_{k=0}^{\infty} | a_k | < \infty$ for each i.

Hence ϕ is continuous.

§3. **Definition 3 [2].** If X is a locally convex topological vector space, then a set $A \subset X$ is said to be total if and only if $\phi \in X'$, such that $\phi(A) = \{0\}$ implies that $\phi \equiv 0$, where X' is the dual space of X.

Lemma. If G be a non-empty subset of ¢ and $f(z) = \sum_{n=0}^{\infty} | a_n z^n | \in (B_M, \|\ldots; x_i\|)$ then $f_\lambda(z) = \sum_{n=0}^{\infty} a_n \lambda^n z^n \in (B_M, \|\ldots; x_i\|)$ for any $\lambda \in G$.

Proof : Since $f \in (B_M, \|\ldots; x_i\|)$ we have

(i) $\lim_{n\to\infty} |a_n|^{1/n} = 0$, (ii) $\limsup_{n\to\infty} \frac{n\log n}{-\log|a_n|} \le 1$ and (iii) if $\limsup_{n\to\infty} \frac{n\log n}{-\log|a_n|} = 1$, then $\frac{1}{e} \limsup_{n\to\infty} n|a_n|^{1/n} < \infty$.

Now, for $\lambda \in G$, $\lim_{n\to\infty} \left|a_n\lambda^n\right|^{1/n} = |\lambda| \lim_{n\to\infty} |a_n|^{1/n} = 0$ (BY (i))

Hence f_λ is an entire function.

Also the order of $f_\lambda = \limsup_{n\to\infty} \frac{n\log n}{-\log|a_n|\,|\lambda|^n}$

$$= \limsup_{n\to\infty} \frac{n\log n}{-\log|a_n| - n\log|\lambda|} \leq 1 \quad \text{(BY (ii)}$$

Again, if $\limsup_{n\to\infty} \frac{n\log n}{-\log|a_n|\,|\lambda|^n} = 1$, then

$$\frac{1}{e} \limsup_{n\to\infty} n\left|a_n\lambda^n\right|^{1/n} = \frac{|\lambda|}{e} \limsup_{n\to\infty} n|a_n|^{1/n} < \infty \text{ (By (iii))}$$

Hence, $f_\lambda \in (B_M, \|\ldots; x_i\|)$ for any $\lambda \in G$.

Theorem 3. Let $f \in (B_M, \|\ldots; x_i\|)$ with $a_n \neq 0$ for all $n \geq 0$ and G be a subset of ¢ containing at least one finite limit point. Again let $A = \{f_\lambda / \lambda \in G\}$ where f_λ is defined as in the lemma. Then A is total in $(B_M, \|\ldots; x_i\|)$.

Proof : By the lemma we have $A \subseteq B_M$. Let ϕ be a continuous linear functional on $(B_M, \|\ldots; x_i\|)$ such that $\phi(A) = 0$.

Now by theorem (2), there is a sequence $\{\mu_n\}$ of complx numbers determined by ϕ, such that

$$\phi(g) = \sum_{n=0}^{\infty} b_n\mu_n, \text{ where } g(z) = \sum_{n=0}^{\infty} b_n z^n \in B_M \text{ and } \left\{\frac{\mu_n}{x_i}\right\}_{n=0}^{\infty} \text{ is bounded for each}$$

x_i. Then, for every $\lambda \in G$,

$$\phi(f_\lambda) = \sum_{n=0}^{\infty} a_n\mu_n\lambda^n = 0. \text{ Let } L(z) = \sum_{n=0}^{\infty} a_n\mu_n z^n.$$

Now, we show that $L \in (B_M, \|\ldots; x_i\|)$.

Since $\left\{\frac{\mu_n}{x_i}\right\}_{n=0}^{\infty}$ is bounded for each x_i there exists a positive constant M such that $|\mu_n| \le Mx_i$ for all n and for each x_i.

Hence, $|a_n\mu_n|^{1/n} \le \left(Mx_i|a_n|^{1/n}\right)$ for all n and for each x_i.

Since $\lim\limits_{n\to\infty}|a_n|^{1/n} = 0$, it follows that $\lim\limits_{n\to\infty}|a_n\mu_n|^{1/n} = 0$ and consequently L is an entire function. Again, since $f \in B_M$, from (2) we get $\limsup\limits_{n\to\infty}\frac{n\log n}{-\log|a_n|} \le 1$.

$$\text{Hence, the order of } L = \limsup_{n\to\infty}\frac{n\log n}{-\log|a_n\mu_n|} \le \limsup_{n\to\infty}\frac{n\log n}{-\log|a_n| - \log Mx}$$

$$= \limsup_{n\to\infty}\frac{n\log n}{-\log|a_n|} \le 1.$$

Now, if $\limsup\limits_{n\to\infty}\frac{n\log n}{-\log|a_n\mu_n|} = 1$, then $\limsup\limits_{n\to\infty}\frac{n\log n}{-\log|a_n|} = 1$

and hence from (3), we have

$\frac{1}{e}\limsup\limits_{n\to\infty} n|a_n|^{1/n} < \infty$ which implies that

$\frac{1}{e}\limsup\limits_{n\to\infty} n|a_n\mu_n|^{1/n} < \infty$ and consequently it follows $L \in B_M$.

Also $L(\lambda) = 0$ for all $\lambda \in G$. This implies that $L(z) \equiv 0$ since G contains a finite limit point. Hence $a_n\mu_n = 0$ for all n and as $a_n \ne 0$ for all $n \ge 0$, it therefore follows that $\mu_n = 0$ for all $n \ge 0$. Hence $\phi \equiv 0$. This completes the proof.

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REFERENCES

1. Fricke, G. H. ; Shah, S. M. ; Sisarcick, W. C., A characterization of entire functions of exponential type and M-bounded index, *Indian University Mathematics Journal,* vol. 23, no. 5 (1973) 405-412.
2. Goffman, C. and Pedrick, G., *First course in functional analysis,* Prentice-Hall, Englewood Cliffs, N. J. (1965).
3. Husain, T., *The open and closed graph theorems in topological vector space,* Clarendon Press, Oxford (1965).
4. Mitra, Rita, *Entire functions of bounded index and the space of entire functions in several complex variables,* Doctoral thesis, Calcutta University, 1994.

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ON THE SECTIONAL CURVATURE OF THE HYPERBOLIC KÄHLERIAN MANIFOLDS OF INDEFINITE METRIC

Raina Ivanova

1. INTRODUCTION

The manifolds of indefinite metric have been studied in the last two decades. In 1979 Kulkarni [1] considered the connected manifolds of indefinite metric and proved that if the sectional curvature of such a manifold is bounded from above or from below, then the manifold is of constant sectional curvature. In 1982 Barros and Romero [2] studied the manifolds of indefinite Kählerian metric and proved that these manifolds have the same property. Kassabov and Borissov [3] proved a similar result for the Kählerian manifolds of indefinite metric. We have considered the hyperbolic and the almost hyperbolic Kählerian manifolds of indefinite metric recently [4]. In the present paper we study the holomorphic sectional curvature of the hyperbolic Kählerian manifolds of indefinite metric. Some conditions for that kind of manifolds to be of constant holomorphic sectional curvature are found out.

2. PRELIMINARIES

Let M be a pseudo-Riemannian manifold with an indefinite metric tensor g and a structure J. Let $T_p(M)$ be the tangent space at a point $p \in M$ and $X, Y \in T_p(M)$.

Definition 2.1. The pesudo-Riemannian manifold (M, g, J) is called *a hyperbolic Kählerian manifold* if the metric tensor g and the structure J of M satisfy the properties

$$J^2 = \mathrm{id}, \quad g(JX, JY) = -g(X,Y), \quad \nabla J = 0$$

with respect to the Levi-Civita connection ∇.

The dimension of any hyperbolic Kählerian manifold is an even number.

Let M be a 2m—dimensional hyperbolic Kählerian manifold and R be the curvature tensor of M. Then R has the properties :

1. $R(X, Y, Z, U) = - R(Y, X, Z, U)$,
2. $R(X, Y, Z, U) = - R(X, Y, U, Z)$,
3. $R(X, Y, Z, U) + R(Y, Z, X, U) + R(Z, X, Y, U) = 0$,
4. $R(X, Y, Z, U) = - R(JX, JY, Z, U)$,

The last equation implies $R(X, Y, Z, U) = - R(X, Y, JZ, JU)$,

Definition 2.2. Let x, y be tangent vectors at a point $p \in M$. The pair $\{x, y\}$ is said to be *an orthonormal pair of vectors of signature* (+, –) if

$$g(x, y) = 0, \; g(x, x) = 1, \; g(y, y) = -1.$$

Pairs of signature (+, +) or (–, –) can be defined analogously.

Definition 2.3. The plane σ is called *holomorphic* if $\sigma = J\sigma$.

Definition 2.4. The plane σ is called *antiholomorphic* if $\sigma \perp J\sigma$, $\sigma \neq J\sigma$. It means that σ is a holomorphic plane if and only if σ has a basis $\{x, Jx\}$ and σ is antiholomorphic one if and only if it has a basis $\{x, y\}$ so that

$$g(x, y) = g(x, Jy) = 0.$$

Definition 2.5. The pair $\{x, y\}$ is *antiholomorphic of signature* (+, –) if

$$g(x, x) > 0, \qquad g(y, y) < 0, \qquad g(x, y) = g(x, Jy) = 0. \qquad \ldots (1)$$

We can define antiholomorphic pairs also of signature (+, +) or (–, –) in a similar way.

Definition 2.6. A 2-plane σ is called *nondegenerate* if the restriction of the metric g on σ is nondegenerate, i.e. its rank is 2.

The sectional curvature of a nondegenerate 2-plane with a basis $\{x, y\}$ is defined by the equation

$$K(\sigma) = \frac{R(x,y,y,x)}{g(x,x)g(y,y) - g^2(x,y)}.$$

We denote $K(x, y) = K(\sigma)$ in case σ is a nondegenerate 2-plane with an orthonormal basis $\{x, y\}$. In particular we denote

$$H(x) = K(x, Jx)$$

if x is unit vector, i.e. $g(x, x) = 1$ or $g(x, x) = -1$.

Definition 2.7. The manifold M is said to be *a manifold of constant holomorphic sectional curvature* μ if the curvature of an arbitrary nondegenerate holomorphic 2-plane σ in $T_p(M)$ is constant, i.e. it does not depend on σ.

As in the case of a definite metric, M is of constant holomorphic sectional curvature if and only if the curvature tensor R has the representation

$$R = \frac{\mu}{4}(\pi_1 + \pi_2), \qquad \text{.. (2)}$$

where

$$\pi_1(x, y, z, u) = g(x, u)\, g(y, z) - g(x, z)\, g(y, u),$$

$$\pi_2(x, y, z, u) = g(x, Ju)\, g(y, Jz) - g(x, Jz)\, g(y, Ju) + 2g(x, Jy)\, g(z, Ju).$$

If M is a connected manifold, then μ is a global constant of the manifold. In the present paper we consider connected manifolds.

3. TWO SUFFICIENT CONDITIONS FOR *M* TO BE OF CONSTANT HOLOMORPHIC SECTIONAL CURVATURE.

Proposition 3.1. *Let M be a $2m$-dimensional hyperbolic Kählerian manifold of indefinite metric, $m \geq 2$. If for any antiholomorphic pair $\{x, y\}$ of signature $(+, -)$ we have*

$$R(x, Jx, Jx, y) = 0, \qquad \text{... (3)}$$

then M is of constant holomorphic sectional curvature.

Proof. Let $\{x, y\}$be an antiholomorphic pair of signature $(+, -)$, where x, y are unit vectors in $T_p(M)$, $p \in M$. According to (1) for any t, $|t| < 1$, the plane $\{x + ty, tx + y\}$ is antiholomoprphic of signature $(+, -)$ too. Having (3) in mind, we get

$$R(x + ty, Jx + tJy, Jx + tJy, tx + y) = 0$$

and then we obtain

$$t[H(x) + K(x, y) + 3K(x, Jy)] + t^3[H(y) + K(x, y) + 3K(x, Jy)] + t(3 + t)\, R(y, Jy, Jy, x) = 0.$$

This equality implies

$$H(x) + K(x, y) + 3K(x, Jy) + t^2[H(y) + K(x, y) + 3K(x, Jy)] = 0 \qquad \text{... (4)}$$

$$R(x, Jy, Jy, x) = 0 \qquad \text{... (5)}$$

for $|t| < 1$, $t \neq 0$. Therefore by continuity the equation (4) holds for any t, $|t| \leq 1$. Putting $t = 0$ into (4), we obtain

$$H(x) + K(x, y) + 3K(x, Jy) = 0.$$

Using the substitution $y \to Jy$ and subtracting the last two equations, we find

$$K(x, y) = K(x, Jy) \qquad \ldots (6)$$

$$H(x) = -4K(x, y). \qquad \ldots (7)$$

Then we get from (4), (6) and (7)

$$H(x) = H(y). \qquad \ldots (8)$$

In the case $m = 2$ we put $H(x) = \mu$ and take account of the equations (3), (5), (6),(7),(8). In such a way we obtain the representation (3), which proves the Proposition 3.1.

Now let $m > 2$ and $x, y \in T_p(M)$. We choose a unit vector $z \in T_p(M)$, such that the span $\{x, y, z\}$ is antiholomorphic.

The relation $H(x) = H(y) = H(z)$, which follows from (8), means that (8) holds also in the case when the pair $\{x, y\}$ is antiholomorphic of signature (+, +) or (–, –). Analogously, (8) holds also for arbitrary unit vectors u, v in $T_p(M)$. Indeed, if x is an unit vector in $T_p(M)$, chosen so that $\{x, u\}$ and $\{x, v\}$ are antiholomorphic, according to (8), we obtain

$$H(x) = H(y) = H(z),$$

and the proof is completed.

Remark 3.2. Proposition 1 holds also in case the signature of $\{x, y\}$ is (–, +).

Propostion 3.3. *Let M be a $2m$-dimensional hyperbolic Kählerian manifold of indefinite metric, $m > 2$. If for any antiholomorphic span $\{x, y, z\}$, $g(x, x) > 0$, $g(y, y) < 0$, we have*

$$R(x, y, y, z) = 0, \qquad \ldots (9)$$

then M is of constant holomorphic sectional curvature.

Proof. Without any loss of generality, we may assume that the signature of M is (–, –, –, ..., +, +, +,...). Let $\{x, y\}$ be an arbitrary antiholomorphic pair of signature (+, –) at a point $p \in M$. One can choose the vector $z \in T_p(M)$, such that the span $\{x, y, z\}$ is antiholomorphic of signature (+, –, –), i.e. $g(z, z) < 0$. It is possible to find t, $t \neq 0$, so that the antiholomorphic pair $\{x + tz, tJx + Jz\}$ is of signature (+, –). Having (9) in mind, we use the equation

$$R(x + tz,\ tJx + Jz,\ tJx + Jz, y) = 0$$

for both cases t and $-t$. Then we sum the obtained two equations up and get

$$R(x,\ Jx,\ Jx,\ y) = 0.$$

According to the Proposition 3.1, M is of constant holomorphic sectional curvature.

Remark 3.4. In case of an appropriate signature of M, the signature of the pair $\{x,\ y\}$ can be chosen $(+,\ +)$, $(-,\ -)$ or $(-,\ +)$.

Remark 3.5. In case of an appropriate signature of M Proposition 3.3 can be formulated also for the antiholomorphic triples $\{x,\ y,\ z\}$ of signature $(+, +, +)$, $(+, +, -)$ etc.

4. HYPERBOLIC KÄHLERIAN MANIFOLDS OF CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

Theorem 4.1. *Let M be a hyperbolic Kählerian manifold of indefinite metric, $m \geq 2$. If for any point $p \in M$ there exists a constant $c(p)$, such that for any unit vector $x \in T_p(M)$ the holomorphic sectional curvature $H(x)$ satisfies the inequality*

$$|\, H(x)\, | \leq c(p), \qquad \dots (10)$$

then M is of constant holomorphic sectional curvature.

Proof. Let $\{x,\ y\}$ be an arbitrary orthonormal antiholomorphic pair of signature $(+,\ -)$ at $p \in M$. For t, $|\, t\, | < 1$, the vector $\dfrac{x+ty}{\sqrt{1-t^2}}$ is a unit vector and then (10) implies

$$\frac{1}{(1-t^2)^2} |\ R(x + ty,\ Jx + tJy,\ Jx + tJy,\ x + ty)| \leq c(p),$$

i.e. we have

$$\begin{aligned} |\ H(x) &+ 4tR(x,\ Jx,\ Jx,\ y) - 2t^2\ [K(x,\ y) - 3tK(x,\ Jy)] \\ &+ 4t^3R(x,\ Jy,\ Jy,\ y) + t^4H(y)\ | \leq c(p)\ (1 - t^2)^2. \end{aligned} \qquad \dots (11)$$

Changing t into $-$ t in (11) and subtracting the results, we obtain

$$|\ H(x) - 2t^2\ [K(x,\ y) - 3K(x,\ Jy)] + tH(y)\ | \leq c(p)\ (1 - t^2)^2. \qquad \dots (12)$$

From (11) by continuity (for $t = \pm\ 1$), we get

$$H(x) + 4tR(x,\ Jx,\ Jx, y) - 2t^2\ [K(x, y) - 3K(x, Jy)] + 4tR(x,\ Jy, Jy,\ y) + tH(y) = 0.$$

The last equation implies

$$R(x, Jx, Jx, y) + R(x, Jy, Jy, y) = 0. \qquad \ldots (13)$$

Putting (13) into (11), we get

$$| H(x) - 2t^2 [K(x, y) - 3K(x, Jy)] + 4t(1 - t^2) R(x, Jx, Jx, y) + t^4 H(y) | \leq c(p)(1 - t^2)^2. \qquad \ldots (14)$$

Now, we can obtain from (12) and (14)

$$| tR(x, Jx, Jx, y) | \leq \frac{1}{2} c(p)(1 - t^2),$$

which by continuity implies

$$R(x, Jx, Jx, y) = 0.$$

According to Proposition 3.1, M is of constant holomorphic sectional curvature.

Let $m > 2$ and the constant $c(p)$ be such that for any antiholomorphic 2-plane σ we have $K(\sigma) \leq c(p)$. Let $\{x, y\}$ be an arbitrary orthonormal antiholomorphic pair of signature $(+, -)$. We choose an unit vector $z \in T_p(M)$ so that the span $\{x, y, z\}$ is antiholomorphic. For $|t| \neq 1$ the pair $\{x, ty, z\}$ is antiholomorphic and then

$$K(x + ty, z) \leq c(p). \qquad \ldots (15)$$

Having in mind that

$$K(x + ty, z) = \varepsilon R\left(\frac{x+ty}{\sqrt{1-t^2}}, z, z, \frac{x+ty}{\sqrt{1-t^2}}\right),$$

where $\varepsilon = g(z, z)$, we obtain from (15)

$$K(x, z) + 2\varepsilon tR(x, z, z, y) - t^2 K(y, z) \leq (1 - t^2)c(p), \quad |t| < 1,$$

$$K(x, z) + 2\varepsilon tR(x, z, z, y) - t^2 K(y, z) \geq (1 - t^2)c(p), \quad |t| > 1,$$

Therefore by continuity, we have

$$K(x, z) + 2\varepsilon tR(x, z, z, y) - t^2 K(y, z) = 0. \qquad \ldots (16)$$

Putting $t = 1$ and $t = -1$ into (16) and subtracting the obtained two equations, we get

$$4tR(x, z, z, y) = 0.$$

According to Proposition 3.3 for the span $\{x, y, z\}$ of signature $(+, +, -)$ or $(+, -, -)$ (see Remark 3.5), M is of constant holomorphic sectional curvature.

In such a way we proved the following

Theorem 4.2. *Let M be a hyperbolic Kählerian manifold of indefinite metric, $m > 2$. If for any point $p \in M$ there exists a constant $c(p)$ such that for any antiholomorphic 2-plane σ in $T_p(M)$ the inequality*

$$K(\sigma) \leq c(p)$$

is satisfied, then M is of constant holomorphic sectional curvature.

Remark 4.3. In Theorem 4.2 the inequality $K(\sigma) \leq c(p)$ can be replaced by $K(\sigma) \geq c(p)$.

Remark 4.4 In the case of an appropriate signature of M, Theorem 4.2 can be proved for the antiholomorphic 2-planes of signature (+, +) or (–, –) too.

REFERENCES

1. Kulkarni R. S., The values of sectional curvature of indefinite metrics. *Comment. Math. Helvetici,* **54** (1979), 173-176.
2. Barros M., Romero A., Indefinite Kähler manifolds. *Math. Ann.,* **261** (1982), 55-62.
3. Kassabov O., Borissov A., On the sectional curvature of Kähler manifolds of indefinite metrics. *Pliska Studia Mathematica Bulgarica,* **9** (1987), 66-70.
4. Ivanova R., Almost hyperbolic Kählerian manifolds of point-wise constant holomorphic sectional curvature, (to appear).

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2. E. Hewitt, Rings of continuous functions I. Trans. Amer Math. Soc. 64(1948), 54-99

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ON THE GENERALIZED CURVATURE TENSOR FIELDS OF CURVATURE DEGREE 2

MAKOTO YAWATA

1. INTRODUCTION

Let (M,g) be a Riemannian manifold and $R(X,Y)$ its Riemannian curvature tensor. Then $R(X,Y)$ satisfies the following conditions :

$$\begin{cases} R(X,Y)Z = -R(Y,X)Z \\ R(X,Y,Z,W) = -R(X,Y,W,Z) \\ R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \end{cases} \qquad \ldots (1.1)$$

where we put $R(X,Y,Z,W) = g(R(X,Y)Z,W)$. In general, a tensor field of type (1.3) satisfying the conditions (1.1) is called generalized curvature tensor or curvature-like tensor.

A generalized curvature tensor field S is called of curvature degree k if it is expressed as a sum of tensor fields each of which contains just k of $R = (R_{hijk})$ (the Riemannian curvature tensor), $Ric = (R_{ij})$ (the Ricci tensor) and r (the scalar curvature).

In this paper, we define a generalized curvature tensor or a curvature like tensor of type (1.5) and study the relations between these generalized curvature tensors and the integral formulas. Let $S = (S_{ij\ldots k})$ and T $(T_{ij\ldots k})$ be the same type form,, then we define the inner product by $(S,T) = S_{ij\ldots k} T^{ij\ldots k}$. Our main results are the followings :

Theorem 1. Let (M,g) be a compact orientable Riemannian manifold. We define a generalized curvature tensor $\tau(H)$ of curvature degree 2 by

$\tau(H)_{hijk} = 4B_{hijk} - R_h^a R_{aijk} - R_i^a R_{hajk} - R_j^a R_{hiak} - R_k^a R_{hija}$, where B_{hijk} is defined by

$$B_{hijk} = -R_{hia}{}^{b} R_{bjk}{}^{a} - R_{bjh}{}^{a} R_{aik}{}^{b} + R_{bji}{}^{a} R_{ahk}{}^{b}.$$

If (M,g) satisfies the conditions

$$4(Ric(\tau(H)),\ Ric) \geq (\tau(H),R),$$
$$\| \nabla r \| \geq 2\ \| Ric \|,$$

then (M,g) is a locally symmetric manifold.

Theorem 2. Let (M,g) be a compact orientable Riemannian manifold. Let H and $\tilde{U}$ be the generalized curvature tensor fields of type (1.5) defined by

$$H(X,Y,Z,W) = (R(X,Y)\ R)\ (Z,W) + (R(Z,W)\ R)\ (X,Y),$$

$$\begin{aligned}\tilde{U}(X,Y,Z,W,U,V) = {} & R(U,Y,Z,W)g(X,V) + R(X,U,Z,W)g(Y,V)\\ & + R(X,Y,U,W)g(Z,V) + R(X,Y,Z,U)g(W,V)\\ & - R(V,Y,Z,W)g(X,U) - R(X,V,Z,W)g(Y,U)\\ & - R(X,Y,V,W)g(Z,U) - R(X,Y,Z,V)g(W,U)\end{aligned}$$

If (M,g) satisfies the conditions

$$8 \| \nabla_a R_{hij}{}^a \|^2 \leq (H, \tilde{U})$$

then (M,g) is a locally symmetric manifild.

2. GENERALIZED CURVATURE TENSORS OF TYPE (1.5)

In this paper, we define a generalized curvature tensor or curvature-like tensor of type (1.5) as a tensor field L of type (1.5) which satisfies the following conditions :

$$\left\{\begin{array}{l} L(X,Y,Z,W) = -L(Y,X,Z,W) \\ L(X,Y,W,Z) = -L(X,Y,Z,W) \\ L(X,Y,Z,W) + L(Y,Z,X,W) + L(Z,X,Y,W) = 0 \end{array}\right. \qquad \ldots (2.1)$$

L is called proper if it satisfies an additional condition

$$L(X,Y,Z,W,U,V) + L(Z,W,U,V,X,Y) + L(U,V,X,Y,Z,W) = 0 \qquad \ldots (2.2)$$

where we put $L(X,Y,Z,W,U,V) = g(L(X,Y,Z,W)U,V)$.

Now, we shall give some examples of the generalized curvature tensor fields of type (1.5)

Example 1. Let $R(X,Y)$ be a Riemannian curvature tensor. We define a tensor H of type (1.5) by

$$H(X,Y,Z,W) = (R(X,Y)\ R)\ (Z,W) + (R(Z,W)\ R)\ (X,Y), \qquad \ldots (2.3)$$

Then H satisfies the conditions (2.1) and (2.2). That is, it is a proper generalized curvature tensor of type (1.5).

Example 2. We define a tensor field K of type (1.5) as follows :

$$K(X,Y,Z,W) = R(X,Y,)R(Z,W) + R(Z,W)R(X,Y) - \frac{1}{2}R(W,Y)R(X,Z) + \frac{1}{2}R(Z,Y,)R(X,W) + \frac{1}{2}R(W,X)R(Y,Z) - \frac{1}{2}R(Z,X,)R(Y,W). \qquad \ldots (2.4)$$

K is a generalized curvature tensor of type (1.5) which is not proper.

Example 3. Let us define a tensor field S of type (1.5) by

$$S(X,Y,Z,W)U = R(U,Y,Z,W)\ Ric(X) + R(X,U,Z,W)\ Ric(Y) + R(X,Y,U,W)\ Ric(Z) + R(X,Y,Z,U)\ Ric(W), \qquad \ldots (2.5)$$

where Ric denotes Ricci curvature tensor and $R(X,Y,Z,W) = g(R(X,Y)\ Z,W)$. Then S is a generalized curvature tensor of type (1.5) which is not proper.

Example 4. The tensor field $\tilde{S}(X,Y.Z,W,U,V)$ is defined by

$$\tilde{S}(X,Y.Z,W,U,V) = g(S(X,Y.Z,W,)U,V) - g(S(X,Y.Z,W,)V,U). \qquad \ldots (2.6)$$

Then $\tilde{S}$ is a generalized curvature tensor of type (1.5) which is proper.

3. GENERALIZED CURVATURE TENSOR OF TYPE (1.3)

In this section, we will induce the generalized curvature tensors of type (1.3) from the generalized curvature tensors of type (1.5) Let $L = (L_{hijka}{}^{b})$, be a generalized curvature tensor of type (1.5), where L $(\partial_h,\partial_i,\partial_j,\partial_k)\partial_a = \sum L_{hijka}{}^{b}\,\partial_b$. As we can easily understand, we have a generalized curvature tensor $\rho\ (L)$ of type (1.3) as follows :

$$\rho(L) = (L_{hijka}{}^{a}) \qquad \ldots (3.1)$$

We can construct another generalized curvature tensor $\tau(L)$ whose components are given by

$$\tau(L)_{hijk} = L_{bhjki}{}^{b} - L_{bijkh}{}^{b} + L_{bjhik}{}^{b} - L_{bkhij}{}^{b}, \qquad \ldots (3.2)$$

Now, we shall give the local components of the above tensors $\rho(L)$ and $\tau(L)$ with respect to the examples given in the section 2. The local components of the tensor H are given by

$$H_{hijka}{}^{b} = -R_{hij}{}^{p}R_{pka}{}^{b} - R_{hik}{}^{p}R_{jpa}{}^{b} - R_{jkh}{}^{p}R_{pia}{}^{b} - R_{jki}{}^{p}R_{hpa}{}^{b}, \qquad \dots (3.3)$$

Therefore, as easily seen, we have $\rho(H) = 0$. The local componets of $H_{bijka}{}^{b}$ are given by

$$H_{bijka}{}^{b} = B_{iajk} - R_{ca}R_{jki}{}^{c}, \qquad \dots (3.4)$$

from which we have the components of $\tau(H)$ as follows :

$$\tau(H)_{hijk} = 4B_{hijk} - R_{h}{}^{a}R_{aijk} - R_{i}{}^{a}R_{hajk} - R_{j}{}^{a}R_{hiak} - R_{k}{}^{a}R_{hija}, \qquad \dots (3.5)$$

Next, we define a generalized curvature tensor $\tilde{U}$ of type (1.5) as follows :

$$\begin{aligned} \tilde{U}(X,Y,Z,W,U,V) = {} & R(U,Y,Z,W)g(X,V) + (R(X,U,Z,W)g(Y,V) \\ & + R(X,Y,U,W)g(Z,V) + R(X,Y,Z,U)g(W,V) \\ & - R(V,Y,Z,W)g(X,U) - R(X,V,Z,W)g(Y,U) \\ & - R(X,Y,V,W)g(Z,U) - R(X,Y,Z,V)g(W,U) \end{aligned} \qquad \dots (3.6)$$

The components of $\tilde{U}$ are given by

$$\begin{aligned} \tilde{U}_{hijkab} = {} & R_{aijk}g_{hb} + R_{hajk}g_{ib} + R_{hiak}g_{jb} + R_{hija}g_{hb} \\ & - R_{bijk}g_{ha} - R_{hbjk}g_{ia} - R_{hibk}g_{ja} - R_{hijb}g_{ha}. \end{aligned} \qquad \dots (3.7)$$

$\tilde{U}$ is a proper generalized curvature tensor of type (1,5).

Contracting h and b, we obtain the following formulas :

$$\tilde{U}_{bijka}{}^{b} = (n-1)\,P_{aijk}, \qquad \dots (3.8)$$

where P_{aijk} is given by

$$P_{aijk} = R_{aijk} - \frac{1}{n-1}(R_{ij}g_{ka} - R_{ik}g_{ja}). \qquad \dots (3.9)$$

Using (3.2) we have

$$\tau(\tilde{U})_{hijk} = -4(n-1)\,P^{*}{}_{hijk}, \qquad \dots (3.10)$$

where $P^{*}{}_{hijk}$ is given by

$$P^{*}{}_{hijk} = R_{hijk} - \frac{1}{n-1}(R_{ij}g_{hk} - R_{ik}g_{hj} + R_{hk}g_{ij} - R_{hj}g_{ik}). \qquad \dots (3.11)$$

P_{hijk} does not satisfy the curvature conditions but P^{*}_{hijk} is a generalized curvature tensor.

4. INTEGRAL FORMULA

In his paper (cf.[3]), T. Sakai obtained the following integral formula :

$$\int_M K dM = \int_M \|\nabla r\|^2 - 4\,\|\nabla Ric\|^2 + \|\nabla R\|^2\, dM, \qquad \dots (4.1)$$

where K is given by

$$\begin{aligned} K = {} & 4R^{hi}R_h{}^aR_{ia} - 4R^{hi}R^{jk}R_{hjki} - 2R^{ab}R_{ahij}R_b{}^{hij} \\ & - 4R^{hijk}R^a{}_{hj}{}^b - R^{hijk}R_{hiab}R_{jk}{}^{ab}. \end{aligned} \qquad \dots (4.2)$$

If we put

$$\frac{1}{4}\tau(H)_{hijk} = B_{hijk} - \frac{1}{4}(R_h{}^a\, R_{aijk} + R_i{}^a\, R_{hajk} + R_j{}^a\, R_{hiak} + R_k{}^a\, R_{hija}), \qquad \dots (4.3)$$

then K can be expressed by

$$K = \frac{1}{2}(R, \tau(\mathrm{H})) - 2(Ric\,(\tau(H)),\ Ric). \qquad \dots (4.4)$$

Therefore if $K \leq 0$ and $4\,\|\nabla Ric\|^2 \leq \|\nabla r\|^2$, then we obtain

$$\int_M \|\nabla R\|^2\, dM \leq 0. \qquad \dots (4.5)$$

Thus, we obtain the Theorem 1.

Next, the following integral formula is well known (cf.[1]).

$$2\int_M (\|\nabla_a R_{hij}{}^a\|^2 + \tilde{K})\, dM = \int_M \|\nabla R\|^2\, dM, \qquad \dots (4.6)$$

where $\tilde{K}$ is given by

$$\tilde{K} = -R^{hijk}\, H_{aijkh}{}^a, \qquad \dots (4.7)$$

Using (3.7), we have

$$H_{hijkab}\, \tilde{U}^{hijkab} = 8R^{hijk}\, H_{aijkh}{}^a, \qquad \dots (4.8)$$

Therefore we can rewrite $\tilde{K}$ as

$$\tilde{K} = -\frac{1}{8}(H, \tilde{U}). \qquad \dots (4.9)$$

If (M,g) satisfies the conditions $8\,\|\nabla_a R_{hij}{}^a\|^2 \le (H, \tilde{U})$, then we have $\int_M \|\nabla R\|^2\, dM \le 0$. Thus, we have Theorem 2.

REFERENCES

1. K. Yano ; Differential geometry on complex and almost complex spaces, Pergamon Press, 1965.
2. K. Nomizu ; On hypersurface satisfying a certain condition on the curvature tensor, Tohoku Math. J.20 (1968), 46-59.
3. T. Sakai ; On eigen-values of Laplacian and curvature of Riemannian manifold, Tohoku Math. J. 23 (1971), 589-603.
4. K. Nomizu ; On the decomposition of generalized curvature tensor fields, Differential Geometry in honor of K. Yano, 1972, 335-345.
5. S. Tanno ; An inequality for 4-dimensional Kählerian manifolds, Proc. Japan Acad., 49(1973), 257-261.
6. S. Tachibana ; A theorem on Riemannian manifolds of positive curvature operator, Proc. Japan Acad., 50(1974), 301-302.

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MORE ON θ-METRISABLE SPACES

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1. INTRODUCTION

Our main interests in this paper are θ-metrisable H-closed spaces and some of their subspaces ; θ-metrisability was introduced by Ganguly and Bandyopadhyay [5] and was further studied in [6] ; a space is said to be θ-metrisable if it is θ-homeomorphic to a metrisable space.

The concept of θ-homeomorphism was primarily introduced by S. Fomin [4] and has been utilised meaningfully by many. A bijective mapping $f : X \to Y$ is said to be a θ-homeomorphism if and only if both f a f^{-1} are θ-continuous ; further f is θ-continuous if, for each $x \varepsilon X$ and each open set V containing f(x) there exists an open set U containing x such that $f(\overline{U}) \subset \overline{V}$. Clearly, such functions need not be continuous but if Y is a regular space then every θ-continuous function is continuous.

Among the concepts, needed for our investigation of θ-metrisability, an important one is that of a θ-closed set. The θ-closure of a set $A \subset X$, denoted by cl_θ A is $\{ x \varepsilon X$: every closed nbd of x meets $A\}$ [9]. The subset A is θ-closed if $cl_\theta(A) = A$. Likewise, the θ-interior of A, denoted by $Int_\theta(A)$ is $\{ x \varepsilon X$: some closed nbd. of x lies in $A\}$. A set A is called θ-open if $Int_\theta(A) = A$. Of course, θ-open sets are open and θ-closed sets are closed. Furthermore, the complement of a θ-open set is a θ-closed set and the complement of a θ-closed set is a θ-open set. Lemma 3 of [9] shows that the collection of θ-open sets in a topological space (X,T) forms a topology for X which we denote by T_θ.

2. PRELIMINARIES

In this section we collect some basic notions and discuss some essential problems with respect to θ-homeomorphism by citing some relevant examples.

2.0. We assume, throughout the paper, that all spaces are Hausdorff ; we further assume the familiarity with H-closed spaces and H-sets ; [9] might very well serve as the necessary background.

2.1. θ-homeomorphism is such a concept which requires careful treatment, as because, it sometimes causes unexpected problems ; for example, if $f : X \rightarrow Y$ is θ-continuous, the mapping $f : X \rightarrow f(X)$ need not be θ-continuous (even if $F(X)$ is a regular subspace of X) [10].

The problem, as mentioned above, poses a positive hindrance to the study of θ-metrisability of subspaces in general ; when Y is a metrisable topological space, $A \subset X$ and $f : X \rightarrow Y$ is a θ-homeomorphism, if is actually continuous since Y is regular and hence $f/A : A \rightarrow f(A)$ is continuous ; but unfortunately, $f^{-1}/f(A) : f(A) \rightarrow A$ is not necessarily θ-continuous ; the following substantiates our statement :—

Example 2.1.1. Let X be the real line ; if T_1 be the Euclidean topology on X an T_2 be the topology of countable complements on X, we define T to be the smallest topology generated by $T_1 \cup T_2$. obviously $T_\theta = T_1$. Now, let $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, \dots\dots\}$, obviously $\{O\}$ is open in T_A (T_A denotes the subspace topology on A) and it is closed in T_A as well ; hence $\{\overline{O}\} \cap \{1, \frac{1}{2}, \frac{1}{3} \dots\} = \Phi$ implying that $B = \{1, \frac{1}{2}, \frac{1}{3} \dots\}$ is θ-closed in A but B is not θ-closed in (X,T), since B is not closed B is not closed in (X, T_1). Since (X, T_θ) is metrisable, obviously (X, T) is θ-homeomorphic to a metrisable space (Y, d) ; but $f/A : A \rightarrow f(A)$ is not a θ-homeomorphism where A is as above.

Note 2.1.2. The problem with the example generates from the following inequality $(T_{A_\theta}) \neq (T_{\theta_A})$. Unless and otherwise inequality is transferred into equality, any study of subspaces become extremely difficult. Fortunately enough we have the following results.

Result 2.1.3. (i) If $D \subset X$ and $\overline{D} = X$, then $(T_{D_\theta}) = (T_{\theta_D})$. Infact, if $f : X \rightarrow Y$ is θ-continuous and $f(X) \subset Z \subset Y$ and Z is dense in Y, then $f : X \rightarrow Z$ is θ-continuous [10].

(ii) If $A \subset X$ and A is θ-open is X ; then $(T_{A_\theta}) = (T_{\theta_A})$; follows from lemma 3.3 [8].

Note 2.1.4. Subsets, other than θ-open and dense can also have the requisite equality ; as can be seen from the following example :

Example 2.1.5. Let $\mathcal{D}$ be any collection of countable (non-closed) discrete subsets with finite number of limit points in a metrisable space (X, T_1). Assume $\mathcal{D}$ to be closed under finite union and subsets of members of $\mathcal{D}$ are also in $\mathcal{D}$. Let T be the $\mathcal{D}$-extension of T_1. Then $T_\theta = T_1$ and (X,T) is θ-homeomorphic to (X,T_θ).

Observations : (i) Let F be T-closed ; then $F = F_0 \cup T$ where F_0 is θ-closed in (X,T) and $T \in \mathcal{D}$.

(ii) Let F be closed in (X,T). Then the θ-closure of F (to be denoted by $cl_\theta F$) is given by $cl_\theta F = F \cup S$; where S is finite.

(iii) Let M be a dense-in-itself subset of (X,T) ; for $x \in M$ consider $U = B(x,r) - T$ (for some $T \in \mathcal{D}$) which is a nbd. of x in (X,T), then $\overline{U \cap M} \cap M = \overline{B(x,r) \cap (M-T)} \cap M = \overline{B(x,r) \cap M} \cap M$ [as M–T is dense in M in (X,T)] $\supset M \cap \overline{B(x,\frac{r}{2})}$. This shows that θ-topology for M coincides with the θ-topology of X relativized to M. [B(x,r) is an open sphere in the metrisable space (x,T_1) (with some metric d' to give T_1) with centre x and radius r].

(iv) In particular, if M be connected in (X,T), M is dense-in-itself (if M is not a singleton) and as such $(T_{M_\theta}) = (T_{\theta_M})$.

Note 2.1.6. We include some of the necessary results which we shall use off and on in this paper.

Result 2.1.6. (i) If X is H-closed and $f : X \to Y$ is θ-continuous and surjective, then Y is H-closed [9].

Result 2.1.6. (ii) Composition of θ-continuous mappings is θ-continuous [1].

Result 2.1.6. (iii) An H-set H in a Urysohn space is θ-closed. [8].

Result 2.1.6. (iv) A θ-metrisable space is almost regular [6].

Result 2.1.6. (v) If $f : X \to Y$ is θ-continuous ; when $A \subset X$ is an H-set, then f(A) is an H-set in Y [9].

Result 2.1.6. (vi) If Y is a Urysohn space and $f : X \to Y$ is a θ-continuous function, then graph of f, G(f) is θ-closed in $X \times Y$ [7].

Result 2.1.6. (vii) Let $f_\alpha : X_\alpha \to Y_\alpha$ be a fucntion for each $\alpha \in \mathcal{A}$ and $f : \pi X_\alpha \to \pi Y_\alpha$ a fucntion defined by $f((x_\alpha)) = (f_\alpha(x_\alpha))$ for each point (x_α) in πX_α. Then f is θ-continuous if and only if f_α is θ-continuous for each $\alpha \in \mathcal{A}$ [1].

Result 2.1.6. (viii) Let $f : X \rightarrow Y$ be a function, where X and Y are almost regular spaces. Then f is θ-continuous if and only if inverse image of every θ-open (respectively, θ-clsoed) subset of Y is θ-open in X [6].

Result 2.1.6. (ix) A θ-metrisable space is Urysohn [6].

3. In this section we shall introduce certain special types of subsets of a θ-metrisable space and obtain some interesting properties

Definition 3.1. A topological space X is said to be P_θ if there exists a 2nd countable metrisable space Z along with a θ-homeomorphism $f : X \rightarrow Z$ and further, there exists a metric compatible with the topology of Z with respect to which Z is complete. It can further be mentioned in this connection, that the topology given by the θ-open sets in X is oviously a separable one.

Note 3.2. By corollary I to theorem I 6.2. [2], Z is homeomorphic to a countable intersection of open sets in the cube I^N, where I is the interval [0,1] of R. Hence X is θ-homeomorphic to a countable intersection of open sets in the cube I^N.

Definition 3.3. A topological space is said to be an S_θ space if it is θ-metrisable and if there exists a P_θ space P and a θ-continuous mapping of P onto X.

A subset A of a topological space X is called an S_θ-set if there exists a P_θ-space Y and a θ-continuous mapping $f : Y \rightarrow X$ such that $f(Y) = A$.

Theorem 3.4. If $A \subset X$ is an S_θ-set such that $\bar{A} = X$ and if T be the toplogy on X, then $(A, (T/A)_\theta)$ is separable.

Proof. Obviously, since $\bar{A} = X$, $(T/A)_\theta = (T_\theta)/A$ ...... (1)

Since A is S_θ, there exists a P_θ space P and a θ-continuous function $f : p \rightarrow X$ such that $f(p) = A$; also, there exists a 2nd countable metric space Q and a θ-homemorphism $g : Q \rightarrow P$. Q is no doubt separable. Let $\hat{T}$ be the topology on Q, then, $fog = (Q, \hat{T}) \rightarrow (X, T_\theta)$ is continuous. But fog (Q) = A and so $fog : (Q,T) \rightarrow (A, T_\theta/A)$ is continuous, from (1) $fog : (Q, \hat{T}) \rightarrow (A, (T/A)_\theta)$ is continuous ...... (2), since (Q,T) is a separable space and fog, as in (2), is continuous $(A, (T/A)_\theta)$ is separable.

Theorem 3.5. Let X be a topological space, $Y \subset X$ and $\bar{Y} = X$. If Y is a P_θ-space, then $Y = \bigcap_n Y_n$ where each Y_n is θ-open in X.

Proof. Let Y be θ-homeomorphic to a complete metric space (z,d) and let f : Y → Z be the θ-homeomorphism. Let $Y_n = \{x \in X : x$ has a θ-open nbd. U in X such that diam f $(U \cap Y) \leq \frac{1}{n}\}$ (with respect to the metric d) [obviously, since $\overline{Y} = X$, $U \cap Y \neq \Phi$].

Now, let $x \in Y_n$, then x has a θ-open nbd. U in X such that diam f $(U \cap Y) \leq \frac{1}{n}$, but, for each $x \in U$, U is a θ-open nbd. of x such that diam f $(U \cap Y) \leq \frac{1}{n}$, hence $U \subset Y_n$; thus Y_n is a union of θ-open sets in X and hence is θ-open in X.

Now, let $x \in Y$ then $f(x) \in Z$ and $S_d\,(f(x), \frac{1}{2n})$, the open sphere with centre f(x), radius $\frac{1}{2n}$ in terms of the metric d, is open in Z and diam $S_d\,(f(x), \frac{1}{2n}) \leq \frac{1}{n}$.

Also, $U = f^{-1}\,(S_d\,(f(x)), \frac{1}{2n})$ is a θ-open set in Y and is a nbd. of x (since f is θ-continuous). Now, since $\overline{Y} = X$, if T is the topology on X, then $(T_Y)_\theta = (T_\theta)_Y$ and thus there exists a θ-open set U in X such that $U \cap Y = f^{-1}\,(S_d\,(f(x), \frac{1}{2n}))$. Hence x has a θ-open nbd. U in X such that diam f $(U \cap Y) \leq \frac{1}{n}$. Thus $x \in Y_n$ (for each n) and so $Y \subset \cap Y_n$ ... (1)

Conversely, let $x \in \bigcap Y_n$; let $\mathcal{U}$ be the collection of all open sets in X each of which contains x. Then $\mathcal{U}$ is a filter base in X and $U \cap Y \neq \Phi$ for each $U \in \mathcal{U}$. Consider $\mathcal{V} = \{f(U \cap Y) : U \in \mathcal{U}\}$. Obviously, $\mathcal{V}$ is a d-cauchyfilter base in Z : infact, for any $\varepsilon > 0$, there is some $U \in \mathcal{U}$ such that d-diam $f(U \cap Y) < \varepsilon$.following Theorem 3.2 Ch. XIV [3], since is d-complete, $\mathcal{V}$ converges to some $z \in Z$. Obviously, $\mathcal{U}$ converges to $f^{-1}(z) = x$. Thus $x \in Y$ and so $\cap Y_n \subset Y$ ... (2).

Hence combining (1) and (2), Y is an intersection of θ-open sets in X.

Theorem 3.6. Let X be an H-closed θ-metrisable space and let $A \subset X$ be an S_θ-set such that $\overline{A} = X$. Then there exists an H-closed θ-metrisable space K and a decreasing sequence (B_n) of subsets of K, each of which is a countable union of H-closed sets and a continuous mapping $f : K \to X$ such that $A = f\,(\bigcap_n B_n)$.

Proof. since A is an S_θ-subset, there is a P_θ-space P and a θ-continuous mapping $g : P \to X$ such that g(P) = A. Now, by 3.2, P is θ-homeomorphic to the intersection of a decreasing sequence (U_n) of open subsets of I^N... (1). Hence, if T denotes the topology of P, then $(P, T_\theta) \subset Y$,where Y is a compact metric space and, in which (P, T_θ) is a dense G_δ-set ... (2) From (1) above.

Let G be the graph of g i.e. $G \subset P \times X$. Consider $h = i \times g : P \to G$ defined by h(x) = (x, g(x)). Then h is a θ-homeomorphism of P onto G.So, G can be expressed as a countable intersection of open sets V_n in $(Y \times X, T_\theta)$ [where T is the topology on $Y \times X$] and it can be taken to be a decending one.

Let $\hat{T}$denote the topology of Y and let $\hat{\hat{T}}$ denote the topology of X ; then for (x, g(x)) ε G, there exist open balls B_1 (x, r_x) ε $\hat{T}$ and B_2 $(g(x), r^x)$ ε $\hat{\hat{T}}_\theta$ such that $\overline{B_1(x,r_x)} \times \overline{B_2(g(x),r^x)} \subset V_n$.

Since (Y, $\hat{T}$) and (X, $\hat{\hat{T}}_\theta$) are each separable as each contains a dense separable subspace (from Th. 3.4) it follows, by hereditary Lindeloff property, that there exists a sequence x_n such that

$$V_n = \bigcup_{n=1}^{\infty} \overline{B_1(x_n,r_{x_n})} \times \overline{B_2(g(x_n),r^{x_n})}.$$ But $\overline{B_1(x_n,r_{x_n})}$ is compact and $\overline{B_2(g(x_n),r^{x_n})}$ is H-closed and hence $\overline{B_1(x_n,r_{x_n})} \times \overline{B_2(g(x_n),r^{x_n})}$ is H-closed for each n. Now, we take $f = \pi : Y \times X \to X$; then π (x, g(x)) = g(x). Hence, since π(G) = A, the result follows ; in fact, we can take $Y \times X$ to be K to be K to keep parity with the statement of the theorem.

Definition 4.1. Let (X,T) be a Urysohn toplogical space. A θ-capacity on X is a mapping f on the power set P(X) of all subsets of X into the extended real line $\overline{R}$ satisfying the following conditions :—

(i) If $A \subset B$, then $f(A) \leq f(B)$;

(ii) If (A_n) is any increasing sequence of subsets of X, then $f(\bigcup_n A_n) = \sup_n f(A_n)$;

(iii) if (K_n) is any decreasing sequence of H-sets in X, then $f(\bigcap_n k_n) = \inf_n f(k_n)$.

Defineirn 4.2. Let f be a θ-capacity of X ; a subset A of X is said to be θ_f-capacitable if f(A) = Sup f(K) ; where K runs through the set of H-sets in X such that $K \subset A$.

Theorem 4.3. Let K be a H-closed space and let f be a θ-capacity on K. If A is the interection of decreasing sequence (A_n) of subsets of K, each of which is a countable union of θ-closed sets, then A is θ_r-capacitable.

Proof. We have to prove (as in definition 4.2.) that

$$f(A) = \text{Sup}\ \{f(\hat{K}) : \hat{K} \text{ is an H-set in K and } \hat{K} \subset A\} ;$$

for this, if we can show that, for each $\varepsilon > 0$, there is an H-set C in X containing A such that $f(C) \geq f(A) - \varepsilon$, then we are done. Put $a = f(A) - \varepsilon$.

We first observe that $A = \cap A_n$; where $A_n = \bigcup\limits_{K=1}^{n} A_{n,k}$ for each n and $A_{n,k}$, for each n and k, is a θ-closed subset of K ; without loss of generality we can take $\{A_{n,k}\}_k$ to be an increasing sequence for each n.

We take $C_0 = A$; since $C_0 \subset \bigcup\limits_{K=1}^{n} A_{1,k}$,

$C_0 = \bigcup\limits_{K=1}^{n} (C_0 \cap A_{1,k})$ and $\{C_0 \cap A_{1,k}\}$ is an increasing sequence. By definition 4.1.

$\text{Sup}_k\ f(C_0 \cap A_{1,k}) = f(C_0) > a$; hence there exists an integer j such that $f(C_0 \cap A_{1,j}) > a$. We take $B_1 = A_{1,j}$. Ovbiously B_1 has been defined with the following properties :

(i) $B_1 \subset A_1$; (ii) if $C_1 = C_0 \cap B_1$ then $C_1 \subset C_0 \subset A$ and $f(c_1) > a$.

Suppose that θ-closed sets B_i have been defined for i < n, so that $B_k \subset K_k$ for $1 < k < n$ and $C_{k-1} = C_{k-2}\ B_{k-1}$ (for $1 < k \leq n$) with $C_{k-1} \subset A$ and $f(c_{k-1}) > a$ for $1 \leq k \leq n$.

Now A_n is the union of an increasing sequence of θ-closed stes $A_{n,k}$ and so, it follows that $f(c_{n-1}) = \text{Sup}_k\ f(c_{n-1} \cap A_{n,k})$

$[\because C_{n-1} \subset A \subset \bigcup\limits_{1}^{n} A_{n,k}]$. Hence, there must exist an index j such that $f(c_{n-1} \cap A_{n,j}) > a$. Put $B_n = A_{n,j}$ and this B_n serves our purpose. In fact, each B_n is a θ-closed set and for each $n \geq 1$, there exists a set C_n with $C_0 = A$, $C_n = C_{n-1} \cap B_n$ and $f(c_n) > a$, for $n \geq 0$.

Let $C = \cap C_n$, Since $A = \cap A_n$ and $B_n \subset A_n$. We have $C = \cap B_n$, incact $C = \cap C_n \subset \cap B_n$ is obvious ; if $\cap B_n \not\subset C$, then there exists some $x \varepsilon \cap B_n$ such that $x \notin C_j$, for some j and so $x \notin C_{j-1}$ and thus $x \notin C_0(=A)$ – a contradiction. C is therefore a θ-closed set in K i.e., an H-set in K and is contained in A.

Let $B'_n = B_1 \cap \ldots \cap B_n$; (B'_n) is a decresing sequence of θ-closed set of K ; as $C_n \subset C_i \subset B_i$ for $i < n$, we also have $C_i \subset B'_n$. Since each B'_n is an H-set in k, by (iii) in definition 4.1, we have $f(C) = \inf_n f(B'_n)$ and since $C \subset C_n \subset B'_n$ we also have $f(C) = \inf_n f(C_n) \geq a$. This completes the proof.

Lemma 4.4. Let Φ be a θ-continuous mapping of a Urysohn H-closed space K into a Urysohn space X, and let f be a θ-capacity on X. If, for each subset A of K we put $g(A) = f(\Phi(A))$, then g is a θ-capacity on K ; moreover, if A is θ_g – capacitable, then Φ (A) is θ_f – capacitable.

Proof. Let A, B $\subset$ K and let A $\subset$ B, then Φ (A), Φ (B) $\subset$ X and since f is a θ-capacity on X, $f(\Phi(A)) \leq f(\Phi(B))$ i.e. $g(A) \leq g(B)$ ... (1)

Let (A_n) be an increasing sequence of subsets of K. Then $(\Phi(A_n))$ is an increasing sequence of subsets of X and since f is a θ-capacity on X, $f\bigcup_n(\Phi(A_n)) = \mathop{\mathrm{Sup}}_n f(\Phi(A_n))$

i.e., $f(\Phi(\cup A_n) = \mathop{\mathrm{Sup}}_n g(A_n)$ i.e. $g(\cup A_n) = \mathop{\mathrm{Sup}}_n g(A_n)$ ... (2)

Now, let (C_n) be a decreasing sequence of H-sets in K and let $C = \bigcap_n C_n$; the sets Φ (C_n) are H-sets and their intersection is Φ (C) ; infact, $\Phi(C) \subset \bigcap_n \Phi(C_n)$ and if $x \varepsilon \Phi(C_n)$ for all n, then the sets $\Phi^{-1}(x) \cap C_n$ is an H-set in K for each n ; infact $\{x\}$ is θ-closed in X and Φ and being θ-continuous, $\Phi^{-1}(\{x\})$ is θ-closed in K ; thus the sets $\phi^{-1}(x) \cap C_n$ form a decreasing sequence of non-empty H-sets of K and therefore, since K is H-closed Urysohn, their intersection is nonempty [10]. We have therefore $f(\Phi(C)) = \inf_n f(\Phi(C_n))$, that is $g(C) = \inf_n g(C_n)$ ... (3).

Combing (1), (2) and (3) we can say (following definition 4.1) that g is a θ-capacity on X.

Now, let A be a subset of K which is θ-capacitable w.r.t. g ; if $a < f(\Phi(A) = g(A)$, there is an H-set $C \subset A$ such that $g(C) \geq a$, thus $\Phi(C)$ is an H-set contained in $\Phi(A)$, and $f(\Phi(C)) \geq a$ this shows that $\Phi(A)$ is capacitable w.r.t.f.

Theorem 4.5. Let X be a θ-metrisable space and let Y be an S_θ subset of X such that $\overline{Y} = X$. Then Y is θ_y-capacitable with respect to every θ-capacity f on X.

Proof. We have seen that there exist an H-closed Urysohn space K, a decreasing sequence (A_n) of subsets of K, each of which is a countable union of H-sets and a continuous mapping $\Phi : K \to X$ such that $Y = \Phi(\cap A_n)$. By theorem 4.3, $\bigcap_n A_n$ is θ_r-capacitable with respect to any θ-capacity f on K. Lemma 4.4 finalises the necessary result.

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REFERENCES

1. Arya, S p. and Dev, M.—On θ-continuous mapping the Math. Student. Vol. XLII No. 1(1974) pp 81-89.
2. Bourbaki—Elements of Mathematics, General Topology Part 2 Addision Wesley Publishing Company 1966.
3. Dugundji, J.—Topology, Prentice-Hall of India Private Ltd. 1975.
4. Fomin, S. V.—Extensions of topological spaces. Ann. of Math (2), 44 (1943), 471-480.
5. Ganguly, S. and Bandopadhyay, T.—Further study of θ-mob, Bull. call. Math. Soc. 86(1943) 73-88.
6. Ganguly, S, Sinha Roy, S and Bandopadhyay, T.—A note on θ-metrisable spaces—Jour. Tri. Math. Soc. I (1999) 71-82.
7. Noire, T.—Properties of θ-continuous functions Accademia Nazianabe Die, Lincei Serie VIII Vol. I VIII pp. 887-891 (1975).
8. Sinha Roy, S and Bandopadhyay, S.—On θ-completely regular and locally θ--closed spaces Bull. Cal. Math. Soc. 87, 19-28 (1995).

9. Velicko, N.V.—H-closed topological spaces Mat. Sb. 79(112) (1966) pp 98-112, Maer. Math. Soc. Transl, 78(2) (1968) pp 103-118.
10. Vermeer, J.—Closed subspaces of H-closed spaces-Pacific J. Math. Soc. Vol. 118 No. 1 (1985) pp 229-246.

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A NEW METHOD OF SOLVING LINEAR HOMOGENEOUS ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT : A general method for solving linear homogeneous ordinary differential equations has been presented. It reduces the given differential equation into a system of first-order ordinary differential equations using some geometrical idea. Obviously the first-order system so obtained is different from the first-order system usually deduced from the given differential equation.

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1. INTRODUCTION

There is no general method for solving a linear homogeneous ordinary differential equation (ODE), even if the order of the differential equation is two. Methods are known only for ODEs with constant coefficients and for Euler's ODEs. In this article a general method is presented that is applicable to any linear homogeneous ODE.

The Method is based upon a very simple geometrical idea. It reduces the given nth order ODE to n first-order ODEs. The success of the method therefore depends upon the success of solving these n first-order equations.

In the next section we explain the geometrical idea and simultaneously develop the method. Section 3 applies the method to a number of examples to exhibit the simplicity and the applicability of the method.

2. THE GEOMETRICAL IDEA

Let us first consider the general linear homogeneous ODE of the second order

$$p_0(x)\; y^{(2)}(x) + p_1(x)\; y^{(1)}(x) + p_2(x)\; y(x) = 0, \qquad \ldots (2.1)$$

where p_0, p_1, p_2 are real-valued functions on $I = [a,b]$, $y^{(r)}(x) = \dfrac{d^r y}{dx^r}$ $(r = 0, 1, 2)$, $y^{(0)}(x) = y(x)$.

If $\varphi : J \to R$ (the set of real numbers) $(J \subset I)$ is a solution of (2.1), we have, for all $x \in J$,

$$p_0(x)\ \varphi^{(2)}(x) + p_1\ (x)\ \varphi^{(1)}(x) + p_2(x)\ \varphi(x) = 0, \qquad \ldots (2.2)$$

We now look upon (2.2) as the statement of the geometrical fact that the two vectors

$$P \equiv (p_0(x),\ p_1(x),\ p_2(x)) \text{ and } \Phi \equiv (\varphi^{(2)}(x),\ \varphi^{(1)}(x).\ \varphi(x)) \qquad \ldots (2.3)$$

are *orthogonal* at each point $x \in J$. In other words, any solution φ of (2.1) has to give rise to a vector Φ, which is orthogonal to P. If Q, R are two *linearly independent* vectors orthogonal to P, the vector Φ must therefore be a linear combination of Q and R :

$$\Phi = f(x)\ Q + g(x)\ R. \qquad \ldots (2.4)$$

It is to be noted that the coefficients f and g are to be dependent on x so that the relation (2.2) is satisfied for all $x \in J$. Equating the respective components from both sides of (2.4) we get three equations. Eliminating φ from these three equations we obtain two first-order ODEs in f and g. The required solution φ is obtained by solving these two first-order ODEs. To illustrate the idea we depict several examples in the next section.

For the nth-order linear homogeneous ODE

$$p_0(x)\ y^{(n)}(x) + p_1(x)\ y^{(n-1)}(x) + \ldots + p_n(x)\ y(x) = 0 \qquad \ldots (2.5)$$

where $p_0, p_1, \ldots p_n$ are real-valued functions on $I = [a,b]$, we find that the inner product of the two vectors

$$P = (p_0,\ p_1,\ \ldots\ p_n) \text{ and } \psi = (\psi^{(n)},\ \psi^{(n-1)} \ldots \psi) \qquad \ldots (2.6)$$

is zero for all $x \in J$, if $\psi : J \to R$ is a solution of (2.5).

In this case we are to choose n linearly independent (n+1)–vectors, $Q_1, Q_2, \ldots Q_n$ so that $P.Q_i = 0$ $(i = 1,2, \ldots n)$.

Then there must exist suitable functions $f_1, f_2, \ldots f_n$ such that

$$\psi = f_1 Q_1 + f_2 Q_2 + \ldots + f_n Q_n, \text{ for all } x \in J. \qquad \ldots (2.7)$$

As before, we equate respective components from the two sides of (2.7) and then eliminate ψ from these n + 1 equations to obtain n first order ODEs in $f_1, f_2, \ldots, f_n$. Solving these n first-order ODEs we shall get the required solution of (2.5).

3. SOME EXAMPLES

Here we exhibit six examples of which the first two are second-order Euler's ODEs, while the third and the fourth examples are second-order ODEs with variable coefficients, but which are not Euler's ODEs. The fifth example depicts a third-order ODE with variable coefficients, which again is not an Euler ODE. The last example is a fourth-order ODE with constant coefficients.

Example 1. $x^2y^{(2)}(x) - xy^{(1)}(x) + y(x) = 0.$ ... (3.1)

Here $P = (x^2, -x, 1)$ and we choose

$$Q = (1, x, 0),\ R = (0, 1, x). \qquad \text{... (3.2)}$$

Then (2.4) implies

$$\varphi^{(2)}(x) = f(x),\quad \varphi^{(1)}(x) = x\, f(x) + g(x),\quad \varphi(x) = x\, g(x). \qquad \text{... (3.3)}$$

Eliminating φ from the equations in (3.3) we get

$$g^{(1)}(x) = f(x),\quad x\, f^{(1)}(x) + g^{(1)}(x) = 0 \qquad \text{... (3.4)}$$

(3.4) implies $x\, f^{(1)}(x) + f(x) = 0$.

Hence $x\, f(x) = A$ (an arbitrary constant). ... (3.5)

Using (3.5) in (3.4) we obtain $g^{(1)}(x) = \dfrac{A}{x}$.

Hence $g(x) = A \ln x + B$ (where B is an arbitrary constant).

Therefore $\varphi(x) = Ax \ln x + Bx$,

which is the required general solution of (3.1).

Example 2. $x^2y^{(2)}(x) - 7xy^{(1)}(x) + 15\, y(x) = 0$ ... (3.6)

Here $P = (x^2, -7x, 15)$ and let us take

$$Q = (7, x, 0),\quad R = (15, 0, -x^2).$$

Then, using (2.4), on elimination of φ, we get

$$xg^{(1)} = -f - 2g,\quad xf^{(1)} = 6f + 15g, \qquad \text{... (3.7)}$$

Clearly (3.7) implies

$$x(f^{(1)} + 5g^{(1)}) = f + 5g.$$

and so $f + 5g = Ax$ (where A is an arbitrary constant). ... (3.8)

Using (3.8) in the first equation of (3.7), we get

$$xg^{(1)} = 3g - Ax.$$

Hence $g = x^3 \left[\frac{A}{2x^2} + B\right]$, where B is an arbitrary constant.

Therefore the required solution φ is given as

$$\varphi(x) = -x^2 f(x) = -\frac{A}{2}x^3 - Bx^5.$$

Example 3. $xy^{(2)}(x) - (x+1)y^{(1)}(x) + y(x) = 0.$ ... (3.9)

Here $P = (x, -x-1, 1)$, and we choose

$Q = (1, 0, -x)$, $R = (0, 1, x+1)$.

Proceeding as before, we find that (2.4) leads to the equations

$$xf^{(1)} - (x+1)g^{(1)} = -f^{(1)}, \quad f = g^{(1)}. \qquad \dots (3.10)$$

(3.10) implies $x(f^{(1)} - f) = 0$. So $f = Ae^x$, where A is an arbitrary constant. Then $g^{(1)} = Ae^x$. So $g = Ae^x + B$, where B is an arbitrary constant.

The required solution φ is therefore given by

$$\varphi(x) = -x f(x) + (x+1) g(x) = Ae^x + B(x+1).$$

Example 4. $x^2y^{(2)}(x) - x(2+x)y^{(1)}(x) + (2+x)y(x) = 0.$ ... (3.11)

Here $P = (x^2, -2x - x^2, 2 + x)$ and we choose

$Q = (0, 1, x)$, $R = (2 + x, x, 0)$.

Proceeding as before we get

$$f^{(1)} = g, \quad g^{(1)} = g. \qquad \dots (3.12)$$

These imply $g = Ae^x$, $f = Ae^x + B$, where A, B are arbitrary constants.

Hence the required solution φ is given by

$$\varphi(x) = x f(x) = Axe^x + Bx.$$

REMARK

The usual method of solving (3.9) or (3.11) is to find one solution φ by inspection, and then a second solution ψ, linearly independent of φ, is obtained from the well-known formula

$$\psi(x) = \phi(x) \int_a^x \frac{1}{\phi^2(t)} \exp[-\int_a^t \frac{p_1(u)}{p_0(u)} du] \, dt,$$

provided ϕ does not vanish on the interval under concern.

In the method described above, there is no need of "inspection' and the general solution is obtained at one go.

Example 5. $x(1-x+x^2)y^{(3)} + (1-x-x^3)y^{(2)} + x(1+x)y^{(1)} - (1+x)y = 0$ ... (3.13)

Here $P = (x-x^2+x^3,\ 1-x-x^3,\ x+x^2,\ -1-x)$

and we choose

$Q_1 = (0, 0, 1, x)$, $Q_2 = (1, 1, 1, 1)$, $Q_3 = (-1+x+x^3,\ x-x^2+x^3,\ 0,\ 0)$.

Then $(\varphi^{(3)}, \varphi^{(2)}, \varphi^{(1)}, \varphi) = f_1Q_1 + f_2Q_2 + f_3Q_3$.

Equating the respective components and then eliminating φ we get

$$\begin{aligned} &xf_1^{(1)} + f_2^{(1)} = f_2, \\ &f_1^{(1)} + f_2^{(1)} = f_2 + f_3\,(x-x^2+x^3), \\ &f_2^{(1)} + f_3^{(1)}\,(x-x^2+x^3) = f_2 + f_3\,(-2+3x-3x^2+x^3). \end{aligned} \quad \text{... (3.14)}$$

Eliminating f_1, f_2 we get

$$(1-x)xf_3^{(1)} + f_3(2-3x) = 0.$$

Hence $f_3x^2\,(1-x) = A$ (an arbitrary constant). ... (3.15)

Using (3.15) in (3.14) we have

$$f_1^{(1)} = A[\frac{1}{x} + \frac{1}{(1-x)^2}]. \quad \text{... (3.16)}$$

Hence $f_1 = A[\ln x + \frac{1}{1-x}] + B$, where B is an arbitrary constant. ... (3.17)\

Then, using (3.17) we get

$$f_2^{(1)} = f_2 - x\,f_1^{(1)},$$

and so $f_2 = -\frac{Ax}{1-x} + Ce^x$, where C is an arbitrary constant. ... (3.18)

Therefore the required solution φ is given by

$\varphi(x) = xf_1(x) + f_2(x) = Ax \ln x + Bx + Ce^x$.

Example 6. $y^{(4)} - 5y^{(2)} + 4y = 0$ ... (3.19)

Here $P = (1, 0, -5, 0, 4,)$ and we choose

$Q_1 = (5, 0, 1, 0, 0)$, $Q_2 = (0, 1, 0, 0, 0)$,

$Q_3 = (0, 0, 0, 1, 0)$ and $Q_4 = (4, 0, 0, 0, -1)$.

Then $(\varphi^{(4)}, \varphi^{(3)}, \varphi^{(2)}, \varphi^{(1)}, \varphi) = f_1Q_1 + f_2Q_2 + f_3Q_3 + f_4Q_4$

implies $f_4^{(1)} = -f_3$, $f_3^{(1)} = f_1$, $f_1^{(1)} = f_2$, $f_2^{(1)} = 5f_1 + 4f_4$, ... (3.20)

From (3.20) we get

$$f_1 + f_2 - 4f_3 + 4f_4 = Ae^x, \quad f_1 - f_2 + 4f_3 + 4f_4 = Be^x, \ldots \quad (3.21)$$

where A, B are arbitrary constants.

Using (3.21) in (3.20) we get, C being an arbitrary constant,

$$f_1 + 2f_3 = -\frac{A}{2}e^x + \frac{B}{6}e^{-x} + C\,e^{2x} \quad \ldots (3.22)$$

Hence $$f_4 = \frac{A}{6}e^x + \frac{B}{6}e^{-x} - \frac{C}{8}e^{2x} + De^{-2x}, \quad \ldots (3.23)$$

where D is an arbitrary constant.

The required solution is then obtained from $\varphi(x) = -f_4(x)$.

REMARKS

The salient features of the method described above are the following :

1. The method straightway yields the general solution of the differential equation.
2. Here is an algorithm suitable for tackling any nth order linear homogeneous ODE with $n > 1$.
3. The underlined concept is extremely simple.
4. The method, being based on a geometric idea, develops a geometric picture of the family of solutions of the given differential equation.

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A NOTE ON AN INFINITESIMAL MOTION IN A SEMI-RICCI SYMMETRIC MANIFOLD

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ABSTRACT : The main object of this note is to show that if the associated vectorfield of a semi-Ricci-symmetric manifold satisfies Killing equation, then it induces a symplectic structure on the manifold and the vectorfield itself is a Hamiltonian vectorfield.

1. INTRODUCTION

A semi-Ricci-symmetric manifold, in short, a $(SRS)_n$ [1], is a Riemannian manifold, where the Ricci tensor S(Y,Z) satisfies the condition,

$$(\nabla_x S)(Y, Z) = 2\alpha(X) S(Y, Z) - \alpha(Y) S(X,Z) - \alpha(Z) S(Y, X) \quad \ldots (1.1)$$

where α is a differentiable 1-form, called the associated 1-form and ∇ is the operator of covariant differentiation with respect to Levi-Civita connection. The vectorfield A, defined by

$$g(X,A) = \alpha(X) \quad \ldots (1.2)$$

is called the associated vectorfield.

The following results have been proved in [1] :

(a) The scalar curvature r is a constant,

(b) $S(X,A) = r\,\alpha(X)$,

(c) $r(\nabla_x \alpha) = S(\nabla_x A,Y) + r\,\alpha(X)\,\alpha(Y) - a\,S(X,Y)$ where $a = \alpha(A)$

(d) The vectorfield A is such that it is not parallel, not concurrent, not recurrent, i.e.,

$\nabla_x A \neq O,\ \nabla_x A \neq \rho x,\ \nabla_x A \neq \mu(X)A.$

2 INFINITESIMAL MOTION IN $(SRS)_n$

Let the associated vectorfield A in a $(SRS)_n$ be an infinitesimal motion. Then it satisfies the Killing equation—

$$g(\nabla_x A,Y) + g(\nabla_y A,X) = 0 \qquad \ldots (2.1)$$

or, equivalently,

$$(\nabla_x \alpha)(Y) + (\nabla_y \alpha)(X) = 0 \qquad \ldots (2.2)$$

Define a 2–form ω in $(SRS)_n$ by

$$\omega = d\alpha \qquad \ldots (2.3)$$

Then $\omega(X,Y) = d\,\alpha\,(X,Y) = (\nabla_x \alpha)(Y) - (\nabla_y \alpha)(X)$

or, $\omega(X,Y) = -\,2(\nabla_y \alpha)(Y)$ $\ldots (2.4)$

by virtue of (2.2). Clearly,

$$d\omega = 0, \text{ and } \omega(X,Y) \neq 0,$$

for otherwise, $(\nabla_y \alpha)(X) = 0$, which is not ture. Therefore ω is a symplectic structure on $(SRS)_n$ [2].

Again, by virtue of (2.4)

$$\omega(A,X) = -2(\nabla_x \alpha)(A) \qquad \ldots (2.5)$$

Now,

$$r(\nabla_x \alpha)(A) = S(\nabla_x A,\ A) + r\ \alpha(X)\ \alpha(A) - a\ S\ (X,\ A)$$

$$= r\ \alpha(\nabla_x A)$$

If $r \neq 0$, then $\nabla_x \alpha\ (A) = \alpha(\nabla_x A)$ and therefore,

$$(\nabla_x \alpha)(A) = 1/2\ \nabla_x a \qquad \ldots (2.6)$$

Substituting in (2.5),

$$\omega(A,X) = -Xa \qquad \ldots (2.7)$$

Equation (2.7) shows that A is a Hamiltonian vectorfield with energy function $-a$. [2].

Further, $\omega(A,X) = (i_A\omega)(X)$. Therefore

$$\begin{aligned} d(i_A\omega)(X,Y) &= X(i_A\omega)(Y) - Y(i_A\omega)(X) - (i_A\omega)[X,Y] \\ &= X(Ya) - Y(Xa) - [X,Y]a \\ &= 0 \end{aligned}$$

which implies

$$d(i_A\omega) = 0. \qquad \text{... (2.8)}$$

Theorem. If the associated vectorfield of a semi-Ricci-symmetric manifold with non-zero scalar curvature, is an infinitesimal motion then it induces a symplectic structure on the manifold and the vectorfield itself is a Hamiltonian vectorfield.

REFERENCES

1. S. Mukhopadhyay and B. Barua, A Riemannian manifold with a semi-symmetric structure, Pure Math. Manuscript 9 (1990-91), 171-180.
2. R. Abraham and J.E. Marsden, Foundations of Mechanics The Benjamin/Cumminge Publ. Com. Inc. (1981)

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ON LEFT SERIAL NEAR RINGS

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ABSTRACT : In this paper we prove some results on a distributively generated left serial near ring N w.r.t. a set $\{e_i\}$, $1 \leq i \leq k$ of orthogonal idempotents with the condition that Σe_i is a right identity. If the radical J of a distributively generated near ring N is D-regular invariant sub-near ring, then the intersection of left integral powers of J cannot have non trivial rps N-group structure. Also if N possesses strictly projective summand and tame N-group $J^{(1,i)}e_i$, $1 \leq i \leq k$, then we find a complete list of ideals of each Ne_i and which finally leads us to prove that N satisfies the module essential descending chain condition.

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1. INTRODUCTION

Oswald'd consideration of Near-rings [10] in which every N-subgroup is principal (prs near-rings) gives rise to many interesting results with chain conditions on annihilators. Baruah's work [1] on the above type of near-rings is directed towards the idea of l.c.m and g.c.d of near-rings. On the other hand in a near-rings, where principal N-subgroups are linearly ordered we get some interesting results of fundamental importance so far some particular types of structure theorems are concerned. These lead us to define what we have proposed to study in some special cases which we observe in near-rings including Klein's four group [11] where the left annihilators of the near-ring (11) and (14) are distributively generated (viz. {o,b}.) In a near-ring N with a right identity e if $N = \sum_{i=1}^{t} L_i$ ($L_i \nabla N$) and $e = \sum_{i=1}^{t} e_i$ ($e_i \varepsilon L_i$) then $e_1, e_2, \ldots, e_t$ are orthogonal idempotents and each e_i is a right identity of each L_i such that $L_i = Ne_i$ [2]. We see more decompositions induced by orthogonal idempotents in Fain [4] and Lyons [5]. In this paper we prove some elegant properties of this concept what we have called a left serial near-ring.

N is said to be a *left serial near-ring* if for a set $\{e_i\}$. $1 \leq i \leq i$ of orthogonal idempotents, $N = Ne_1 \oplus \dots \oplus Ne_k$, where N-subgroups of each Ne_i ($i = 1, 2, \dots, k$) are linearly ordered by set inclusion. Strictly projectiveness of some substructure of such near-rings with well behaved radical gives us a complete list of ideals of these structures of such near-rings as well as some typical type of descending chain condition.

Here we prove that if the radical J of a distributively generated near-ring (d.g.n.r.) N is D-regular invariant sub near-ring, then the intersection of left integral powers of J can not have non-trival rps-N-group structure.

Finally if N with the acc on left N-subgroups in which principal N subgroups are linearly ordered, is a left serial near-ring w.r.t. a set $\{e_i\}$, $1 \leq i \leq k$ of right complete orthogonal idempotents such that each summand is a strictly projective N-group in case of invariant radical J(N) in the sense J(N) = J(N)N, and also for $t \geq 0$, each $J^{(1,t)}e_i$ is a tame N-group then we find a complete list of ideals of each Ne_i, $1 \leq i \leq k$ which contain $J^{(1,t)}e_i$, $t \in Z^+$ and lastly, N satisfies modulo essential descending chain condition (medcc) if intersection of left integral powers of J is an rps-N-group.

2. DEFFINITIONS AND NOTATIONS

The basic concepts which are used in this paper can be found in Pilz [11]. We now begin our discussion with some preliminary definitions used in the paper.

An N-group E is *irreducible* (simple) if it has no proper N-subgroup (ideal) of it [7]. The radical J_2 (N) is defined to be the intersection of all the annihilators of irreducible N-groups and J(N) is the intersection of all the maximal left ideals of N which are also maximal as left N-subgroups [7,8,9]. If $1 \in N$, it is seen that $J_2(N) = J(N)$ (=J) (Lemma 4 in [9]).

The radical of E is defined as

$\hat{J}(E) = \cap' \{M \mid M$ is an ideal maximal as a N-subgroup of N$\}$.

If A and B are two left N-subgroups of N such that $A \subseteq B$, then A is *essential left N-subgroup* of B (denoted by $A \subseteq_e B$) when any non-zero left N-ubgroup $C (C \subseteq B)$ of N has non-zero intersection with A (equivalently, B is an essential extension of A).

A left ideal I of N is said to be a *strictly essential* left ideal of N (denoted $I \underset{e}{\subseteq} N$), if for any left N-subgroup K ($\neq 0$) on N, $I \cap k \neq 0$.

A left N-subgroup A of N is *closed* if A has no proper essential extension. Thus $A \underset{e}{\subseteq} B \subseteq N \Rightarrow A = B$.

An ideal is said to be *closed* if it is closed as a left N-subgroup.

An idempotent e is called *co-central* if both e and (1–e) are central.

A set $\{e_i\}$ of arthogonal idempotents of N is called *right complete arthogonal idempotents* if $e = \Sigma e_i$ is a right identity of N.

An N-group E is said to be *uniform* if each non-zero ideal of E is essential in E.

Two subsets A and B of an N-group are said to be *linearly ordered* if either $A \subseteq B$ or $B \subseteq A$.

A finite sequence $E = E_0 \supseteq E_1 \ldots \supseteq E_t = 0$ of ideals is said to be a *principal sequence* if each $\frac{E_i}{E_{i+1}}$ is simple and such a finite sequence is said to be a *restricted principal sequence* if each $\frac{E_i}{E_{i+1}}$ is irreducible.

An N-group E is a *ps-N-group* if E has a principal sequence and is an *rps-N-group* if E has a restricted principal sequence. Clearly an rps-N-group is a ps-N-group. It is noted (p. 52 in [11]) that a ps-N-group thus defined [11] satisfies both the chain conditions on its ideals. In alternating group A_5, 0 is the only proper ideal. Clearly A_5 is not irreducible. Hence A_5 is not an rps-N-group. But it satisfies both the chain conditions on its N-subgroups. Hence it is a ps-N-group.

Left annihilator of a ε N is defined as $1(a) = \{n \in N \mid na = 0\}$.

Note 1. If N is with distributively generated left annihilators, then for two orthogonal idempotents $e_1, e_2 \in N$, we have $1(e_1) \cap 1(e_2) = 1(e_1 + e_2)$.

For, $x \in 1(e_1) \cap 1(e_2) \Rightarrow xe_1 = xe_2 = 0$

Now $1(e_1) = <S_1>$ where S_1 is a set of distributive elements of N.

$$\Rightarrow x = \Sigma \pm s_i,\ s_i \in S_1$$

$$\Rightarrow (\Sigma \pm s_i)(e_1 + e_2) = \Sigma (\pm s_i) e_2 = xe_2 = 0$$

$$\Rightarrow x \in 1(e_1 + e_2) \Rightarrow 1(e_1) \cap 1(e_2) \subseteq 1(e_1 + e_2)$$

and $y \varepsilon l(e_1 + e_2) = \langle s \rangle$, s is a set of distributive elements of N.

$$\Rightarrow y = \Sigma \pm d_i, \varepsilon s$$
$$\Rightarrow d_i (e_1 + e_2) = 0, \text{ for each } i$$
$$\Rightarrow d_i e_1 + d_i e_2 = 0$$
$$\Rightarrow d_i e_1^2 + d_i e_2 e_1 = 0$$
$$\Rightarrow d_i e_1 = 0$$
$$\Rightarrow d_i \varepsilon l(e_1).$$

Similarly, $d_i \varepsilon l(e_2)$.

$$\Rightarrow d_i \varepsilon l(e_1) \cap l(e_2) \text{ for each } i$$
$$\Rightarrow y \varepsilon l(e_1) \cap l(e_2)$$
$$\Rightarrow l(e_1 + e_2) \subseteq l(e_1) \cap l(e_2).$$

Therefore, $l(e_1) \cap l(e_2) = l(e_1 + e_2)$

Thus for any set $\{e_1,\ldots,e_n\}$ of orthogonal idempotents in a near-ring N with distributively generated left annihilators we have, $l(e_1) \cap \ldots \cap l(e_n) = l(e_1 + \ldots + e_n)$.

Radical J is called *D-regular* if for each $a \varepsilon J$, we have a $x \varepsilon J$ such that $a = xa$.

An N-group E is called *tame N-group* [11] if any N-subgroup of E is an ideal of E.

An N-group E is called *semi-simple* if it is the direct sum (or sum) of simple ideals [7].

The *near-ring N* is called *semi-simple* if N is semi-simple as an N-group.

A near-ring N is called *strictly semi-simple* if it is a direct sum of irreducible left ideals [7]. Clearly strictly semi-simple near-rings are semi-simple.

N-group E is *projective* if every diagram (I) can be embedded in the commutative diagram (II).

$$\begin{tikzcd}
 & E \arrow[d] & \\
A \arrow[r] & B \arrow[r] & O
\end{tikzcd}$$

(I)

$$\begin{tikzcd}
 & E \arrow[d] \arrow[dl] & \\
A \arrow[r] & B \arrow[r] & O
\end{tikzcd}$$

(II)

A short exact sequence (s.e.s.) $0 \to A \to B \xrightarrow{f} C \to 0$ *almost splits* if there is a splitting homomorphism $g : C \to B$ such that $fg = I_c$ (identity). The s.e.s. splits if there is a normal splitting homomorphism. N-group E is *strictly projective* if every s.e.s. splits. A strictly projective N-group is projective.

Near-ring N satisfies the *modulo essential descending chain condition (medcc)* if $\frac{N}{K}$ satisfies the d.c.c. on its N-subgroups for each essential left-ideal K of N.

Example. Consider $N = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a,b,c \in F\right\}$

Then N is a near-ring of 2×2 upper triangular matrices over a near-field F.

Here $A = \left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in F\right\}$, $D = \left\{\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \mid b,c \in F\right\}$ are left ideals of N.

Also $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are orthogonal idempotents of N.

We easily get $A = Ne$ and $D = Nf$ and thus $N = Ne \oplus Nf$. Also N-subgroups of Ne and Nf are linearly ordered.

Thus N is a left serial near-ring

Also we note that, when $Z = \{0, \pm 1, \pm 2, \ldots\}$, $Q = \{\frac{a}{b} \mid a,b \in Z, b \neq 0\}$ and $S = \{\frac{a}{b} \mid a,b \in Z, b \notin 2Z\}$ then S is a sub near-ring of Q. It can be easily seen that the proper ideals of S are of the form 2^kS $(k \in Z)$. The ideals of S satisfies the acc but not the dcc. Here 0 and 1 are the only orthogonal idempotents of S and $S = S.1 + S.0$. Thus S is a left serial near-ring. As Q is the field of fractions of S, $Q = \{\frac{a}{b} \mid a,b \in S\}$. Suppose A and B are any two S-subgroups of Q. If $A \not\subseteq B$, then $a \in A$ such that $a \notin B$.

Now b ε B $\Rightarrow$ a s $\not\subseteq$ bs ($\subseteq$ B). Again a,b, ε Q $\Rightarrow$ $\frac{a}{b}$ ε Q or $\frac{b}{a}$ ε Q which gives $\frac{a}{b}$ ε S or $\left(\frac{b}{a}\right)^{-1}$ ε S. Since $\frac{a}{b}$ ε S $\Rightarrow$ aS $\subseteq$ bS which is not the case, it follows that $\left(\frac{b}{a}\right)^{-1}$ ε S, which gives bS $\subseteq$ aS ($\subseteq$ A). And thus b = b.1 ε aS $\subseteq$ A (clearly 1 ε S). Hence S-subgroups of Q are linearly ordered.

In Klein's four group [11, p. 340], (1) is a near-ring w.r.t. the multiplication defined as in [11]. Here a set of orthogonal idempotents is {0, b}. Then e = 0 + b = b is a right identity but not a left identity. Thus {0, b} is a set of right completment orthogonal idempotents.

Also we see in dihedral group N(=D_8) = {0, a, 2a, 3a, b, a + b, 2a + b, 3a + b} [11] a set of orthogonal idempotents is {a, b}. Then e = a + b is a right identity not a left identity. Thus {a, b} is a set of right complete orthogonal idempotents.

Throughout our discussion, unless otherwise specified, N will denote a zero-symmetric right near-ring with 1 ε N.

3. PRELIMINARIES

If N is a d.g.n.r and M $\subseteq$ N, then we write $\langle M\rangle = \left\{ \sum_{finite} n_i m_i \mid n_i \; \varepsilon \; N, \; m_i \; \varepsilon \; M \right\}$.

Then ⟨M⟩ is a left N-subgroup generated by M and in the sense ⟨M⟩ is a finately generated left N-subgroup of N if M is finite.

Lemma 3.1. In a d.g.n.r N with the acc on its left N-subgroups, if principal N-subgroup are linearly ordered, then its left N-subgroups are principal.

Proof. Let K be a left N-subgroup of N. Then x_1 ε K gives $\langle x_1\rangle = Nx_1 \subseteq$ K. If $\langle x_1\rangle \subseteq$ K, then we have x_2 ε K such that $x_2 \notin \langle x_1\rangle$ and then $\langle x_1\rangle \subset \langle x_1,\ldots x_2\rangle \subseteq$ K. Thus we get a strictly accending chain of left N-subgroups of N, viz., $\langle x_1\rangle \subsetneq \langle x_1,x_2\rangle \subset \langle x_1,x_2,x_3\rangle \subset \ldots$, each one of these contained in K. As N satisfies the acc on some t ε Z^+, $\langle x_1,x_2,\ldots,x_t\rangle$ = K. Again since w.l.o.g. we can write $Nx_t \subseteq \ldots \subseteq Nx_1$. Now x_1 ε K gives $Nx_1 \subseteq$ K and clearly $\sum_{finite} n_i\, x_i$ ε Nx_1 (n_i ε N). Thus K = Nx_1.

Lemma 3.2. If $\{e_j\}$, $1 \leq j \leq k$ is a set of right complete orthogonal idempotents of N with distributively generated left annihilators, then $I \cong Ie_1 \oplus \ldots \oplus Ie_k$, where I is an ideal of N.

Proof. define $\phi : I \to Ie_1 \oplus Ie_2 \oplus \ldots \oplus Ie_k$

by $\phi(i) = (ie_1, ie_2, \ldots, ie_k)$

Then clearly ϕ is N-homomorphism

Let $\phi(i_1) = \phi(i_2)$, $i_1, i_2 \in I$

$\Rightarrow (i_1e_1, \ldots, i_1e_k) = (i_2e_1, \ldots, i_2e_k)$

$\Rightarrow i_1e_1 = i_2e_j$, $J = 1,2,\ldots,k$

$\Rightarrow (i_1 - i_2)\, e_j = 0$, $J = 1,2,\ldots,k$

$\Rightarrow i_1 - i_2 \in 1(e_1) \cap \ldots \cap 1(e_k) = 1(e_1 + \ldots + e_k)$

[by note 1] $= 1(e)$, where $e = \sum_{j=1}^{k} e_j$

$\Rightarrow (i_1 - i_2)\, e = 0$

$\Rightarrow i_1 - i_2 = 0$ $\quad$ [$e = \sum_{j=1}^{k} e_j$ is a right identity]

$\Rightarrow i_1 = i_2$

Thus ϕ is a monomorphism and so I is embedded in $Ie_1 \oplus \ldots \oplus Ie_k$ when left annihilators of N are distributively generated.

Then we can have $I \cong I' \subseteq Ie_1 \oplus \ldots \oplus \ldots \oplus Ie_k$

Again $Ie_1 \oplus Ie_2 \oplus \ldots \oplus Ie_k \subseteq I$.

Thus $I \cong Ie_1 \oplus \ldots \oplus e_k$

The radical J of a near-ring is a left ideal. If we define J.J as $J.J = \left\{ \sum_{\text{fin}} a_ib \mid a_i, b_i \in J \right\}$, then J.J can be given a left N-subgroup structure if N is a d.g.n.r. Because of the possible non-associative character we write J.(J.J) as left 3-power of J written as $J^{(1,3)}$. In general we say $J^{(1,k)}$ as J left k-power of J. We consider $J^{(1,t)} = J.J^{(1,t-1)}$.

Now the following lemma is of considerable importance to what we have proposed.

Lemma 3.3. If principal N-subgroups of a d.g.n.r. N with the acc on its left N-subgroups, are linearly ordered and radical J is a left ideal invariant as a sub-near ring, then $J^{(1,t)}$ is an ideal of $J^{(1,t-1)}$.

Proof. We have $J^{(1,t)} = J.J^{(1,t-1)} = \left\{\sum_{\text{fin}} a_i b_i \mid a_i \in J,\ b_i \in J^{(1,t-1)}\right\}$ is a left N-subgroup of N. By lemma (3.1), $J^{(1,t-1)} = Ny$ for some $y \in N$. It can be easily seen that Jy is an ideal of Ny. Thus we get the N-group structure of $\frac{Ny}{Jy}\ (= \frac{J^{(1,t-1)}}{Jy})$

The mapping $\alpha : J^{(1,t-1)} \to \frac{Ny}{Jy}$ defined by $\alpha(x) = x + Jy$, $x \in J^{(1,t-1)} = Ny$ is a natural epimorphism.

$$\frac{J^{(1,t-1)}}{Jy} \cong \frac{J^{(1,t-1)}}{\ker \alpha} \Rightarrow \ker \alpha = Jy$$

$$\begin{aligned} \text{Again, } \ker \alpha &= \{x \in J^{(1,t-1)} \mid x + Jy = Jy\} \\ &= \{x \mid x \in J^{(1,t)} \cap J^{(1,t-1)}\} \subseteq J^{(1,t)} \end{aligned}$$

$$\Rightarrow \ker \alpha \subseteq J^{(1,t)}$$

Suppose $x' \in J^{(1,t)} = J.J^{(1,t-1)}$

Then $x' = \sum_{\text{fin}} a_i b_i,\ a_i \in J,\ b_i \in J^{(1,t-1)}$

As each $a_i b_i \in Jy = \ker \alpha$, we get

$$J^{(1,t)} \subseteq \ker \alpha$$

$$\Rightarrow \ker \alpha = J^{(1,t)} = Jy$$

Thus we get $J^{(1,t)}$ is an ideal of $J^{(1,t-1)}$.

That is we get the N-group structure $\frac{J^{(1,t-1)}}{J^{(1,t)}}$ and is true for all $t \in Z^{+}$ (assuming $J^0 = N$).

If $\mathcal{F} = \{E' \mid E'$ is an ideal of $E, E' \neq E\}$ and $\mathcal{F}$ has a maximal element, say E'', then E'' is an ideal of E which is maximal as a left N-subgroup. So, the N-group E/E'' is irreducible.

Also we know that $J(N) = \cap$ ann (F), F is irreducible N-group. Now $K \subseteq J(N)$ gives us $K \subseteq \cap$ Ann (E/E'') which gives $KE \subseteq E' \subset E$. Hence $KE = E$ gives us $E = 0$.

Note 2. Thus we note that if N is a d.g.n.r. and E is an rps-N-group, then for any left ideal K of N with $K \subseteq J$, $KE = E$ gives $E = 0$.

4. THE MAIN THEOREM

Theorem 4.1. If N a d.g.n.r. where $T = \bigcap_{t=1}^{\infty} J^{(1,t)}$ is an rps-N-group and J is D-regular invariant sub near-ring, then $\bigcap_{t=1}^{\infty} J^{(1,t)} = 0$.

Proof. Here $JT = \left\{ \sum_{finite} j_i t_i \mid j_i \varepsilon J,\ t_i \varepsilon T \right\}$ is a left N-subgroup of N.

Now $JT \subseteq T \cap J$ as J is invariant and T is a left N-sub-group of N.

If $a \varepsilon T \cap J$ $(\subseteq J)$, J being D-regular, we have $a = xa(\varepsilon JT)$ for some $x \varepsilon J$.

Hence, $T \cap J \subseteq JT \Rightarrow JT = T \cap J = T$ and $T = 0$ [Note 2] $T = O$. i.e. $\bigcap_{t=1}^{\infty} J^{(1,t)} = 0$.

In the following result N is a d.g.n.r. in which principal N-subgroups are linearly ordered and left annihilators are distributively generated. Also N satisfies the acc on left N-subgroups and each summand of N is a strictly projective N-group with invariant radical J(N) such that $J(N) = J(N)N$.

Theorem 4.2. If N is a left serial near-ring w.r.t. a set $\{e_i\}$, $1 \leq i \leq k$ of right complete orthogonal idempotents such that for $t \geq 0$, each $j^{(1,t)}e_i$ is inherited strongly regular N-group, then for each $t \varepsilon Z^+$, a complete list of ideals of Ne_i which contain $J^{(1,t)} e_i$ is $Ne_i \supseteq Je_i \supseteq J^{(1,2)} e_i \supseteq \ldots \supseteq J^{(1,t)} e_i$.

Proof. N being a left serial, $N = Ne_1 \oplus \ldots \oplus Ne_k$, where N-subgroups of each Ne_i as an N-group are linearly ordered.

Then $J(Ne_i) = J(N)Ne_i$ [12] $= (J(N)N)e_i = J(N)\, e_i = Je_i$. Thus Je_i is the intersection of ideals of Ne_i maximal as N-subgroup of Ne_i. Then Je_i is the unique ideal maximal as N-subgroup of Ne_i.

Therefore, each $\frac{Ne_i}{Je_i}$ is an irreducible N-group.

Define $\phi : Ne_1 \oplus \ldots \oplus Ne_k \rightarrow \frac{Ne_1}{Je_1} \oplus \ldots \oplus \frac{Ne_k}{Je_k}$

by $\phi\, (N_1e_1, \ldots, n_ke_k) = (\overline{n_1e_1}, \ldots, \overline{n_ke_k})$ Clearly ϕ is onto.

Also it can be easily seen that

For an $x, y\ \varepsilon\ Ne_1 \oplus \ldots \oplus Ne_k$ and $r\ \varepsilon\ N$,

$$\phi(x + y) = \phi(x) + \phi(y) \text{ and } \phi(rx) = r\, \phi(x).$$

$$\begin{aligned} \text{Again } \ker \phi &= \{(n_1e_1, \ldots, n_ke_k) \mid \phi(n_1e_1, \ldots, n_ke_k) = (\bar{o}, \ldots, \bar{o})\} \\ &= \{(n_1e_1, \ldots, n_ke_k) \mid n_ie_i\ \varepsilon\ Je_i\},\ 1 \leq i \leq k \\ &= Je_1 \oplus \ldots \oplus Je_k \cong J \text{ (by lemma 3.2)} \end{aligned}$$

$$\text{So } \frac{N}{J} \cong \frac{Ne_1}{Je_1} \oplus \ldots \oplus \frac{Ne_k}{Je_k}$$

J invariant gives $\frac{N}{J}$ is a near-ring also.

We define $\alpha : \frac{N}{J} \times \frac{Ne_i}{Je_i} \rightarrow \frac{Ne_i}{Je_i}$ by

$$\alpha\, ((n + J)\, (n_ie_i + Je_i)) = nn_ie_i + Je_i$$

Then α is well defined because of the N-group structure of $\frac{Ne_i}{Je_i}$.

Also, $(\bar{n}_1 + \bar{n}_2)\ (n_ie_i + Je_i) = \bar{n}_1(n_ie_i + Je_i) + \bar{n}_2(n_ie_i + Je_i)$ and $\bar{n}_1\, (\bar{n}_2 \overline{n_ie_i}) = (\bar{n}_1\, \bar{n}_2)\ \overline{n_ie_i}$

Thus each $\frac{Ne_1}{Je_1}$ are $\frac{N}{J}$ – group also which are irreducible as $\frac{N}{J}$ group. Because, if not irreducible, then suppose $\frac{Ke_1}{Je_1}$ is a proper $\frac{N}{J}$ subgroup of $\frac{Ne_1}{Je_i}$.

Then $Je_i \subseteq Ke_i \underset{\neq}{\subseteq} Ne_i$

Now, $N \times \frac{Ke_1}{Je_1} \to \frac{Ne_i}{Je_i}$ by $n\,(\overline{Ke_1}) = \overline{nke_1}$ is well defined such that $(n_1 + n_2)\,\overline{Ke_1} = n_1\,(\overline{Ke_1}) + n_2\,(\overline{Ke_1})$ and $n_1\,(n_2\,(\overline{Ke_1})) = (n_1 n_2(\overline{Ke_1})$

Therefore $\frac{Ke_i}{Je_1}$ is a N-subgroup of $\frac{Ne_1}{Je_1}$ which gives $\frac{Ne_1}{Je_1}$ is not irreducible as N-group, a contradiction.

Hence $\frac{Ne_1}{Je_1}$ is an irreducible $\frac{N}{J}$ group.

By lemma (3.3), $J^{(1,t+1)}$ is an ideal of $J^{(1,t)}$.

Now $J^{(1,t+1)} \subseteq J^{(1,t)} \Rightarrow J^{(1,t+1)}\, e_i \subseteq J^{(1,t)}\, e_i$.

Also $-(ae_i) + je_i + (ae_i) = (-a + j + a)\, e_i \;\varepsilon\; J^{(1,t+1)}\, e_i$, $ae_i \;\varepsilon\; J^{(1,t)}$, $je_i \;\varepsilon\; J^{(1,t+1)}$ and $n(je_i + ae_i) - n(ae_i) = (n(j + a) - na)\, e_i \;\varepsilon\; J^{(1,t+1)}\, e_i$, $n \;\varepsilon\; N$.

Therefore, $J^{(1,t+1)}e_i$ is an ideal of $J^{(1,t)}e_i$.

Thus $\frac{J^{(1,t)}e_1}{J^{(1,t+1)}e_1}$ exists as an N-group.

Hence exists as an $\frac{N}{J}$ – group also.

Now suppose, $\frac{Me_1}{J^{(1,t+1)}e_1}$ be an $\frac{N}{J}$ subgroup of $\frac{J^{(1,t)}e_1}{J^{(1,t+1)}e_i}$.

Then Me_i is an $\frac{N}{J}$ subgroup of $J^{(1,t)}\, e_i$ which gives Me_i is an N-subgroup of $J^{(1,t)}\, e_i \subseteq N$.

Since for each $t \geq 0$, $J^{(1,t)}e_i$ is a tame N-group therefore Me_i is an ideal of $J^{(1,t)}e_i$.

$$\Rightarrow \frac{Me_i}{J^{(1,t+1)}e_i} \text{ is an ideal of } \frac{J^{(1,t)}e_i}{J^{(1,t+1)}e_i}$$

$$\Rightarrow \frac{Me_i}{J^{(1,t+1)}e_i} \text{ is an } \frac{N}{J} \text{ ideal of } \frac{J^{(1,t)}e_i}{J^{(1,t+1)}e_i}$$

Thus $\frac{N}{J}$ subgroups of $\frac{J^{(1,t)}e_i}{J^{(1,t+1)}e_i}$ are $\frac{N}{J}$ ideals, $t \geq 0$.

We have $\frac{N}{J} \cong \frac{Ne_1}{Je_1} \oplus \ldots \oplus \frac{Ne_k}{Je_k}$

Then $\frac{N}{J}$ subgroups of $\frac{N}{J}$ are left ideals (because $\frac{N}{J}$ subgroups of each $\frac{Ne_i}{Je_i}$ are ideals).

But each $\frac{Ne_i}{Je_i}$ are $\frac{N}{J}$ subgroups of $\frac{N}{J}$.

Hence we get, $\frac{N}{J}$ is the direct sum of irreducible left ideals. Therefore $\frac{N}{J}$ is strictly semi-simple which gives $\frac{N}{J}$ is semi-simple and has the d.c.c. on left ideals [7].

Thus $\frac{N}{J}$ is semi-simple Artinian near-ring. Therefore, $\frac{J^{(1,t)}e_i}{J^{(1,t+1)}e_i}$ is semi-simple as $\frac{N}{J}$ group [7].

But as N-subgroups of each Ne_i are linearly ordered, therefore $\frac{J^{(1,t)}e_i}{J^{(1,t+1)}e_i}$ is either zero or simple.

Hence in either case a complete list of ideals of Ne_i which contain $J^{(1,t)}e_i$ is $Ne_i \supseteq Je_i \supseteq J^{(1,2)}e_i \supseteq \dots \supseteq J^{(1,t)}e_i$.

Theorem 4.3. If N is as above and $T = \bigcap_{t=1}^{\infty} J^{(1,t)}$ is an rps-N-group, then N satisfies medcc.

Proof. Here $N = Ne_1 \oplus \dots \oplus Ne_k$ where N-subgroups of each Ne_i are linearly ordered.

Let K be a strictly essential left ideal of N.

Each $Ne_i \cap K \subseteq K$, therefore

$(Ne_1 \cap K) + (Ne_2 \cap K) + \dots + (Ne_k \cap K) \subseteq K \subseteq N \; (= Ne_1 \oplus \dots \oplus Ne_k)$

Hence $(Ne_1 \cap K) \oplus \dots \oplus (Ne_k \cap K)$ is direct.

That is $(Ne_1 \cap K) \oplus \dots \oplus (Ne_k \cap K) \subseteq K$.

It can be easily seen that $(Ne_i \cap K)$ is an ideal of Ne_i. Now define $\phi : Ne_1 \oplus \dots \oplus Ne_k \to \frac{Ne_1}{Ne_1 \cap K} \oplus \dots \oplus \frac{Ne_k}{Ne_k \cap K}$ by $\phi\,(n_1e_1, \dots, n_ke_k) = (\overline{n_1e_1}, \dots, \overline{n_ke_k})$

Clearly ϕ is an epimorphism and

$$\begin{aligned}
\operatorname{Ker} \phi &= \{(n_1e_1, \dots, n_ke_k) \mid (n_1e_1, \dots, n_ke_k) = (\bar{o}, \dots, \bar{o})\} \\
&\ \{(n_1e_1, \dots, n_ke_k) \mid n_ie_i \in Ne_i \cap K\},\ i = 1, \dots, k \\
&= \{(n_1e_1, \dots, n_ke_k) \mid (n_1e_1, \dots, n_ke_k) \in \oplus\,(Ne_i \cap K)\} \\
&= (Ne_1 \cap K) \oplus \dots \oplus (Ne_k \cap K)
\end{aligned}$$

Therefore, $\frac{N}{(Ne_1 \cap K) \oplus \dots \oplus (Ne_k \cap K)} = \frac{Ne_1}{Ne_1 \cap K} \oplus \dots \oplus \frac{Ne_k}{Ne_k \cap K}$

As K is strictly essential left ideal, therefore

$$Ne_i \cap K \neq 0,\ i = 1, \dots, k.$$

Suppose e be one of the e_i 's and $L(\neq 0)$ be a N-subgroup of Ne.

By theorem (4.1), $\bigcap_{t=1}^{\infty} J^{(1,t)} = 0$. Then we have

$$\bigcap_{t=1}^{\infty} J^{(1,t)}e = 0$$

Therefore, $L \not\subseteq J^{(1,m)} e$ for some $m \varepsilon Z^+$.

Since N-subgroups of Ne are linearly ordered, therefore, $J^{(1,m)}e \subseteq L$.

Again by theorem (4.2.), a complete list of ideals of Ne which contain $J^{(1,m)} e$ is

$$Ne \supseteq Je \supseteq J^{(1,2)}e \supseteq \ldots \supseteq J^{(1,m)}e$$

Therefore $L = J^{(1,m)}e$, for some $n \varepsilon \{1,2 \ldots, m\}$

Thus the only ideals of Ne which contain L are

$$Ne \supseteq Je \supseteq J^{(1,2)}e \supseteq \ldots \supseteq J^{(1,n-1)}e \text{ and } J^{(1,n)}e \ (= L)$$

That is $\frac{Ne}{L}$ satisfies the d.c.c. on left ideals

$$\Rightarrow \frac{Ne}{L} \text{ is Artinian which gives}$$

$$\frac{N}{(Ne_1 \cap K) \oplus \ldots \oplus (Ne_k \cap K)} \text{ is an Artinian N-group}$$

Again we have, $$\frac{N}{K} = \frac{\dfrac{N}{(Ne_1 \cap K) \oplus \ldots \oplus (Ne_k \cap K)}}{\dfrac{K}{(Ne_1 \cap K) \oplus \ldots \oplus (Ne_k \cap K)}}$$

Therefore $\frac{N}{K}$ is Artinian N-group which proves N satisfies medcc.

REFERENCES

1. M. N. Barua, Near-rings and Near-ring modules, some special types, Naya Prakash, Calcutta, 1984.

2. J. C. Belidleman, Non semi-simple distributively generated near-rings with minimum condition, Math Annalen 170 (1967), 206-213.

3. K. C. Chowdhury, A. Masum and H. K. Saikia : FSD N-group with acc on annihilators, Indian J. Pure and Applied Maths. 24(2), (1993), 747-755.

4. C. G. Fain, Some structure theorems for near-rings Doctoral dissertation, University of Oklahama, 1968.
5. C. G. Lyons, Endomorphism Near-rings, Oberwallfach, 1972.
6. C. G. Lyons, On decompositions of E(G), Rocky Mountain J. Math. 3 (1973), 575-582.
7. G. R. Mason, Injective and Projective near-ring modules, composite Mathematica, 33, Fasc. 1 (1976), 43-54.
8. G. R. Mason and A. Oswald, Injective and Projective near-ring modules. Teeside polytechnique Mathematical reports (1981).
9. A. Oswald, Semi-simple Near-rings having the maximal condition on subgroups, J. London Math. Soc. (2) 11 (1975), 408-412.
10. A. Oswald, Near-rings in which every N-subgroup is principal, Proc. London Math. Soc. (3) 23 (1974), 67-88.
11. G. Pilz, Near-rings, North Holland, Amsterdam. 1977.
12. Helen K. Saikia and K. C. Chowdhury. On near-ring subgroups of projective and d near-ring groups, Bull, Calcutta Math. Soc. 88 (1996), 63-70.

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DIRECT *k*-SUMMANDS IN SEMIRINGS

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ABSTRACT : Idea of direct summands in semirings was introduced by Padmanavan and Subramanian [4] for a commutative semiring. Later Cornish [1] developed the idea to non-commutative semirings and established a sort of relation between a direct summand and a complemented both-sided ideals of a semiring. In this paper we try to extend this concept in the light of *k*-ideals of a semiring.

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1. INTRODUCTION

A *semiring* is a non-empty set S together with two binary operations '+' and '·' such that, $(S,+)$ is a commutative semigroup and $(S,\cdot)$ is a semigroup which are related by ring like distributivity. A semiring is called *commutative* if both the operations are commutative. An additively cancellative semiring is called a *halfring.* Call an element $s \in S$ a *zeroid* if $s + a = a$, for some $a \in S$. We donete by $Z(S)$, the set of all zeroids of S. If S is a semiring with zero 0, then $0 \in Z(S)$. The zero element of S, is called an *absorbing zero* if $a.0 = 0.a = 0$ for all $a \in S$. If $Z(S) = \{0\}$, then S is called *plain* [2]. An identity element of S, if it exists, is an element $1_s \in S$ such that $1_s.x = x.1_s = x$ for all $x \in S$. A *k*-ideal [3] I of a semiring S is an ideal of S such that if $a \in I$ and $x \in S$ and $a + x \in I$ then $x \in I$. Let A be a subsemiring of a semiring S. Then $\overline{A} = \{a \in S \mid a + x \in A$ for some $x \in A\}$ is called the *k-closure* [5] of A. It can be verified that if A is an ideal of S then $\overline{A}$, as defined above, is a *k*-ideal of S. Infact, it is the smallest *k*-ideal of S containing A and $A = \overline{A}$ if and only if A is a *k*-ideal. Throughout this paper we assume that S has an absorbing zero 0 and an identity element 1_s.

2. DIRECT *k*-SUMMAND

Let S be a semiring. We recall that, a pair of elements (e_1,e_2), for some $e_1,e_2 \in S$ is called an *idempotent pair* [5] in the semiring S, if $e_2 + e_2^2 = e_2e_1$ and $e_1 + e_1e_2 = e_1^2$. We now give the definition of a complemented pair of elements in a semiring S as follows :

Definition 2.1. In a semiring S, a pair of elements (s_1,s_2) for some $s_1,s_2 \in S$ is said to be *complemented*, with another pair, say, $(r_1,r_2) \in S \times S$ as its complement, if

$$1_s + s_2 + r_2 = s_1 + r_1 \text{ and } r_is_j = 0 = s_ir_j \text{ for } i = 1,2, \; ; j = 1,2 \; ;$$

We see that, $(1_s,0)$ is a complemented pair in S with the coresponding pair being $(0,0)$. Following result is an automatic consequence of definitions of idempotent pair and Definition 2.1.

Proposition 2.2. *If* (e_1,e_2) *is a complemented pair in* S, *with* (f_1,f_2) *as its complement, then* (e_1,e_2) *and* (f_1,f_2) *are idempotent pairs.*

We now define the concept of a *central complemented pair* in S as follows :

Definition 2.3. In a semiring S, a complemented pair (e,f) is said to be *central*, if for all $x \in S$, we have,

$$ex + xf + (e + f)\, x\, (e + f) = xe + fx + (e + f)\, x\, (e + f).$$

Towards defining a both-sided k-ideal generated by a central complemented pair we now introduce the following :

Definition 2.4. For a central complemented pair $(e_1,e_2) \in S \times S$, we define,

$$\langle e_1,e_2\rangle_k = \{x \in S \mid x + xe_2 + s = xe_1 + s\},$$

where, $s = \sum\limits_{i,j} s_ie_1s_j + \sum\limits_{i,j} t_ie_2t_j \in S$ for some $s_i,s_j,t_i,t_j \in S$, for all $i,j \in N$ and summations are all finite.

Proposition 2.5. $\langle e_1,e_2\rangle_k$ *is a k-ideal of S.*

Proof. Indeed, for any $x,y \in \langle e_1,e_2\rangle_k$ and any $s_1 \in S$, $x + y$, $s_1x \in \langle e_1,e_2\rangle_k$ follows immediately. We prove that, $xs_1 \in \langle e_1,e_2\rangle_k$ also. Now, $x \in \langle e_1,e_2\rangle_k$ implies $x + xe_2 + s = xe_1 + s$, where s is as defined in Definition 2.4. This gives us,

$$xs_1 + xe_2s_1 + ss_1 = xe_1s_1 + ss_1 \qquad \ldots (2.1)$$

Now, since (e_1,e_2) is central, for all $x \in S$,

$$e_1x + xe_2 + \bar{s} = xe_1 + e_2x + \bar{s},$$

where,

$$\bar{s} = (e_1 + e_2)\, x\, (e_2 + e_1) \in S.$$

This gives us,

$$e_1xs_1 + xe_2s_1 + \bar{s}s_1 = xe_1s_1 + e_2xs_1 + \bar{s}s_1. \qquad \dots (2.2)$$

Now adding $e_1xs_1 + \bar{s}s_1$ in both sides of (3.2.1) we have,

$$xs_1 + xe_2s_1 + e_1xs_1 + \bar{s}s_1 + ss_1 = xe_1s_1 + e_1xs_1 + \bar{s}s_1 + ss_1$$

This gives us,

$$xs_1 + xe_1s_1 + e_2xs_1 + \bar{s}s_1 + ss_1 = xe_1s_1 + e_1xs_1 + \bar{s}s_1 + ss_1 \dots (2.3)$$

[by (2.2)]

Now as, $xs_1 \in S$, and (e_1,e_2) is central, we also have,

$$e_1(xs_1) + (xs_1)e_2 + \tilde{s} = (xs_1)e_1 + e_2(xs_1) + \tilde{s}, \qquad \dots (2.4)$$

where, $\tilde{s} = (e_1 + e_2)\, xs_1\, (e_1 + e_2)$ by Definition 2.3. Now on adding $(xs_1e_1 + \tilde{s})$ on both sides of (2.3) we get,

$$xs_1 + xe_1s_1 + e_2xs_1 + xs_1e_1 + \bar{s}s_1 + ss_1 + \tilde{s} = xe_1s_1 + e_1xs_1 + xs_1e_1 + \bar{s}s_1 + ss_1 + \tilde{s}$$

which implies on using (2.4),

$$xs_1 + xe_1s_1 + e_1xs_1 + xs_1e_2 + \tilde{s} + \bar{s}s_1 + ss_1 = xe_1s_1 + e_1xs_1 + xs_1e_1 + \bar{s}s_1 + ss_1 + \tilde{s}$$

i.e.,

$$xs_1 + xs_1e_2 + \hat{s} = xs_1e_1 + \hat{s}$$

where,

$$\hat{s} = xe_1s_1 + e_1xs_1 + \bar{s}s_1 + ss_1 + \tilde{s}$$

and through a somewhat routine calculation it can be shown that, $\hat{s}$ is in accordance with the requirements of Definition 2.4. Hence with an appeal to the Definition 2.4 we conclude that, $xs_1 \in \langle e_1,e_2\rangle_k$. Consequently, $\langle e_1,e_2\rangle_k$ is an ideal of S. Next, let $x + y, y \in \langle e_1,e_2\rangle_k$, so that, $(x + y) + (x + y)e_2 + r_1 = (x + y)e_1 + r_1$ and $y + ye_2 + r_2 = ye_1 + r_2$, where, r_1,r_2 are elements of S of the form stipulated as in Definition 2.4. Now,

$$x + y + xe_2 + ye_2 + r_1 = xe_1 + ye_1 + r_1$$

i.e.,

$$x + xe_2 + (y + ye_2 + r_2) + r_1 = xe_1 + ye_1 + r_2 + r_1$$

i.e.,

$$x + xe_2 + ye_1 + r_2 + r_1 = xe_1 + ye_1 + r_2 + r_1$$

which implies $x \in \langle e_1,e_2\rangle_k$; infact, as $ye_1 = ye_1 1_s$ we see that, $ye_1 + r_2 + r_1$ can be expressed in the required form of Definition 2.4, this proves that, $\langle e_1,e_2\rangle_k$ is a k-ideal of S.

Now the stage is set to define a direct k-summand and a complemented k-ideal of a semiring and to check their interrelation.

Definition 2.6. A *direct k-summand* of a semiring S is a *k-ideal* A of S, for which there exists another k-ideal B of S, called *k-cosummand* of A, such that, each $x \in S$ can be expressed as, $x + a + b = a' + b'$, where, $a,a' \in A$ and $b,b' \in B$; and such an expression is unique in the sense that, if,

$$x + a_1 + b_1 = a_2 + b_2 \text{ and } x + a_3 + b_3 = a_4 + b_4$$

then, $a_1 + b_4 = a_2 + a_3$ and $b_1 + b_4 = b_2 + b_3$, where, $a_i \in A, b_i \in B$ for all $i = 1,2,3,4$.

Definition 2.7. A k-ideal A in a semiring S is called *complemented* if there exists another k-ideal B such that, $A \cap B = (0)$ and $\overline{A+B} = S$, where, $\overline{A+B}$ is the k-closure of $A + B$.

Lemma 2.8. *In a semiring S, if a k-ideal J be complemented, then there exist some,* $e_1,e_2 \in S$ *such that,* (e_1,e_2) *is a central complemented pair and* $J = \langle e_1,e_2\rangle_k$.

Proof. As the k-ideal J is complemented, we have another k-ideal, say, K such that, $J \cap K = (0)$ and $\overline{J+K} = S$. As $1_s \in S$, $1_s + e_2 + f_2 = e_1 + f_1$, for some $e_1,e_2 \in J$ and $f_1 f_2 \in K$ which implies,

$$e_1 + e_1e_2 + e_1f_2 = e_1^2 + e_1f_1$$

i.e.,

$$e_1 + e_1e_2 = e_1^2,$$

as e_1f_2, $e_1f_1 \in J \cap K = (0)$ i.e., $e_1f_2 = e_1f_1 = (0)$. Proceeding in an essentially similar manner, we may show that the other required criteria are fulfilled, whence (e_1,e_2) is a complemented pair with $(f_1 f_2)$ as a complement. It is indeed a central complemented pair.

Towards proving this, let us consider any $y \in S$. As we have $1_s + e_2 + f_2 = e_1 + f_1$ then on multiplying it by y, from the left and from the right respectively, we get,

$$y + ye_2 + yf_2 = ye_1 + yf_1 \text{ and } y + e_2y + f_2y = e_1y + f_1y$$

where, $ye_2, ye_1, e_1y, e_2y \in J$ and $f_1y, f_2y, yf_1, yf_2 \in K$ since J and K are ideals. Again $1_s + e_2 + f_2 = e_1 + f_1$ also gives that,

$$e_2y + e_2ye_2 + e_2yf_2 = e_2ye_1 + e_2yf_1,$$

whence by virtue of $e_2yf_2, e_2yf_1 \in J \cap K = (0)$ we get,

$$e_2y + e_2ye_2 = e_2ye_1. \qquad \ldots (2.5)$$

Similarly, we may show,

$$e_1y + e_2ye_2 = e_1ye_1, \qquad \ldots (2.6)$$

$$ye_2 + e_2ye_2 = e_1ye_2, \text{ and} \qquad \ldots (2.7)$$

$$ye_1 + e_2ye_1 = e_1ye_1, \qquad \ldots (2.8)$$

so that,

$$e_2y + e_2ye_2 + e_2ye_1 = e_1y + e_1ye_2 + e_2ye_1 \quad \text{[by (2.5) and (2.6)]}$$

which implies, by (2.8) that,

$$e_2y + e_2ye_2 + e_1ye_1 + (ye_1 + e_2ye_1) = e_1y + e_1ye_2 + e_2ye_1 + e_1ye_1$$

which implies, by (2.7) that,

$$e_2y + ye_1 + e_2ye_2 + e_1ye_1 + e_1ye_2 + e_2ye_1 = e_1y + e_1ye_2 + e_2ye_1 + e_1ye_1 + (ye_2 + e_2ye_2)$$

i.e.,

$$e_2y + ye_1 + (e_2ye_2 + e_1ye_1 + e_1ye_2 + e_2ye_1) = e_1y + ye_2 + (e_2ye_2 + e_1ye_1 + e_1ye_2 + e_2ye_1)$$

i.e.,

$$e_2y + ye_1 + (e_1 + e_2)\, y\, (e_1 + e_2) = e_1y + ye_2 + (e_1 + e_2)\, y\, (e_1 + e_2)$$

whence (e_1,e_2) is a central complemented pair.

Now, let $x \in J$; then, $x + xe_2 + xf_2 = xe_1 + xf_1$ which implies $x + xe_2 = xe_1$, as $xf_1, xf_2 \in J \cap K = (0)$ which indicates that, $x \in \langle e_1,e_2\rangle_k$ as $0 \in S$, whence $J \subseteq \langle e_1,e_2\rangle_k$. Again, if $x \in \langle e_1,e_2\rangle_k$ then $x + xe_2 + s = xe_1 + s$, where s is as prescribed in Definition 2.4. Now, as $e_1,e_2 \in J$, and J is a k-ideal of S, hence $x \in J$ so that $\langle e_1,e_2\rangle_k \subseteq J$. Consequently, $J = \langle e_1,e_2\rangle_k$.

Lemma 2.9. *in a plain semiring S, if $J = \langle e_1,e_2\rangle_k$ such that (e_1,e_2) is a central complemented pair, then J is a complemented k-ideal of S.*

Proof. Since (e_1,e_2) is a complemented pair, there exists its complement pair, say, (f_1,f_2) for some $f_1,f_2 \in S$. We claim that,

(i) $\langle e_1,e_2\rangle_k \cap \langle f_1,f_2\rangle_k = (0)$ and,

(ii) $\overline{\langle e_1,e_2\rangle_k + \langle f_1,f_2\rangle_k} = S.$

To prove our claim, let, $x \in \langle e_1,e_2\rangle_k \cap \langle f_1,f_2\rangle_k$, then,

$$x + xe_2 + s_1 = xe_1 + s_1, \text{ and,} \qquad \dots (2.9)$$

$$x + xf_2 + s_2 = xf_1 + s_2 \; ; \qquad \dots (2.10)$$

where $s_1,s_2 \in S$ and are in accordance with Definition 2.4. Now, multiplying (2.10) by e_1 in the right, we get,

$$xe_1 + xf_2e_1 + s_2e_1 = xf_1e_1 + s_2e_1$$

i.e.,

$$xe_1 + s_2e_1 = s_2e_1$$

and, multiplying (2.10) by e_2 in the right, we get,

$$xe_2 + xf_2e_2 + s_2e_2 = xf_1e_2 + s_2e_2$$

i.e.,

$$xe_2 + s_2e_2 = s_2e_2 \text{ [by Definition 2.1]}$$

whence we have,

$$\begin{aligned} & xe_2 + xe_1 + s_2e_2 + s_2e_1 \\ = {} & s_2e_1 + s_2e_2 \\ = {} & s_2e_1 + xe_1 + s_2e_2 \\ = {} & xe_1 + xe_1 + s_2e_1 + s_2e_2 \qquad [\text{as } s_2e_1 = xe_1 + s_2e_1] \end{aligned}$$

From (2.9) we know, $x + xe_2 + s_1 = xe_1 + s_1$ which implies, on adding $xe_1 + s_2e_1 + s_2e_2$ in both sides,

$$x + xe_2 + xe_1 + s_1 + s_2e_2 + s_2e_1 = xe_1 + xe_1 + s_1 + s_2e_1 + s_2e_2$$

i.e.,

$$x + xe_1 + xe_1 + s_2e_2 + s_2e_1 + s_1 = xe_1 + xe_1 + s_2e_2 + s_2e_1 + s_1$$

i.e.,

$$x + \hat{s} = \hat{s},$$

where, $\hat{s} = xe_1 + xe_1 + s_2e_2 + s_2e_1' + s_1 \in S$ whence $x \in Z(S) = \{0\}$ as S is plain. This implies, $x = 0$. Consequently,

$$\langle e_1,e_2\rangle_k \cap \langle f_1,f_2\rangle_k = (0).$$

Now, to prove (ii), it will be sufficient to show,

$$S \subseteq \overline{\langle e_1,e_2\rangle_k + \langle f_1,f_2\rangle_k}.$$

Indeed, as (e_1,e_2) and (f_1,f_2) are mutually complemented pairs, we have, $1_s + e_2 + f_2 = e_1 + f_1$. If $x \in S$ be arbitrary, then, $x + xe_2 + xf_2 = xe_1 + xf_1$. Now, as (e_1,e_2) is an idempotent pair [by Proposition 2.2], so $e_1 + e_1e_2 = e_1^2$ and $e_2 + e_2^2 = e_2e_1$ show that, $e_2e_1 \in \langle e_1,e_2\rangle_k$ and since it is an ideal, so xe_1, $xe_2 \in \langle e_1,e_2\rangle_k$ and a similar argument would show that, $xf_1, xf_2 \in \langle f_1,f_2\rangle_k$ so that, we have,

$$x \in \overline{\langle e_1,e_2\rangle_k + \langle f_1,f_2\rangle_k}.$$

Hence the lemma follows.

Finally, we conclude this paper with the following theorem.

Theorem 2.10. *In a halfring S, if a k-ideal J be complemented then it is a direct k-summand.*

Proof. As k-ideal J is complemented in S, by Lemma 2.8. we get, $J = \langle e_1,e_2\rangle_k$ and by Lemma 2.9. there must exist another k-ideal, $\langle f_1,f_2\rangle_k = K$ (say) for some mutually complemented central complemented pairs (e_1,e_2) and (f_1,f_2) such that, $J \cap K = (0)$ and $\overline{J+K} = S$. Now, suppose $x \in S$ be some arbitrary element of S such that, $x + a_1 + b_1 = a_2 + b_2$ and $x + a_3 + b_3 = a_4 + b_4$ where $a_i \in J$ and $b_i \in K$ for $i = 1,2,3,4$. Now, $xe_1 + a_1e_1 + b_1e_1 = a_2e_1 + b_2e_1$, and, $xe_2 + a_1e_2 + b_1e_2 = a_2e_2 + b_2e_2$ which imply $xe_1 + a_1e_1 + b_1e_1 + a_2e_2 + b_2e_2 = xe_2 + a_1e_2 + b_1e_2 + a_2e_1 + b_2e_1$, so that, $xe_1 + a_1e_1 + a_2e_2 = xe_2 + a_1e_2 + a_2e_1$ (since $J \cap K = (0)$). In a similar way, it can be shown that, $xe_1 + a_3e_1 + a_4e_2 = xe_2 + a_3e_2 + a_4e_1$. Now, $1_s + e_2 + f_2 = e_1 + f_1$ implies $a_1 + a_1e_2 + a_1f_2 = a_1e_1 + a_1f_1$ i.e., $a_1 + a_1e_2 = a_1e_1$ (as $a_1 \in J$; $f_1,f_2 \in K$ and $J \cap K = (0)$). Similarly, we can show that,

$$a_2 + a_2e_2 = a_2e_1$$

$$a_3 + a_3e_2 = a_3e_1$$

$$a_4 + a_4e_2 = a_4e_1$$

Now, we have, $xe_1 + a_1e_1 + a_2e_2 = xe_2 + a_1e_2 + a_2e_1$. Adding $a_1 + a_2$ in both sides we get,

$$xe_1 + a_1e_1 + (a_2e_2 + a_2) + a_1 = xe_2 + a_2 + a_2e_1 + (a_1e_2 + a_1)$$

$$\text{i.e., } xe_1 + a_1e_1 + a_2e_1 + a_1 = xe_2 + a_2e_1 + a_1e_1 + a_2$$

$$\text{i.e., } xe_1 + a_1 = xe_2 + a_2 \qquad \ldots \text{(i)}$$

Again, we have, $xe_1 + a_3e_1 + a_4e_2 = xe_2 + a_3e_2 + a_4e_1$;

On adding $a_3 + a_4$ on both sides we get,

$$xe_1 + a_3e_1 + (a_4e_2 + a_4) + a_3 = xe_2 + (a_3e_2 + a_3) + a_4e_1 + a_4$$

$$\text{i.e., } xe_1 + a_3e_1 + a_4e_1 + a_3 = xe_2 + a_3e_1 + a_4e_1 + a_4$$

$$\text{i.e., } xe_1 + a_3 = xe_2 + a_4 \qquad \ldots \text{(ii)}$$

Combining (i) and (ii) we get,

$$xe_2 + xe_1 + a_1 + a_4 = xe_2 + xe_1 + a_2 + a_3$$

whence, $a_1 + a_4 = a_2 + a_3$.

In an essentially similar manner it can be shown that, $b_1 + b_4 = b_2 + b_3$. Consequently, the required nature of uniqueness of representation of x, as prescribed in Definition 2.6. is attained. Hence, $J = \langle e_1,e_2\rangle_k$ is a direct k-summand, with $K = \langle e_1,e_2\rangle_k$ as a k-cosummand. Hence the theorem follows.

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REFERENCES

1. W. H. Cornish, Direct summands in semirings ; *Math. Japon.* 16 (1971), 13-19.
2. J. S. Golan, The theory of semirings with applications in mathematics and theoretical computer science, *Pitman Monographs and Surveys in Pure and Applied Mathematics,* 54, Longman Sci. Tech., Harlow, 1992.
3. D. R. Latorre, On h-ideals and k-ideals in hemirings ; *Publ. Math. Debrecen* 12 (1965), 219-226.

4. R. Padmanavan and H. Subramanian, Ideals in semirings ; *Math. Japon.* 13 (1968), 123-128.

5. M. K. Sen and P. Mukhopadhyay, Von Neumannian regularity in semirings ; *Kyungpook Math. J.* 35 (No.2) (1995), 249-258.

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ON ENTIRE FUNCTIONS OF BOUNDED INDEX IN SEVERAL COMPLEX VARIABLES

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ABSTRACT : In this paper we have obtained a necessary and sufficient condition for an entire function in $\mathbb{C}^n$ to be of bounded index and also we have proved that the set of all entire functions of bounded index in $\mathbb{C}^n$ is closed with respect to usual multiplication of functions.

AMS Subject Classification : 32A

Key words : Bounded index, Unbounded index, minimal element

1. INTRODUCTION

In [5], Krishna and Shah introduced the concept of bounded index for analytic functions in several complex variables with respect to an arbitrary given point α in $\mathbb{C}^n$. Thereafter Chakraborty and Mitra [2] introduced the concept of entire functions of bounded index in $\mathbb{C}^n$ without any reference point.

In the case of one complex variable, Fricke [3] obtained "a necessary and sufficient condition for an entire function to be of bounded index in $\mathbb{C}$"

Further, in [3] he proved "If f is of bounded index, then $g(z) = f(az + b)$ is of bounded index for any $a, b, c \in \mathbb{C}$".

Further, in [3] he proved "the class of all entire functions of bounded index in $\mathbb{C}$ is closed under multiplication".

Considering the concept of Chakraborty and Mitra [2], here we have generalised the above three results in the case of several complex variables. The results which we have obtained are as follows :

Theorem 1. An entire function f in $\mathbb{C}^2$ is of bounded index iff for each $(r_1,r_2) \geq (0,0)$ $\exists$ non-negative integers $n_1 = n_1(r_1,r_2)$ and $n_2 = n_2(r_1,r_2)$ and constants $M_1 = M_1(r_1,r_2) > 0$, $M_2 = M_2(r_1,r_2) > 0$, such that for each $z = (z_1,z_2) \in \mathbb{C}^2$ $\exists$ integers $l = l(z_1,z_2)$, $m = m(z_1,z_2)$ with $0 \leq l \leq n_1$ and $0 \leq m \leq n_2$ and

$$\max_{\substack{|\xi_i - z_i| = r_i \\ i=1,2}} \left\{ \left| \frac{\partial^{l+m} f(\xi_1,\xi_2)}{\partial z_1^l \partial z_2^m} \right| \right\} \leq M_1 M_2 \left| \frac{\partial^{l+m} f(z_1,z_2)}{\partial z_1^l \partial z_2^m} \right|$$

Theorem 2. If f is of bounded index in $\mathbb{C}^2$, then $g(z_1,z_2) = f(a_1 z_1 + b_1,\ a_2 z_2 + b_2)$ is of bounded index for any (a_1,a_2), $(b_1,b_2) \in \mathbb{C}^2$.

Theorem 3. If $f(z_1,z_2)$ and $g(z_1,z_2)$ are of bounded index, then $h(z_1,z_2) = f(z_1,z_2)\, g(z_1,z_2)$ is of bounded index.

2. NOTATIONS

We denote complex and real n-spaces by $\mathbb{C}^n$ and R^n respectively and the set of non-negative integers by I. $I^n_+ = \{x / x \in I^n,\ x_i > 0,\ i = 1,2, \ldots, n\}$. We indicate the points $(z_1,\ldots,z_n)$, $(w_1,\ldots,w_n)$, etc. of $\mathbb{C}^n$ or R^n by their corresponding unsuffixed symbols z, w etc. For z, $w \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$, we define $\alpha z = (\alpha z_1,\ldots, \alpha z_n)$, $z + w = (z_1 + w_1,\ldots, z_n + w_n)$, $z.w = z_1 w_1 + \ldots + z_n w_n$, $|z| = (|z_1|^2 + \ldots + |z_n|^2)^{1/2}$. The positive hyperoctant R^n_+ in R^n will be $R^n_+ = \{x/x \in \mathrm{R}^n,\ x_i \geq 0,\ i = 1, \ldots, n\}$.

For $t \in R^n_+$, we set $\| t \| = t_1 + \ldots + t_n$ and for $m \in I^n$, $m! = m_1!, \ldots, m_n!$, $z^m = z_1^{m_1} \ldots z_n^{m_n}$ for $z \in \mathbb{C}^n$, ($z_i^0 = 1$ even if $z_i = 0$). For any $k \in R$, $\bar{k} = (k, \ldots, k)$. In particular, $\bar{o}$ and $\bar{\infty}$ will respectively denote the n-tuples $(o,\ldots,o)$ and $(\infty,\ldots,\infty)$. Also for $x,y \in \mathrm{R}^n$, we say that (i) $x \leq y$ iff $x_i \leq y_i$ (ii) $x < y$ iff $x \leq y$ but $x \neq y$ (iii) $x << y$ iff $x_i < y_i$, $i = 1,\ldots,n$. For an entire function f with domain $\mathbb{C}^n$, $f^{k)}$ will denote the partial derivative $\dfrac{\partial^{|k|} f}{\partial z_1^{k_1} \ldots \partial z_n^{k_n}}$ where $k \in I^n$ and $f^{(0)} = f$. Also we denote $\dfrac{1}{k!} f^{k)}$ by f_k.

§3. An entire function f in $\mathbb{C}^n$ has a Taylor expansion about any point $w \in \mathbb{C}^n$ of the form : $f(z) = \sum_{m=\bar{0}}^{\bar{\infty}} \alpha_m (z - w)^m$, where $a_m = \dfrac{f^{(m)}(w)}{w!}$.

Definition 1. Let f be an entire function in $\mathbb{C}^n$. Then for each point $w \in \mathbb{C}^n$ there exists an n-tuple $m(w) \in I^n$ such that

$$\max_{p \leq m} \left\{ \frac{\left|f^{(p)}(w)\right|}{p!} \right\} \geq \left\{ \frac{\left|f^{(q)}(w)\right|}{q!} \right\} \ \forall q \in I^n.$$

There may be several tuples m for which the above inequality holds. A minimal element of such tuples is called an index of f at the point w.

Definition 2. An entire function f in $\mathbb{C}^n$ is said to be of **bound index** iff there exists an $m \in I^n$ such that for all $z \in \mathbb{C}^n$,

$$\max_{p \leq m} \left\{ \frac{\left|f^{(p)}(z)\right|}{p!} \right\} \geq \left\{ \frac{\left|f^{(q)}(z)\right|}{q!} \right\} \ \forall q \in I^n. \qquad \ldots (3.1)$$

There may have several tuples satisfying (3.1). A minimal element of such tuples is called an **index** of f and is denoted by $I(f)$. An entire function which is not of bounded index is said to be of **unbounded index.**

§4. In this section we prove theorems for an entire function in $\mathbb{C}^2$ instead of $\mathbb{C}^n$, only to avoid complicated notations. However, the proofs of the theorems in $\mathbb{C}^n$ are immediate.

Proof of theorem 1. For $r = (r_1, r_2) \geq (o, o)$ and $z = (z_1, z_2) \in \mathbb{C}^2$ and an entire function f, let us define

$$M_{l+m}(f, z, r) = \max_{\substack{|\xi_i - z_i| = r_i \\ i=1,2}} \left\{ \left| \frac{\partial^{l+m} f(\xi_1, \xi_2)}{\partial z_1^l \partial z_2^m} \right| \right\} \ for \ l, \ m = 0, 1, 2, \ldots$$

Necessary condition. Let us assume that f is of bounded index and also let (p,q) be an index of f.

For $(r_1, r_2) \geq (o, o)$, let us choose $n_1 = n_1(r_1, r_2) = p$ and $n_2 = n_2(r_1, r_2) = q$ and $M_1 = M_1(r_1, r_2) = 2^{h_1} \ p!$ and $M_2 = M_2(r_1, r_2) = 2^{h_2} \ q!$, where h_1, h_2 are positive integers such that

$$\frac{r_1}{h_1} \leq \frac{1}{4(p+1)} \text{ and } \frac{r_2}{h_2} \leq \frac{1}{4(q+1)}$$

For $z^{(0)} = \left(z_1^0, z_2^0\right) \in C^2$, let us choose $l = \left(l\left(z_1^0, z_2^0\right),\ m = m\ \left(z_1^0, z_2^0\right)\right)$ to be an index of f at $z^{(0)}$. Then obviously $l \leq p$ and $m \leq q$. Now by the lemma [2, pp 10] we get

$$\max_{\substack{(k_1,k_2)\leq(p,q)\\ \left|z_i - z_i^{(0)}\right| = r_i\\ i=1,2}} \left\{\frac{1}{k_1!k_2!}\left|\frac{\partial^{k_1+k_2} f(z_1,z_2)}{\partial z_1^{k_1}\partial z_2^{k_2}}\right|\right\} \leq 2^{h_1+h_2}\frac{1}{l!m!}\left\{\left|\frac{\partial^{l+m} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^l \partial z_2^m}\right|\right\}$$

Now, $$M_{l+\mathrm{m}}(f, z_{(0)}, r) \leq p!q!\max_{\substack{(l,m)\leq(p,q)\\ \left|z_i - z_i^{(0)}\right| = r_i\\ i=1,2}}\left\{\frac{1}{l!m!}\left|\frac{\partial^{l+m} f(z_1,z_2)}{\partial z_1^l \partial z_2^m}\right|\right\}$$

$$\leq p!q!\,2^{h_1+h_2}\frac{1}{l!m!}\left|\frac{\partial^{l+m} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^l \partial z_2^m}\right| \leq M_1 M_2 \left|\frac{\partial^{l+m} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^l \partial z_2^m}\right|$$

Since $z^{(0)}$ is arbitrary, the necessary condition follows.

Sufficient condition. From the given conditions we may assume that for $r = (2,2)$ $\exists$ non-negative integers $n_1 = n_1(2,2)$, $n_2 = n_2(2,2)$ and constants $M_1 = M_1(2,2) > 0$, $M_2 = M_2(2,2) > 0$ such that for each $z = (z_1, z_2) \in \mathbb{C}^2$ $\exists$ integers $l = l(z_1, z_2)$, $m = m(z_1, z_2)$ with $0 \leq l \leq n_1$ and $0 \leq m \leq n_2$ and

$$M_{l+m}(f, z, r) \leq M_1 M_2 \left|\frac{\partial^{l+m} f(z_1,z_2)}{\partial z_1^l \partial z_2^m}\right| \qquad \ldots (4.1)$$

We propose to show that f is of bounded index. Now, $\exists$ integers $m_1, m_2 > 0$ such that

$$\begin{cases} n_1! M_1 2^{-m_1} < 1 \\ n_2! M_2 2^{-m_2} < 1 \end{cases} \qquad \ldots (4.2)$$

Here we take m_1, m_2 as the least positive integers for which (4.2) is satisfied. We now estimate $\dfrac{1}{p_1!p_2!}\left|\dfrac{\partial^{p_1+p_2} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{p_1}\partial z_2^{p_2}}\right|$ in the following three cases :

Case (i) $p_1 \geq m_1 + n_1$ and $p_2 \geq m_2 + n_2$

Case (ii) $p_1 \geq m_1 + n_1$ and $p_2 < m_2 + n_2$

Case (iii) $p_1 < m_1 + n_1$ and $p_2 \geq m_2 + n_2$

Case (i). For an arbitrary point $z^{(0)} = \left(z_1^{(0)}, z_2^{(0)}\right) \in \mathbb{C}^2$ $\exists$ integers $l_0 = l_0\left(z_1^{(0)}, z_2^{(0)}\right)$, $m_0 = m_0\left(z_1^{(0)}, z_2^{(0)}\right)$ with $0 \leq l_0 \leq n_1$ and $0 \leq m_0 \leq n_2$ such that

$$M_{l_0+m_0}(f,\ z^{(0)},\ r) \leq M_1 M_2 \left|\frac{\partial^{l_0+m_0} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{l_0} \partial z_2^{m_0}}\right| \qquad \ldots (4.3)$$

Now, by Cauchy's inequality for an entire function g of two variables, we have,

$$\left|\frac{\partial^{k_1+k_2} g(z_1, z_2)}{\partial z_1^{k_1} \partial z_2^{k_2}}\right| \leq \frac{k_1! k_2!}{R_1^{k_1} R_2^{k_2}} \max_{\substack{|\xi_i - z_i| = R_i \\ i=1,2}} \{|g(\xi_1, \xi_2)|\}$$

for k_1, $k_2 = 0, 1, 2, \ldots$ and any R_1, $R_2 > 0$. Then, substituting $g(z_1, z_2) = \dfrac{\partial^{l_0+m_0} f(z_1, z_2)}{\partial z_1^{l_0} \partial z_2^{m_0}}$ and $R_1 = 2 = R_2$ in the above inequality, we obtain

$$\frac{1}{k_1! k_2!} \left|\frac{\partial^{(l_0+k_1)+(m_0+k_2)} f(z_1, z_2)}{\partial z_1^{l_0+k_1} \partial z_2^{m_0+k_2}}\right| \leq 2^{-k_1} 2^{-k_2} \max_{\substack{|\xi_i - z_i| = 2 \\ i=1,2}} \left\{\left|\frac{\partial^{l_0+m_0} f(\xi_1, \xi_2)}{\partial z_1^{l_0} \partial z_2^{m_0}}\right|\right\}$$

$$= 2^{-k_1} 2^{-k_2} M_{l_0+m_0}(f, z, \overline{2}) \qquad \ldots (4.4)$$

Hence,

$$\frac{1}{p_1! p_2!} \left|\frac{\partial^{p_1+p_2} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{p_1} \partial z_2^{p_2}}\right| = \frac{1}{p_1! p_2!} \left|\frac{\partial^{(l_0+p_1-l_0)+(m_0+p_2-m_0)} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{p_1} \partial z_2^{p_2}}\right|$$

$$= \frac{(p_1 - l_0)!}{p_1!} \frac{(p_2 - m_0)!}{p_2!} \frac{1}{(p_1 - l_0)!(p_2 - m_0)!} \left|\frac{\partial^{(l_0+p_1-l_0)+(m_0+p_2-m_0)} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{p_1} \partial z_2^{p_2}}\right|$$

$$\leq \frac{1}{(p_1 - l_0)!(p_2 - m_0)!} \left|\frac{\partial^{l_0+(p_1-l_0)+m_0+(p_2-m_0)} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{p_1} \partial z_2^{p_2}}\right|$$

$$\leq 2^{-(p_1-l_0)}\ 2^{-(p_2-m_0)}\ M_{l_0+m_0}(f, z^{(0)}, \overline{2}) \qquad \text{[By 4.4]}$$

$$\leq 2^{-(p_1-l_0)}\, 2^{-(p_2-m_0)} M_1 M_2 \left|\frac{\partial^{l_0+m_0} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{l_0} \partial z_2^{m_0}}\right| \qquad \text{[By 4.3]}$$

$$\leq 2^{n_1-p_1}\, 2^{n_2-p_2} M_1 M_2 \left|\frac{\partial^{l_0+m_0} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{l_0} \partial z_2^{m_0}}\right| \leq 2^{-m_1} M_1\ 2^{-m_2} M_2 \left|\frac{\partial^{l_0+m_0} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{l_0} \partial z_2^{m_0}}\right|$$

$$\leq \frac{1}{n_1! n_2!}\left|\frac{\partial^{l_0+m_0} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{l_0} \partial z_2^{m_0}}\right| \text{[By 4.2]}$$

$$\leq \frac{1}{l_0! m_0!}\left|\frac{\partial^{l_0+m_0} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{l_0} \partial z_2^{m_0}}\right| \forall p_i \geq n_i + m_i,\ i = 1,2.$$

Case (ii). $\dfrac{1}{p_1! p_2!}\left|\dfrac{\partial^{p_1+p_2} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{p_1} \partial z_2^{p_2}}\right|$

$$= \frac{(p_1-l_0)!}{p_1!}\,\frac{1}{(p_1-l_0)!\,p_2!}\left|\frac{\partial^{(l_0+p_1-l_0)+p_2} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{p_1} \partial z_2^{p_2}}\right| \leq \frac{1}{(p_1-l_0)!\,p_2!}$$

$$\left|\frac{\partial^{l_0+(p_1-l_0)+p_2} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{p_1} \partial z_2^{p_2}}\right|$$

Now keeping z_2 fixed in f so that f becomes a function of single variable z_1 and applying Cauchy's inequality we get,

$$\frac{1}{p_1! p_2!}\left|\frac{\partial^{p_1+p_2} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{p_1} \partial z_2^{p_2}}\right| .$$

$$\leq 2^{-(p_1-l_0)}\,\frac{1}{p_2!} \max_{|\xi_l - z_l^{(0)}|=2}\left\{\left|\frac{\partial^{l_0+p_2} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{l_0} \partial z_2^{p_2}}\right|\right\} \text{ for all } p_2 < m_2 + n_2.$$

$$= \frac{2^{-(p_1-l_0)}}{p_2!} M_{l_0}\left(\frac{\partial^{p_2} f}{\partial z_2^{p_2}}, z^{(0)}, 2\right) \leq \frac{2^{-(p_1-l_0)}}{p_2!} M_1 \left|\frac{\partial^{l_0+p_2} f(z_1^0, z_2^0)}{\partial z_1^{l_0} \partial z_2^{p_2}}\right|$$

$$\leq 2^{-m_1} M_1 \frac{1}{p_2!} \left|\frac{\partial^{l_0+p_2} f(z_1^0, z_2^0)}{\partial z_1^{l_0} \partial z_2^{p_2}}\right| \leq \frac{1}{n_1!} \frac{1}{p_2!} \left|\frac{\partial^{l_0+p_2} f(z_1^0, z_2^0)}{\partial z_1^{l_0} \partial z_2^{p_2}}\right| \qquad \text{[By 4.2]}$$

$$\leq \frac{1}{l_0! p_2!} \left|\frac{\partial^{l_0+p_2} f(z_1^0, z_2^0)}{\partial z_1^{l_0} \partial z_2^{p_2}}\right| \text{ for all } p_1 \geq m_1 + n_1 \text{ and } p_2 < m_2 + n_2.$$

Case (iii). Proceeding as in the case (ii) we get,

$$\frac{1}{p_1! p_2!} \left|\frac{\partial^{p_1+p_2} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{p_1} \partial z_2^{p_2}}\right| \leq \frac{1}{p_1! m_0!} \left|\frac{\partial^{p_1+m_0} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{p_1} \partial z_2^{m_0}}\right| \text{ for all } p_1 < m_1 + n_1 \text{ and } p_2 \geq m_2 + n_2.$$

Combining the cases (i), (ii) and (iii) we see that for $k_1, k_2 = 0, 1, 2, \ldots$

$$\frac{1}{k_1! k_2!} \left|\frac{\partial^{k_1+k_2} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{k_1} \partial z_2^{k_2}}\right|$$

$$\leq \max \left\{ \frac{1}{t_1! t_2!} \left|\frac{\partial^{t_1+t_2} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{t_1} \partial z_2^{t_2}}\right| \text{ (for } 0 \leq t_1 \leq m_1 + n_1 \text{ and } 0 \leq t_2 \leq n_2 + m_2\text{)},\right.$$

$$\frac{1}{l_0! m_0!} \left|\frac{\partial^{l_0+m_0} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{l_0} \partial z_2^{m_0}}\right|, \frac{1}{l_0! p_2!} \left|\frac{\partial^{l_0+p_2} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{l_0} \partial z_2^{p_2}}\right| \text{ (for } p_2 < m_2 + n_2\text{)},$$

$$\left.\frac{1}{p_1! m_0!} \left|\frac{\partial^{p_1+m_0} f(z_1^{(0)}, z_2^{(0)})}{\partial z_1^{p_1} \partial z_2^{m_0}}\right| \text{ (for } p_1 < m_1 + n_1\text{)}\right\}$$

$\Rightarrow f$ is of bounded index at $(z_1^{(0)}, z_2^{(0)})$ and an index pair of f does not depend on the point under consideration. Since $(z_1^{(0)}, z_2^{(0)})$ is arbitrary, f is of bounded index.

Proof of Theorem 2. We know that g is an entire function for all (a_1, a_2), (b_1, b_2), $\in \mathbb{C}^2$.

Case (i). The theorem is trivial if we take $a_1 = a_2 = 0$, because in this case g is a constant function and hence is of bounded index.

Case (ii). We now suppose $a_1 \neq 0$ and $a_2 \neq 0$.

Subcase (i). First we suppose $b_1 = b_2 = 0$.

Then $g(z_1, z_2) = f(a_1z_1, a_2z_2)$. Since f is of bounded index, we have by theorem (1), for each $(r_1, r_2) \geq (0, 0)$ $\exists$ non-negative integers $n_1 = n_1(r_1, r_2)$, $n_2 = n_2(r_1, r_2)$ and constants $M_1 = M_1(r_1, r_2) > 0$, $M_2 = M_2(r_1, r_2) > 0$ such that for each $z = (z_1, z_2) \in \mathbb{C}^2$ $\exists$ integers $l = l(z_1, z_2)$, $m = m(z_1, z_2)$ with $0 \leq l \leq n_1$ and $0 \leq m \leq n_2$ and

$$M_{l+m}(f,z,r) \leq M_1M_2\left|\frac{\partial^{l+m}f(z_1,z_2)}{\partial z_1^l \partial z_2^{m_2}}\right| \qquad \ldots (4.5)$$

We now choose $(z_1^{(0)}, z_2^{(0)})$ such that $z_1 = a_1 z_1^{(0)}$ and $r_1 = |a_1| \cdot r_1^{(0)}$ where $r_1^{(0)} = |z_1^{(0)}|$, $z_2 = a_2 z_2^{(0)}$, and $r_2 = |a_2| r_2^{(0)}$ where $r_2^{(0)} = |z_2^{(0)}|$. Then,

$$M_{l+m}(g,z^{(0)},r^{(0)}) = \max_{\substack{|\xi_i - z_i^{(0)}| = r_i^0 \\ i=1,2}} \{|\, a_1^l a_2^m f^{(l+m)}(a_1\xi_1, a_2\xi_2)|\}$$

$$= |a_1|^l\, |a_2|^m \max_{\substack{\left|\frac{\gamma_i}{a_i} - z_i^{(0)}\right| = r_i^0 \\ i=1,2}} \{|f^{(l+m)}(\gamma_1, \gamma_2)|\} \text{ [where } \gamma_i = a_i\,\xi_i,\ i = 1,2]$$

$$= |a_1|^l\, |a_2|^m\, M_{l+m}(f,z,r) \leq |a_1|^l\, |a_2|^m\, M_1M_2\left|\frac{\partial^{l+m}f(z_1,z_2)}{\partial z_1^l \partial z_2^{m_2}}\right| \qquad \text{[By (4.5)]}$$

$$= M_1M_2\, |g^{(l+m)}(z_1^{(0)}, z_2^{(0)})|$$

Hence by theorem (1), g is of bounded index.

Subcase (ii). We now suppose that $(b_1, b_2) \neq (O, O)$. Let (p, q) be an index of f at $(a_1 z_1^{(0)} + b_1, a_2 z_2^{(0)} + b_2)$. Then $\frac{1}{p!q!}\left|\frac{\partial^{p+q}f(a_1z_1^{(0)}+b_1, a_2z_2^{(0)}+b_2)}{\partial z_1^p \partial z_2^q}\right|$

$$\geq \frac{1}{l!m!}\left|\frac{\partial^{l+m}f(a_1z_1^{(0)}+b_1, a_2z_2^{(0)}+b_2)}{\partial z_1^l \partial z_2^m}\right| \text{ for all } l,\ m = 0,\ 1,\ 2,\ \ldots$$

$$\Rightarrow \frac{1}{p!q!}\left|\frac{\partial^{p+q}g(z_1^{(0)},z_2^{(0)})}{\partial z_1^p \partial z_2^q}\right| \geq \frac{1}{l!m!}\left|\frac{\partial^{l+m}g(z_1^{(0)},z_2^{(0)})}{\partial z_1^l \partial z_2^m}\right| \quad \text{for all } l,\ m = 0,\ 1,\ 2,\ ..$$

$\Rightarrow$ g is of bounded index at $z^{(0)}$. Since f is of bounded index, it follows that g is of bounded index.

Case (iii). We now suppose that only one of a_1 and a_2 is equal to zero and without any loss of generality we assume that $a_2 = 0$ but $a_1 \neq 0$. Then $g(z_1, z_2) = f(a_1 z_1 + b_1, b_2)$ for all $(z_1, z_2) \in \mathbb{C}^2$. In this case g reduces to a function of single variable z_1 only. So, g is of bounded index [3]. Hence the theorem.

§5. Let $f(z_1, z_2)$, $g(z_1, z_2)$ be of bounded index where $M = (M_1, M_2)$ is an index of f and $N = (N_1, N_2)$ is an index of g. Also let $L = (L_1, L_2) = (M_1 + N_1, M_2 + N_2)$. We define constants $A_0, A_1, \ldots, A_{L_1}$, and $B_0, B_1, \ldots, B_{L_2}$, by $A_{L_1} = 1$ and $A_n = \frac{1}{2L_1}(A_{n+l})^2$ for $0 \leq n \leq L_1$ and $B_{L_2} = 1$ and $B_n = \frac{1}{2L_2}(B_{n+l})^2$ for $0 \leq n \leq L_2$

Furthermore, let $f_{k_1,k_2}(z_1, z_2) = \dfrac{1}{k_1!k_2!}\dfrac{\partial^{k_1+k_2}f(z_1,z_2)}{\partial z_1^{k_1}\partial z_2^{k_2}}$

We now prove the following lemma.

Lemma. If $h(z_1, z_2) = f(z_1, z_2)\, g(z_1, z_2)$, then for each $(z_1^{(0)},z_2^{(0)}) \in \mathbb{C}^2$ $\exists$ integers $k_1 = k_1(z_1^{(0)},z_2^{(0)}) \leq L_1$, $k_2 = k_2(z_1^{(0)},z_2^{(0)}) \leq L_2$ such that $\left|h_{k_1,k_2}(z_1^{(0)},z_2^{(0)})\right| \geq \frac{1}{4} A_{k_1} B_{k_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right|$ where (i_1,i_2) is an index of f at $(z_1^{(0)},z_2^{(0)})$ and (j_1,j_2) is an index of g at $(z_1^{(0)},z_2^{(0)})$.

Proof. Let (k_1,k_2) be a minimal element of the set $(n_1, n_2) \in I^2$ for which

$$\left|f_{l_1,l_2}(z_1^{(0)},z_2^{(0)})\, g_{n_1-l_1,n_2-l_2}(z_1^{(0)},z_2^{(0)})\right| \geq A_{n_1} B_{n_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \quad \ldots (5.0)$$

for some l_1, l_2 where $0 \leq l_1 \leq n_1$ and $0 \leq l_2 \leq n_2$. If we take $n_1 = i_1 + j_1$, $n_2 = i_2 + j_2$ and $l_1 = i_1$, $l_2 = i_2$, we have

$$\left|f_{l_1,l_2}(z_1^{(0)},z_2^{(0)})\, g_{n_1-l_1,n_2-l_2}(z_1^{(0)},z_2^{(0)})\right| \geq A_{n_1}\, B_{n_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right|$$

Therefore such a (k_1,k_2) exists where $k_1 \leq L_1$ and $k_2 \leq L_2$ and, we have for this (k_1,k_2),

$$\left|f_{l_1,l_2}(z_1^{(0)},z_2^{(0)})\, g_{k_1-l_1,k_2-l_2}(z_1^{(0)},z_2^{(0)})\right| \geq A_{k_1}\, B_{k_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \ldots \quad (5.1)$$

where $0 \leq l_1 \leq k_1$ and $0 \leq l_2 \leq k_2$. Again we have

$$\left\{ \begin{array}{l} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\right| \geq \left|f_{l_1,l_2}(z_1^{(0)},z_2^{(0)})\right| \\ \\ \left|g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \geq \left|g_{k_1-l_1,k_2-l_2}(z_1^{(0)},z_2^{(0)})\right| \end{array} \right. \quad \ldots (5.2)$$

Therefore from (5.1) and (5.2) we obtain

$$\left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\right| g_{k_1-l_1,k_2-l_2}(z_1^{(0)},z_2^{(0)})\Big| \geq \left|f_{l_1,l_2}(z_1^{(0)},z_2^{(0)})\, g_{k_1-l_1,k_2-l_2}(z_1^{(0)},z_2^{(0)})\right|$$

$$\geq A_{k_1}\, B_{k_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right|$$

and

$$\left|f_{l_1,l_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \geq \left|f_{l_1,l_2}(z_1^{(0)},z_2^{(0)})\, g_{k_1-l_1,k_2-l_2}(z_1^{(0)},z_2^{(0)})\right|$$

$$\geq A_{k_1}\, B_{k_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right|$$

i.e.

$$\left\{ \begin{array}{l} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\right| g_{k_1-l_1,k_2-l_2}(z_1^{(0)},z_2^{(0)})\Big| \geq A_{k_1}\, B_{k_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \\ \text{and} \\ \left|f_{l_1,l_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \geq A_{k_1}\, B_{k_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \end{array} \right. \quad \ldots (5.3)$$

Again, clearly

$$\left\{ \begin{array}{l} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\right| \neq 0 \\ \text{and} \\ \left|g_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\right| \neq 0 \end{array} \right. \quad \ldots (5.4)$$

Therefore (5.3) and (5.4) implies that

$$\begin{cases} \left|f_{l_1,l_2}(z_1^{(0)},z_2^{(0)})\right| \geq A_{k_1}\, B_{k_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\right| \\ \text{and} \\ \left|g_{k_1-l_1,k_2-l_2}(z_1^{(0)},z_2^{(0)})\right| \geq A_{k_1}\, B_{k_2} \left|g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \end{cases} \qquad \ldots (5.5)$$

Now, by our choice of (k_1, k_2) we must have,

$$\left|f_{u_1,u_2}(z_1^{(0)},z_2^{(0)})\, g_{w_1,w_2}(z_1^{(0)},z_2^{(0)})\right| < A_{u_1+w_2}\, B_{u_1+w_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \qquad \ldots (5.6)$$

for all (u_1, u_2), (w_1, w_2), with $(u_1 + w_1, u_2 + w_2) << (k_1, k_2)$.

In particular, for $u_1 = l_1$, $u_2 = l_2$, we get

$$\left|f_{l_1,l_2}(z_1^{(0)},z_2^{(0)})\, g_{w_1,w_2}(z_1^{(0)},z_2^{(0)})\right| < A_{l_1+w_1}\, B_{l_2+w_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right|$$

$< A_{k_1-1}\, B_{k_2-1} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right|$ for $0 \leq w_1 < k_1 - l_1$ and $0 \leq w_2 < k_2 - l_2$. and hence by (5.5) we have,

$$A_{k_1} B_{k_2} \left|g_{w_1,w_2}(z_1^{(0)},z_2^{(0)})\right| < A_{k_1-1}\, B_{k_2-1} \left|g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| = \frac{A_{k_1}^2 B_{k_2}^2}{4L_1L_2} \left|g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right|$$

$$\left|g_{w_1,w_2}(z_1^{(0)},z_2^{(0)})\right| < \frac{A_{k_1} B_{k_2}}{4L_1L_2} \left|g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \text{ for } 0 \leq w_1 < k_1 - l_1 \text{ and } 0 \leq w_2 < k_2 - l_2. \qquad \ldots (5.7)$$

Again in (5.6), putting $w_1 = k_1 - l_1$ and $w_2 = k_2 - l_2$ and then using (5.5), we obtain

$$\left|f_{u_1,u_2}(z_1^{(0)},z_2^{(0)})\right| < \frac{1}{4L_1L_2} A_{k_1}\, B_{k_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\right| \text{ for } 0 \leq u_1 < l_1 \text{ and } 0 \leq u_2 < l_2 \ldots (5.8)$$

Let us now replace l_2 by $2t_2 - l_2$ and $k_2 - l_2$ by $2v_2 + l_2 - k_2$ (where $0 \leq l_2, t_2, v_2 \leq k_2$) in (5.1), and we obtain

$$\left|f_{l_1,2t_2-l_2}(z_1^{(0)},z_2^{(0)})\, g_{k_1-l_1,2v_2+l_2-k_2}(z_1^{(0)},z_2^{(0)})\right| \geq A_{k_1}\, B_{k_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right|$$

for $0 \leq l_1 \leq k_1$ and $0 \leq l_2, t_2, v_2 \leq k_2$ and it implies

$$\left\{\begin{array}{l} \left|f_{l_1,2t_2-l_2}(z_1^{(0)},z_2^{(0)})\right| \geq A_{k_1}\,B_{k_2}\left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\right| \\ \text{and} \\ \left|g_{k_1-l_1,2v_2+l_2-k_2}(z_1^{(0)},z_2^{(0)})\right| \geq A_{k_1}\,B_{k_2}\left|g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \end{array}\right. \qquad \ldots (5.9)$$

In particular, putting $u_1 = v_1$, $u_2 = k_2 - v_2$, $w_1 = k_1 - l_1$, $w_2 = 2v_2 + l_2 - k_2$ in (5.6), we get

$$\left|f_{v_1,k_2-v_2}(z_1^{(0)},z_2^{(0)})\,g_{k_1-l_1,2v_2+l_2-k_2}(z_1^{(0)},z_2^{(0)})\right| < A_{v_1+k_1-l_1}B_{v_2+l_2}\left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\,g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right|$$

$\leq A_{k_1-1}\,B_{k_2-1}\left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\,g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right|$ for $0 \leq v_1 < l_1$ and $0 \leq v_2 < k_2 - l_2$ and hence by (5.9) we obtain

$$A_{k_1}\,B_{k_2}\left|f_{v_1,k_2-v_2}(z_1^{(0)},z_2^{(0)})\,g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \leq \left|f_{v_1,k_2-v_2}(z_1^{(0)},z_2^{(0)})\,g_{k_1-l_1,2v_2+l_2-k_2}(z_1^{(0)},z_2^{(0)})\right|$$

$< A_{k_1-1}\,B_{k_2-1}\left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\,g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right|$ for $0 \leq v_1 < l_1$ and $0 \leq v_2 < k_2 - l_2$.

This implies

$$\left|f_{v_1,k_2-v_2}(z_1^{(0)},z_2^{(0)})\right| < \frac{1}{4L_1L_2}A_{k_1}\,B_{k_2}\left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\right|$$

for $0 \leq v_1 < l_1$ and $0 \leq v_2 < k_2 - l_2$ ....(5.10)

Again in (5.6), putting $u_1 = l_1$, $u_2 = 2t_2 - l_2$, $w_1 = t_1$, $w_2 = k_2 - t_2$, we obtain

$$\left|f_{l_1,2t_2-l_2}(z_1^{(0)},z_2^{(0)})\,g_{t_1,k_2-t_2}(z_1^{(0)},z_2^{(0)})\right| < A_{l_1+t_1}B_{t_2+k_2-l_2}\left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\,g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right|$$

$\leq A_{k_1-1}\,B_{k_2-1}\left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\,g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right|$ for $0 \leq t_1 < k_1 - l_1$ and $0 \leq t_2 < l_2$ and hence by (5.9),we obtain

$$A_{k_1}\,B_{k_2}\left|g_{t_1,k_2-t_2}(z_1^{(0)},z_2^{(0)})\right| < A_{k_1-1}\,B_{k_2-1}\left|g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| = \frac{1}{4L_1L_2}A_{k_1}^2B_{k_2}^2\left|g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right|$$

This implies

$$\left|g_{t_1,k_2-t_2}(z_1^{(0)},z_2^{(0)})\right| < \frac{1}{4L_1L_2}A_{k_1}\,B_{k_2}\left|g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \text{ for } 0 \leq t_1 < k_1 - l_1 \text{ and } 0 \leq t_2 < l_2 \ldots (5.11)$$

Again, replacing l_2 by $l_2 - 1$ and $k_1 - l_1$ by $k_1 - l_1 - 1$ and $k_2 - l_2$ by $2s_2 + l_2 - k_2$ (where $0 \leq l_1 \leq k_1$ and $0 \leq l_2,\ s_2 \leq k_2$) in (5.1), we get

$$\left|f_{l_1,l_2-1}(z_1^{(0)},z_2^{(0)})\, g_{k_1-l_1-1,2s_2+l_2-k_2}(z_1^{(0)},z_2^{(0)})\right| \geq A_{k_1} B_{k_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \text{ for}$$

$0 \leq l_1 \leq k_1$ and $0 \leq l_2,\ s_2 \leq k_2$. This implies

$$\begin{cases} \left|f_{l_1,l_2-1}(z_1^{(0)},z_2^{(0)})\right| \geq A_{k_1} B_{k_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\right| \\ \text{and} \\ \left|g_{k_1-l_1-1,2s_2+l_2-k_2}(z_1^{(0)},z_2^{(0)})\right| \geq A_{k_1} B_{k_2} \left|g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \end{cases} \qquad \ldots (5.12)$$

In particular, putting $u_1 = l_1$, $u_2 = k_2 - s_2$, $w_1 = k_1 - l_1 - 1$ and $w_2 = 2s_2 + l_2 - k_2$ in (5.6), and then using (5.12), we get,

$$\left|f_{l_1,k_2-s_2}(z_1^{(0)},z_2^{(0)})\right| < \frac{1}{4L_1L_2} A_{k_1} B_{k_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\right| \text{ for } 0 \leq s_2 < k_2 - l_2. \qquad \ldots (5.13)$$

Again in (5.6), putting $u_1 = l_1$, $u_2 = l_2 - 1$, $w_2 = k_2 - l_2$ and then using (5.12), we get,

$$\left|g_{w_1,k_2-l_2}(z_1^{(0)},z_2^{(0)})\right| < \frac{1}{4L_1L_2} A_{k_1} B_{k_2} \left|g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \text{ for } 0 \leq w_1 < k_1 - l_1 \qquad \ldots (5.14)$$

Further replacing l_1 by $2p_1 - l_1$, l_2 by $l_2 - 1$, and $k_1 - l_1$ by $k_1 - l_1 - 1$ (where $0 \leq p_1,\ l_1 \leq k_1$, $0 \leq l_2 \leq k_2$) in (5.1), we get

$$\left|f_{2p_1-l_1,l_2-1}(z_1^{(0)},z_2^{(0)})\, g_{k_1-l_1-1,k_2-l_2}(z_1^{(0)},z_2^{(0)})\right| \geq A_{k_1} B_{k_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \text{ for}$$

$0 \leq p_1,\ l_1 \leq k_1$ and $0 \leq l_2 \leq k_2$ This implies

$$\begin{cases} \left|f_{2p_1-l_1,l_2-1}(z_1^{(0)},z_2^{(0)})\right| \geq A_{k_1} B_{k_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\right| \\ \text{and} \\ \left|g_{k_1-l_1-1,k_2-l_2}(z_1^{(0)},z_2^{(0)})\right| \geq A_{k_1} B_{k_2} \left|g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \end{cases} \qquad \ldots (5.15)$$

In particular, putting $u_1 = l_1$, $w_1 = k_1 - l_1 - 1$, $w_2 = k_2 - l_2$ in (5.6) and then using (5.15), we get

$$\left|f_{l_1,u_2}(z_1^{(0)},z_2^{(0)})\right| < \frac{1}{4L_1L_2} A_{k_1} B_{k_2} \left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\right| \text{ for } 0 \leq u_2 < l_2. \qquad \ldots (5.16)$$

Again in (5.6), putting $u_1 = 2p_1 - l_1$, $u_2 = l_2 - 1$, $w_1 = k_1 - p_1$, $w_2 = k_2 - l_2$, and then using (5.15), we get

$$\left|g_{k_1-p_1,k_2-l_2}(z_1^{(0)},z_2^{(0)})\right| < \frac{1}{4L_1L_2} A_{k_1} B_{k_2} \left|g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| \text{ for } 0 \le p_1 < l \qquad \ldots (5.17)$$

Now,

$$h^{(k_1+k_2)}(z_1, z_2) = \frac{\partial^{k_2}}{\partial^{k_2}_{z_2}}\left\{\frac{\partial^{k_1} h(z_1,z_1)}{\partial^{k_1}_{z_1}}\right\}$$

$$\Rightarrow \frac{h^{(k_1+k_2)}(z_1,z_1)}{k_1!k_2!} = \sum_{r_1=0}^{k_1}\sum_{r_2=0}^{k_2}\left[\frac{1}{r_1!r_2!}\frac{\partial^{r_1+r_2} f}{\partial z_1^{r_1}\partial z_2^{r_2}}\,\frac{1}{(k_1-r_1)!(k_2-r_2)!}\frac{\partial^{(k_1-r_1)+(k_2-r_2)}g}{\partial z_1^{k_1-r_1}\partial z_2^{k_2-r_2}}\right] \qquad \ldots (5.18)$$

$$\Rightarrow h_{k_1,k_2}(z_1, z_2) = \sum_{r_1=0}^{k_1}\sum_{r_2=0}^{k_2}\left[f_{r_1,r_2}(z_1, z_2)\; g_{k_1-r_1,k_2-r_2}(z_1, z_2)\right] \ldots \text{(A)}$$

$$\Rightarrow \left|h_{k_1,k_2}(z_1^{(0)},z_2^{(0)})\right| = \left|\sum_{r_1=0}^{k_1}\sum_{r_2=0}^{k_2}\left[f_{r_1,r_2}(z_1^{(0)},z_2^{(0)})\, g_{k_1-r_1,k_2-r_2}(z_1^{(0)},z_2^{(0)})\right]\right|$$

$$\ge \left|f_{l_1,l_2}(z_1^{(0)},z_2^{(0)})\; g_{k_1-l_1,k_2-l_2}(z_1^{(0)},z_2^{(0)})\right| - \left|\sum_{\substack{r_1=0\\(r_1,r_2)\ne(l_1,l_2)}}^{k_1}\sum_{r_2=0}^{k_2}\left[f_{r_1,r_2}(z_1^{(0)},z_2^{(0)})\, g_{k_1-r_1,k_2-r_2}(z_1^{(0)},z_2^{(0)})\right]\right| \qquad \ldots (5.19)$$

$$\ge A_{k_1}B_{k_2}\left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right|$$

$$- \left|\sum_{\substack{r_1=0\\(r_1,r_2)\ne(l_1,l_2)}}^{k_1}\sum_{r_2=0}^{k_2}\left[f_{r_1,r_2}(z_1^{(0)},z_2^{(0)})\, g_{k_1-r_1,k_2-r_2}(z_1^{(0)},z_2^{(0)})\right]\right| \quad \text{[By (5.1)]} \qquad \ldots (5.20)$$

$$= A_{k_1}B_{k_2}\left|f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)})\right| - \sum_{r_1=0}^{l_1-1}\sum_{r_2=0}^{l_2-1}\left|f_{r_1,r_2}(z_1^{(0)},z_2^{(0)})\, g_{k_1-r_1,k_2-r_2}(z_1^{(0)},z_2^{(0)})\right|$$

$$- \sum_{r_1=l_1+1}^{k_1} \sum_{r_2=l_2+1}^{k_2} \left| f_{r_1,r_2} (z_1^{(0)}, z_2^{(0)})\, g_{k_1-r_1,k_2-r_2} (z_1^{(0)}, z_2^{(0)}) \right|$$

$$- \sum_{r_1=0}^{l_1} \sum_{r_2=l_2+1}^{k_2} \left| f_{r_1,r_2} (z_1^{(0)}, z_2^{(0)})\, g_{k_1-r_1,k_2-r_2} (z_1^{(0)}, z_2^{(0)}) \right|$$

$$- \sum_{r_1=l_1+1}^{k_1} \sum_{r_2=0}^{l_2} \left| f_{r_1,r_2} (z_1^{(0)}, z_2^{(0)})\, g_{k_1-r_1,k_2-r_2} (z_1^{(0)}, z_2^{(0)}) \right|$$

$$- \sum_{r_1=0}^{l_1-1} \left| f_{r_1,l_2} (z_1^{(0)}. z_2^{(0)})\, g_{k_1-r_1,k_2-l_2} (z_1^{(0)}, z_2^{(0)}) \right|$$

$$- \sum_{r_2=0}^{l_2-1} \left| f_{l_1,r_2} (z_1^{(0)}. z_2^{(0)})\, g_{k_1-l_1,k_2-r_2} (z_1^{(0)}, z_2^{(0)}) \right| \qquad \ldots (5.21)$$

$$\Rightarrow \left| h_{k_1,k_2} (z_1^{(0)}, z_2^{(0)}) \right| \geq A_{k_1} B_{k_2} \left| f_{i_1,i_2} (z_1^{(0)}, z_2^{(0)})\, g_{j_1,j_2} (z_1^{(0)}, z_2^{(0)}) \right|$$

$$- \sum_{r_1=0}^{l_1-1} \sum_{r_2=0}^{l_2-1} \left| f_{r_1,r_2} (z_1^{(0)}, z_2^{(0)})\, g_{j_1,j_2} (z_1^{(0)}, z_2^{(0)}) \right| -$$

$$\sum_{r_1=l+1}^{k_1} \sum_{r_2=l_2+1}^{k_2} \left| f_{i_1,i_2} (z_1^{(0)}, z_2^{(0)})\, g_{k_1-r_1,k_2-r_2} (z_1^{(0)}, z_2^{(0)}) \right|$$

$$- \sum_{r_1=0}^{l_1} \sum_{r_2=l_2+1}^{k_2} \left| f_{r_1,r_2} (z_1^{(0)}, z_2^{(0)})\ \ g_{j_1,j_2} (z_1^{(0)}, z_2^{(0)}) \right| -$$

$$\sum_{r_1=l+1}^{k} \sum_{r_2=0}^{l_2} \left| f_{i_1,i_2} (z_1^{(0)}, z_2^{(0)})\, g_{k_1-r_1,k_2-r_2} (z_1^{(0)}, z_2^{(0)}) \right|$$

$$- \sum_{r_1=0}^{l_1-1} \left| f_{i_1,i_2} (z_1^{(0)}. z_2^{(0)})\, g_{k_1-r_1,k_2-l_2} (z_1^{(0)}, z_2^{(0)}) \right| - \sum_{r_2=0}^{l_2-1} \left| f_{l_1,r_2} (z_1^{(0)}, z_2^{(0)})\ \ g_{j_1,j_2} (z_1^{(0)}, z_2^{(0)}) \right|$$

$$\ldots (5.22)$$

i.e.,

$$\geq A_{k_1} B_{k_2} \left| f_{i_1,i_2} (z_1^{(0)}, z_2^{(0)})\, g_{j_1,j_2} (z_1^{(0)}, z_2^{(0)}) \right| - \sum_{r_1=0}^{l_1-1} \sum_{r_2=0}^{l_2-1} \left| f_{r_1,r_2} (z_1^{(0)}, z_2^{(0)})\, g_{j_1,j_2} (z_1^{(0)}, z_2^{(0)}) \right|$$

$$- \sum_{r_1=0}^{k_1-l_1-1} \sum_{r_2=0}^{k_2-l_2-1} \left|f_{l_1,i_2}\,(z_1^{(0)},z_2^{(0)})\, g_{r_1,r_2}\,(z_1^{(0)},z_2^{(0)})\right| -$$

$$\sum_{r_1=0}^{l_1-1} \sum_{r_2=0}^{k_2-l_2-1} \left|f_{r_1,k_2-r_2}\,(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}\,(z_1^{(0)},z_2^{(0)})\right|$$

$$- \sum_{r_2=0}^{k_2-l_2-1} \left|f_{l_1,k_2-r_2}\,(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}\,(z_1^{(0)},z_2^{(0)})\right| -$$

$$\sum_{r_1=0}^{k_1-l_1-1} \sum_{r_2=0}^{l_2-1} \left|f_{i_1,l_2}\,(z_1^{(0)},z_2^{(0)})\, g_{r_1,k_2-r_2}\,(z_1^{(0)},z_2^{(0)})\right|$$

$$- \sum_{r_1=0}^{k_1-l_1-1} \left|f_{l_1,l_2}\,(z_1^{(0)},z_2^{(0)})\, g_{r_1,k_2-l_2}\,(z_1^{(0)},z_2^{(0)})\right| - \sum_{r_1=0}^{l_1-1} \left|f_{l_1,l_2}\,(z_1^{(0)},z_2^{(0)})\, g_{k_1-r_1,k_2-l_2}\,(z_1^{(0)},z_2^{(0)})\right|$$

$$- \sum_{r_2=0}^{l_2-1} \left|f_{l_1,r_2}\,(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}\,(z_1^{(0)},z_2^{(0)})\right| \qquad \ldots (5.23)$$

Now using (5.8), (5,7), (5.12), (5.17), (5.13), (5.18), (5.23), (5.22), we obtain

$$\left|h_{k_1,k_2}\,(z_1^{(0)},z_2^{(0)})\right| \geq A_{k_1} B_{k_2} \left|f_{l_1,l_2}\,(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}\,(z_1^{(0)},z_2^{(0)})\right|$$

$$- \frac{A_{k_1} B_{k_2}}{4L_1L_2} \left|f_{l_1,l_2}\,(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}\,(z_1^{(0)},z_2^{(0)})\right| \{l_1 l_2 + (k_1 - l_1)\,(k_2 - l_2) + l_1\,(k_2 - l_2) +$$

$$(k_2 - l_2) + l_2(k_1 - l_1) + (k_1 - l_1) + l_1 + l_2\}$$

$$= A_{k_1} B_{k_2} \left|f_{l_1,l_2}\,(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}\,(z_1^{(0)},z_2^{(0)})\right| \left\{1 - \frac{k_1k_2 + k_1 + k_2}{4L_1L_2}\right\}$$

$$\geq \frac{1}{4} A_{k_1} B_{k_2} \left|f_{l_1,l_2}\,(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}\,(z_1^{(0)},z_2^{(0)})\right|$$

This completes the proof.

Proof of theorem 3. Theorem (2) implies that $f(a_1z_1, a_2z_2)$ and $g(a_1z_1, a_2z_2)$ are of bounded index for all $(a_1, a_2) \in \mathbb{C}^2$ and, in particular, for $a_1 > 1$ and $a_2 > 1$.

Let $M = (M_1, M_2)$ be an index of f and $N = (N_1, N_2)$ be an index of g. Also, let $L = (L_1, L_2) = (M_1 + N_1, M_2 + N_2)$. Let us now introduce the constants $A_0, A_1, \ldots, A_{L_1}$ and $B_0, B_1, \ldots, B_{L_1}$ as defined in the previous lemma. Furthermore, let $Ma = (Ma_1, Ma_2)$ be an index of $u(z_1, z_2) = f(a_1z_1, a_2z_2)$ and $Na = (Na_1, Na_2)$ be an index of $w(z_1, z_2) = g(a_1z_1, a_2z_2)$. Since $a_1 > 1$, we have

$a_1^{-n_1} = O(n_1^{m_1})$ for any non-negative integer m_1. Therefore, $\exists$ an integer $n_1' \geq 0$ such that

$$(n_1 + 1)\, a_1^{-n_1} < \frac{1}{2} A_0 B_0 a_1^{-Ma_1 - Na_1} \text{for any } n_1 \geq n_1'. \qquad \ldots (5.24)$$

Similarly,

$$(n_2 + 1)\, a_2^{-n_2} < \frac{1}{2} A_0 B_0 a_2^{-Ma_2 - Na_2} \text{for any } n_2 \geq n_2'. \qquad \ldots (5.25)$$

Let (t_1, t_2) be an index of $u(z_1, z_2)$ at $\left(\frac{z_1^{(0)}}{a_1}, \frac{z_2^{(0)}}{a_2}\right)$ and (s_1, s_2) be an index of $w(z_1, z_2)$ at $\left(\frac{z_1^{(0)}}{a_1}, \frac{z_2^{(0)}}{c_2}\right)$.. clearly,

$$t_1 \leq Ma_1,\ s_1 \leq Na_1 \text{ and} \qquad \ldots (5.26)$$

$$t_2 \leq Ma_2,\ s_2 \leq Na_2 \qquad \ldots (5.27)$$

$$\therefore\ u_{t_1,t_2}(z_1, z_2) = a_1^{t_1} a_2^{t_2} f_{t_1,t_2}(a_1z_1, a_2z_2)$$

$$\Rightarrow u_{t_1}\left(\frac{z_1^{(0)}}{a_1}, \frac{z_2^{(0)}}{a_2}\right) = a_1^{t_1} a_2^{t_2} f_{t_1,t_2}(z_1^{(0)}, z_2^{(0)}) \qquad \ldots (5.28)$$

$$\therefore\ a_1^{t_1} a_2^{t_2} \left|f_{t_1,t_2}(z_1^{(0)}, z_2^{(0)})\right| = \left|u_{t_1,t_2}\left(\frac{z_1^{(0)}}{a_1}, \frac{z_2^{(0)}}{a_2}\right)\right|$$

$$\geq \left|u_{l_1,l_2}\left(\frac{z_1^{(0)}}{a_1}, \frac{z_2^{(0)}}{a_2}\right)\right| = a_1^{l_1} a_2^{l_2} \left|f_{l_1,l_2}(z_1^{(0)}, z_2^{(0)})\right| \ \forall (l_1, l_2) \in I^2.$$

$$\Rightarrow \left|f_{l_1,l_2}\,(z_1^{(0)},z_2^{(0)})\right| \le a_1^{t_1-l_1}\,a_2^{t_2-l_2}\,\left|f_{t_1,t_2}\,(z_1^{(0)},z_2^{(0)})\right| \text{ and similarly} \qquad \dots (5.29)$$

$$\left|g_{l_1,l_2}\,(z_1^{(0)},z_2^{(0)})\right| \le a_1^{s_1-l_1}\,a_2^{s_2-l_2}\,\left|g_{s_1,s_2}\,(z_1^{(0)},z_2^{(0)})\right| \;\forall (l_1,\, l_2) \in I^2. \qquad \dots (5.30)$$

Thus, for $n_1 \ge n_1'$, $n_2 \ge n_2'$, using the relation (A) we get

$$\left|h_{n_1,n_2}\,(z_1^{(0)},z_2^{(0)})\right| = \left|\sum_{p_1=0}^{n_1}\sum_{p_2=0}^{n_2} f_{p_1,p_2}\,(z_1^{(0)},z_2^{(0)})\,g_{n_1-p_1,n_2-p_2}\,(z_1^{(0)},z_2^{(0)})\right|$$

$$\le \sum_{p_1=0}^{n_1}\sum_{p_2=0}^{n_2} \left[a_1^{t_1-p_1}\,a_2^{t_2-p_2}\left|f_{t_1,t_2}\,(z_1^{(0)},z_2^{(0)})\right| \times a_1^{s_1-(n_1-p_1)}\,a_2^{s_2-(n_2-p_2)}\left|g_{s_1,s_2}\,(z_1^{(0)},z_2^{(0)})\right|\right]$$

[By using (5.29) and (5.30)]

$$= \left|f_{t_1,t_2}\,(z_1^{(0)},z_2^{(0)})\right|\left|g_{s_1,s_2}\,(z_1^{(0)},z_2^{(0)})\right| \sum_{p_1=0}^{n_1}\sum_{p_2=0}^{n_2} (a_1^{t_1+s_1-n_1}\,a_2^{t_2+s_2-n_2})$$

$$= (n_1 + 1)\,a_1^{t_1+s_1-n_1}(n_2 + 1)\,a_2^{t_2+s_2-n_2}\left|f_{t_1,t_2}\,(z_1^{(0)},z_2^{(0)})\right|\left|g_{s_1,s_2}\,(z_1^{(0)},z_2^{(0)})\right|$$

[By (5.26) and (5.27)]

$$\le (n_1 + 1)\,a_1^{Ma_1+Na_1-n_1}(n_2 + 1)\,a_2^{Ma_2+Na_2-n_2}\left|f_{t_1,t_2}\,(z_1^{(0)},z_2^{(0)})\right|\left|g_{s_1,s_2}\,(z_1^{(0)},z_2^{(0)})\right|$$

$$\le \frac{1}{4}\,A_0^2\,B_0^2\left|f_{t_1,t_2}\,(z_1^{(0)},z_2^{(0)})\right|\left|g_{s_1,s_2}\,(z_1^{(0)},z_2^{(0)})\right| \text{ [By (5.24) and (5.25)]}$$

$$\le \frac{1}{4}A_0\,B_0\left|f_{t_1,t_2}\,(z_1^{(0)},z_2^{(0)})\right|\left|g_{s_1,s_2}\,(z_1^{(0)},z_2^{(0)})\right| \;[\because \mathrm{A}_0,\ \mathrm{B}_0 < 1]$$

$\Rightarrow$ for $\mathrm{n}_l \ge n_1'$ and $n_2 \ge n_2'$

$$\left|h_{n_1,n_2}\,(z_1^{(0)},z_2^{(0)})\right| \le \frac{1}{4}A_0\,B_0\left|f_{i_1,i_2}\,(z_1^{(0)},z_2^{(0)})\right|\left|g_{j_1,j_2}\,(z_1^{(0)},z_2^{(0)})\right| \qquad \dots (5.31)$$

where $(i_1,\, i_2)$ is an index of f at $(z_1^{(0)},z_2^{(0)})$ and $(j_1,\, j_2)$ is an index of g at $(z_1^{(0)},z_2^{(0)})$. Again by the above lemma $\exists$ integers $k_1 \le L_1$ and $k_2 \le L_2$ such that

$$\left|h_{k_1,k_2}\,(z_1^{(0)},z_2^{(0)})\right| \ge \frac{1}{4}\,A_{k_1}\,B_{k_2}\left|f_{i_1,i_2}\,(z_1^{(0)},z_2^{(0)})\,g_{j_1,j_2}\,(z_1^{(0)},z_2^{(0)})\right|$$

$$\geq \frac{1}{4} A_0 B_0 \left| f_{i_1,i_2}(z_1^{(0)},z_2^{(0)})\, g_{j_1,j_2}(z_1^{(0)},z_2^{(0)}) \right|$$

$$\geq \left| h_{n_1,n_2}(z_1^{(0)},z_2^{(0)}) \right| \text{ [By (5.31)] for all } n_1 \geq n_1' \text{ and } n_2 \geq n_2'$$

$$\Rightarrow \max_{\substack{k_1 \leq L_1 \\ k_2 \leq L_2}} \left\{ \frac{1}{k_1!k_2!} \left| \frac{\partial^{k_1+k_2} h(z_1^{(0)},z_2^{(0)})}{\partial z_1^{k_1} \partial z_2^{k_2}} \right| \right\} \geq \left\{ \frac{1}{n_1!n_2!} \left| \frac{\partial^{n_1+n_2} h(z_1^{(0)},z_2^{(0)})}{\partial z_1^{n_1} \partial z_2^{n_2}} \right| \right\}$$

for all $n_1 \geq n_1'$ and $n_2 \geq n_2'$

$\Rightarrow$ h is of bounded index and its indices do not exceed (max (L_1, n_1'), (max (L_2, n_2')). This completes the proof.

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REFERENCES

1. B. A. Fucks, Introduction to the theory of analytic functions of several complex variables, Amer. Math. Soc. Providence, R. I. (1963).
2. B. C. Chakraborty and Rita Mitra, Entire functions of bounded index in several complex variables, Jour. Pure Math. 9 (1992), 5-13.
3. G. H. Fricke, A characterization of functions of bounded index, Indian J. Math. 14, no. 3 (1972), 207-212.
4. G. H. Fricke and R. E. Powell, Some product theorems of functions of bounded index. Ind. J. Pure Appl. Math. 3, No. 5 (1972), 795-800.
5. J. G. Krishna and S. M. Shah, Functions of bounded indices in one and several complex variables. Math. Essays dedicated to A. J. Macintyre. Ohio Univ. Press. Athens, Ohio (1970), 223-235.

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UNIFICATION OF INTEGRAL TRANSFORMS

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1. INTRODUCTION

An integral transform is a linear operator T on a suitable function space $\mathcal{F}$, defined with the help of an integral :

$$(Tf)(u) = \int_I K(t, u)\, f(t)\, dt, \ \forall f \in \mathcal{F} \qquad \dots (1.1)$$

where I is an interval, f : I→R (the set of real numbers), and K : I × J → R, J being an interval; the function K is called the kernel of the transform T. By varying K, I, T we get all the known integral transforms, like Laplace, Fourier, Bessel, Hankel etc. What is to be noted here is that, in each case, the kernel K satisfies a second order linear homogeneous ordinary differential equation together with one initial condition. Our aim here is to exhibit a special ill-posed problem that generates an integral transform from which one can derive all the known integral transforms. Further it will be noted that the integral transforms so generated plays the vital role in the problem of the convergence of the eigen function expansion of an arbitrary function corresponding to the eigen value problem associated with the ill-posed problem.

The ill-posed initial-value problem π_0 under concern is exhibited in §2. We then define a transform T on the space G of truncated functions in §3, and let H = {T f : f ∈ G}. §4 defines the Sturm-Liouville problem π_b on [0,b] and discusses some of its properties. This helps us to define an inner product in H as shown in §5. Let $\hat{E}$ denote the completion of H. We are now ready to define the transform T on $L^2\,[0,\infty)$ [the set of all functions f : [0, ∞) → R such that $\int_0^\infty |f(t)|^2 dt < \infty$]. One can easily verify that the wellknown transforms are only special cases of T (vide §7.). Finally in §8 it is shown that the convergence of the eigen function expansion of arbitrary $f \in L^2$ corresponding to the singular eigenvalue problem π on $[0, \infty)$ generated by the ill-posed problem π_0 can be equivalently dealt with in $\hat{H}$ considering the corresponding expansion of Tf.

2. THE INITIAL-VALUE PROBLEM π_0

The initial-value problem π_0 under concern consists of the differential equation

$$l\,[y] \equiv -\,(p(t)y^{(1)}(t))^{(1)} + q\,(t)\,y\,(t) = \lambda\,y\,(t) \qquad \ldots (2.1)$$

and the initial condition

$$p\,(0)\,y\,(0)\cos\alpha + y^{(1)}\,(0)\sin\alpha = 0 \qquad \ldots (2.2)$$

where (i) $p, q : [0, \infty) \rightarrow R$,

(ii) p is absolutely continuous on all compact subintervals of $[0, \infty)$,

(iii) q is continuous on $[0, \infty)$,

(iv) $p\,(t) > 0,\ t \in [0, \infty)$,

(v) $\lambda = \mu + i\gamma\ (\mu, \gamma, \in R)$

(vi) $0 \leq \alpha < \pi$

(vii) $0 \leq t < \infty$

3. THE TRANSFORM $\mathcal{T}$ OF TRUNCATED FUNCTIONS

For any b>0, we define the class G_b as a subset of $L^2 = L^2\ [0, \infty)$:

$G_b = \{f \in L^2 :$ (a) $f\,(t) = 0\ \forall\ t > c \geq b$, for some $c \in R$,

(b) $p\,(0)\,f\,(0)\cos\alpha + f^{(1)}\,(0)\sin\alpha = 0$,

$p\,(b)\,f\,(b)\cos\beta + f^{(1)}\,(b)\sin\beta = 0,\ 0 \leq \alpha, \beta < \pi$,

(c) $\tilde{f} = l\,[f] \in L^2\}$

and $G = \cup\ \{G_b : b \in R^+\}$ ... (3.2)

Let $\phi\,(., \lambda)$ denote the solution of (2.1), that satisfies

$$p\,(0)\,\phi\,(0, \lambda) = \sin\alpha, \qquad \phi^{(1)}\,(0, \lambda) = -\cos\alpha \qquad \ldots (3.2)$$

For any $f \in G$, we define its transform $\mathcal{T} f$ by stipulating

$$(\mathcal{T} f)(u) = \int_0^\infty f(t)\, \phi(t, u)\, dt,\ u \in R. \qquad \ldots (3.3)$$

We note that the integral on the right hand side of (3.3) is only a finite integral and that $\mathcal{T} f$ is continuous.

Let $H = \{\mathcal{T} f : f \in G\}$. It is to be noted that H is a real linear space.

4. THE BOUNDARY-VALUE PROBLEM π_b

For any b>o, let π_b denote the boundary-value problem consisting of (2.1), (2.2) and the boundary condition

$$p(b)\, y(b) \cos\beta + y^{(1)}(b) \sin\beta = 0. \qquad \ldots (4.1)$$

Let $\{\lambda_{nb}\}_n$ denote the eigenvalues of the boundary–value porblem π_b.

We know that the eigen function of π_b corresponding to the eigenvalue λ_{nb} is a multiple of $\phi(., \lambda_{nb})$; let $(r_{nb})^{1/2}\, \phi(., \lambda_{nb})$ be the corresponding normalised eigenfunction.

For each b>0, we define a step function $\rho_b : (-\infty, \infty) \to R$ by requiring

(i) $\rho_b(.)$ is constant in $(\lambda_{n-1b}, \lambda_{nb})$, $n \in \mathbb{N}$,

(ii) $\rho_b(0) = 0$,

(iii) $\rho_b(\lambda_{nb}) = \frac{1}{2}\{\rho_b(\lambda_{nb}-) + \rho_b(\lambda_{nb}+\}$,

(iv) if $\lambda_{n-1b} < t_1 < \lambda_{nb} < t_2 < \lambda_{n+1b}$, $\rho_b(t_2) - \rho_b(t_1) = r_{nb}$,

(1) – (4) will define ρ_b uniquely.

The following property of ρ_b can be easily derived using Helly's selection theorem and Helly-Bray theorem :

Lemma 1. There exists a sequence $\{b_n\}_n$ of real numbers such that

(i) $b_n \to \infty$ as $n \to \infty$,

(ii) $\rho_b(u) = \lim_{n\to\infty} \rho_{b_n}(u)$ exists for all $u \in R$

(iii) for all finite u_1, u_2, and for all continuous $f : [u_1, u_2] \to R$,

$$\lim_{n\to\infty}\int_{u_1}^{u_2} f(t)\, d\rho_{b_n}(t) = \int_{u_1}^{u_2} f(t)\, d\rho(t).$$

Further, we have

Theorem 1. For each $f \in G_b$,

$$\text{(A)} \quad \int_0^b \{f(t)\}^2\, dt = \int_{-\infty}^{\infty} \{(\mathcal{T}f)(u)\}^2 d\rho_b(u),$$

$$\text{(B)} \quad \int_0^b \{l\, [f]\, (t)\}^2 dt = \int_{-\infty}^{\infty} u^2\{(\mathcal{T}f)(u)\}^2 d\rho_b(u),$$

This follows from the Parsevals relation connected with π_b. We next prove the following;

Theorem 2. For any $f \in G$,

$$\int_0^{\infty} \{f(t)\}^2\, dt = \int_{-\infty}^{\infty} \{(\mathcal{T}f)(u)\}^2 d\rho(u), \qquad \ldots (4.2)$$

$\mathcal{T}f$ being defined by (3.3).

Proof. If $f \in G$, then $f \in G_b$ for some b. By (iii) of Lemma 1, we have, for each fixed $U>0$, and for each $f \in G$,

$$\lim_{n\to\infty}\int_{-U}^{U} \{\mathcal{T}f)(u)^2\}^2 d\rho_{b_n}(u) = \int_{-U}^{U} \{\mathcal{T}f)(u)\}^2 d\rho(u), \qquad (4.3)$$

Also, by (B) of Theorem 1,

$$\Big[\int_{-\infty}^{-U} + \int_{U}^{\infty}\Big]\{(\mathcal{T}f)(u)\}^2 d\rho_b(u) \le \frac{1}{U^2}\int_{-\infty}^{\infty} u^2\{\mathcal{T}f)(u)\}^2 d\rho_b(u) = \frac{1}{U^2}\int_0^{\infty}\{l[f](t)\}^2 dt \qquad (4.4)$$

Using (4.3) and (4.4), the required result can now be derived from (A) of Theorem–1.

5. THE HILBERT SPACE $\hat{H}$

An innerproduct in $H = \{\mathcal{T}f : f \in G\}$ is now defined by stipulating, for all $\mathcal{T}f$, $\mathcal{T}g \in H$,

$$<\mathcal{T}f, \mathcal{T}g> = \int_{-\infty}^{\infty} (\mathcal{T}f)\,(u)\,(\mathcal{T}g)\,(u)\, d\rho\,(u). \qquad \ldots (5.1)$$

That the integral on the right hand side of (5.1) exists follows from Schwarz's inequality and (A) of Theorem–1. We next show that

Theorem 3. H is an inner–product space.

Proof. Let us define

$$\|\mathcal{T}f\|' = +\,[\int_{-\infty}^{\infty} \{(\mathcal{T}f)\ (u)\}^2\ d\rho\ (u)]^{1/2} = [< \mathcal{T}f,\ \mathcal{T}f >]^{1/2}$$

(5.1) will define an inner product on H if

$$\|\mathcal{T}f + \mathcal{T}g\|' \leq \|\mathcal{T}f\|' + \|\mathcal{T}g\|' \quad \forall,\ g \in G \qquad \ldots (5.2)$$

It follows immediately on using (4.1), since L^2 is a normed linear space.

6. THE TRANSFORM T ON L^2

Since G is dense in $L^2 = L^2\ [0, \infty)$, given any $f \in L^2$. We can find a sequence $\{f_n\}_n$ in G that converges to f in L^2. If $\{f_n\}_n$, $\{g_n\}_n$ are two sequences in L^2, that converge to f in L^2, we shall first show that, for almost all $u \in R$,

$$\lim_{n\to\infty} (\mathcal{T}f_n)\ (u) = \lim_{n\to\infty} (\mathcal{T}g_n)\ (u) \qquad \ldots (6.1)$$

Since $\| f_n - f_m \| = \|\mathcal{T}f_n - \mathcal{T}f_m\|'$, any Cauchy sequence in L^2 generates a Cauchy sequence in H. Hence $\{Tf_n\}_n$ and $\{Tg_n\}_n$ are Cauchy sequences in H. If they converge to F and G respectively, we have.

$$\|F - G\|' \leq \|F - Tf_n\|' + \|Tf_n - Tg_n\|' + \|Tg_n - G\|'$$

$$= \|F - Tf_n\|' + \|f_n - g_n\| + \|Tg_n - G\|' \to 0 \text{ as } n \to \infty$$

Hence (6.1) follows.

So, if $\hat{H}$ denotes the completion of the innerproduct space H, we can define

$T : L^2\ [0, \infty) \to \hat{H}$ by stipulating,

$$(Tf)\ (u) = \lim_{n\to\infty} (Tf_n)\ (u)$$

$$= \lim_{n\to\infty} \int_0^\infty f_n (y)\ \phi (t, u)\ dt,$$

where $\{f_n\}_n$ is any sequence in G that converges to f in $L^2 [0, \infty)$. The extension of Theorem 2 to all $f \in L^2$ now follows immediately.

7. SPECIAL CASES OF T

The following table will indicate some of the known transforms as special cases of T :

Differential equation	Initial condition	Transform
$y^{(2)}(t) + \lambda^2 y(t) = 0$	$y(0) = 0$	Fourier sine
$y^{(2)}(t) + \lambda^2 y(t) = 0$	$y^{(1)}(0) = 0$	Fourier cosine
$y^{(2)}(t) - \lambda^2 y(t) = 0$	$\lambda y(0) + y^{(1)}(1) = 0$	Laplace
$-(t^2 y^{(1)}(t)^{(1)} + \lambda(\lambda-1)\, y(t) = 0$	$(\lambda-1)\, y(1) - y^{(1)}(1) = 0$	Mellin
$(ty^{(1)}(t))^{(1)} - \frac{\gamma^2 - \frac{1}{4}}{t^2} y(t) = \lambda\, y(t)$	$y(0) = 0$	Hankel

8. THE IMPACT OF THE TRANSFORMATION T ON THE EIGENFUNCTION EXPANSION

Let π denote the eigenvalue problem consisting of (2.1) – (2.2) and some suitable boundary condition at infinity, so that the spectrum of π is discrete. Let $\{\lambda_n\}_n$ denote the eigenvalues of π and ψ_n denote the normalised eigenfunction corresponding to the eigenvalue λ_n. The eigenfunction expansion of any $f \in L^2$ corrosponding to the eigenvalue problem π is

$$\sum_{n=0}^{\infty} C_n\ \psi_n (t), \quad \text{where } C_n = \int_0^\infty f(t)\ \psi_n (t)\ dt.$$

The advantage that is obtained from the transformation T is exhibited in the following theorem :

Theorem 4. $\sum_{n=0}^{\infty} C_n\ \psi_n (t)$ converges in $L^2 [0, \infty)$ if and only if $\sum_{n=0}^{\infty} C_n\ (T\psi_n)(u)$ converges in $\hat{H}$.

Proof . Let $S_n(t) = \sum_{R=0}^{n} C_R \psi_R(t)$, $T_n(u) = \sum_{R=0}^{n} C_R (T\psi_R)(u)$, $\forall n \in N$

Clearly $T_n(u)) = T(S_n(u)$, $\forall\ n \in N$,

Further $\| S_n - S_m \|^2 = \| T(S_n) - T(S_m) \|^2$, and the theorem follows.

REFERENCES

1. E. C. Titchmarsh, Eigenfunction expansions associated with second order differential equations, 2nd edition 1962, Oxford University Press.
2. D.V. Widder, The Laplace Transform, Princetion University Press, 1946.

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ALMOST HYPERBOLIC KÄHLERIAN MANIFOLDS OF POINT-WISE CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

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1. INTRODUCTION

The sectional curvature of the manifolds of indefinite metric has been studied by many authors. R. Kulkarni [1] considered a Riemannian manifold of indefinite metric and proved that if the sectional curvature is bounded (from above or from below), then the manifold is of constant sectional curvature. In 1982 M. Barros and A. Romero [2] proved an analogous theorem in case of a Kählerian manifold of indefinite metric. Some similar results are given in [3]. In [4] this problem is solved in the case of the hyperbolic Kählerian manifold of indefinite metric. L. Graves [5] and M. Dajczer [6] studied the properties of the sectional curvature of the pseudo-Riemanian manifolds of indefinite metric in respect to the isotropic vectors and the isotropic planes. It was shown in [7] how the properties of the curvature of an n-dimensional isotropic tangent space ($n = 1,2$) determine the properties of the curvature of the hyperbolic Kählerian manifold. A Borisov and O. Kassabov [8] studied the sectional curvatures (both the holomorphic and the antiholomorphic ones) of the almost Hermitian manifold of definite as well as of indefinite metric.

In the present paper we consider the almost hyperbolic Kählerian manifolds of indefinite metric. We prove that if the holomorphic sectional curvature of an almost hyperbolic Kählerian manifold is bounded, then the manifold is of point-wise constant holomorphic sectional curvature.

2. PRELIMINARIES

Let us first of all give some definitions which are necessary for our further considerations.

Definition 1. Let M be a pseudo-Riemannian manifold of indefinite metric g and almost product structure J, i.e.

$$J^2 = id, \qquad g(JX,\ JY) = -\ g(X,Y)$$

for the arbitrary vector fields X, Y and $\nabla J \neq 0$. Then the triple (M,g,J) is called ***an almost hyperbolic Kählerian manifold.***

The Dimension of any hyperbolic Kählerian manifold is an even number.

Let (M,g,J) be a $2m$-dimensional almost hyperbolic Kählerian manifold with a curvature tensor R and a Levi-Civita connection ∇. Let $T_p(M)$ be the tangent space at an arbitrary point $p \in M$.

Definition 2. The pair $\{x,a\}$ is called ***an orthonormal pair of vectors of signature*** $(+,-)$ if

$$g(x,x) = 1, \qquad g(a,a) = -\ 1, \qquad g(x,a) = 0,$$

where $x,a \in T_p(M)$.

Analogously we can define ***an orthonormal pair of vectors of signature*** $(+,+)$ ***or*** $(-,-)$ as well as ***an orthonormal triple of vectors of signature*** $(+,+,+)$, $(-,-,-)$ etc.

For the convenience' sake we shall denote the unit vectors of signature $(+)$ by x,y,z ; the unit vectors of signature $(-)$ by a,b,c and the unit vectors of an arbitrary signature by X,Y,Z.

Definition 3. The plane σ is called ***holomorphic plane*** if $\sigma \perp J\sigma$, $(\sigma \neq J\sigma)$. It means, that σ is a holomorphic plane if and only if σ has a basis $(x,\ Jx)$.

Definition 4. The plane σ is called ***antiholomorphic*** if and only if it has a basis (x,y) so that

$$g(x,y) = g(x,Jy) = 0.$$

Definition 5. The pair $\{x,y\}$ is *antiholomorphic of signature* $(+,-)$ if

$$g(x,x) > 0, \qquad g(y,y) < 0, \qquad g(x,y) = g(x,Jy) = 0.$$

We can define antiholomorphic pairs also of signature $(+,+)$ or $(-,-)$ in a similar way.

Definition 6. A 2-plane is called ***nondegenerate*** if the restriction of the metric g on σ is nondegenerate, i.e. it is of rank 2.

Definition 7. ***The sectional curvature*** of a nondegenerate 2-plane σ spanned by the vectors X,Y is given by the formula

$$K(\sigma) = \frac{R(X,Y,Y,X)}{g(X,Y)g(Y,Y)-g(X,Y)^2}$$

In case the basis (x,y) is orthonormal we have

$$K(\sigma) = R(x,y,y,x).$$

Let us notice

$$K(x,y) = K(\sigma)$$

when σ is spanned by (x,y) and

$$H(x) = K(x,Jx)$$

in case σ is spanned by (x,Jx). The vector x is a unit vector, i.e.

$$g(x,x) = 1 \quad \text{or} \quad g(x,x) = -1.$$

Definition 8. The manifold M is called *a manifold of point-wise constant holomorphic sectional curvature* if the curvature of an arbitrary nondegenerate holomorphic plane σ of $T_p(M)$ is constant, i.e. it does not depend on σ.

Then the manifold M is a manifold of point-wise constant holomorphic sectional curvature if for any point $p \in M$ there exists a constant $c(p)$ such that for any unit vector $X \in T_p(M)$ the holomorphic sectional curvature

$$H(X) = K(X,JX)$$

does not depend on X, i.e.

$$H(X) = c(p).$$

We can define *a manifold of point-wise constant antiholomorphic sectional curvature* in a similar way.

3. ON THE POINT-WISE CONSTANCY OF THE HOLOMORPHIC SECTIONAL CURVATURE

Before the characterization of the holomorphic sectional curvature by a condition for its point-wise constancy we prove the following

Lemma 3.1. ***Let M be a 2m-dimensional almost hyperbolic Kählerian manifold of indefinete metric. If for any antiholomorphic pair {x, a} the equation***

$$R(x, Jx, Jx, a) + R(x, Jx, Ja, x) = 0 \qquad \dots (1)$$

holds, then M is a manifold of point-wise constant holomorphic sectional ccurvature.

Proof. Let $\{x,a\}$ be an orthonormal antiholomorphic pair of vectors and $|t| < 1$. We consider the pair $\{x + ta,\ tx + a\}$ which is also antiholomorphic. Using the equation (1), we get

$$R(x + ta,\ Jx + tJa,\ Jx + tJa,\ tx + a) + R(x + ta,\ Jx + tJa,\ tJx + Ja.x + ta) = 0,$$

The last equation gives us

$$2H(x) + 2R(x,\ Jx,\ Ja,\ a) + 2R(x,\ Ja,\ Jx,\ a) + K(x,\ Ja) + K(Jx,\ a) = 0, \qquad \dots (2)$$

$$2H(a) + 2R(x,\ Jx,\ Ja,\ a) + 2R(x,\ Ja,\ Jx,\ a) + K(x,\ Ja) + K(Jx,\ a) = 0, \qquad \dots (3)$$

$$R(a,\ Ja,\ Ja,\ x) + R(a,\ Ja,\ Jx,\ a) = 0, \qquad \dots (4)$$

for any t, $|t| < 1$. The equations (2) and (3) imply

$$H(x) = H(a)$$

and the proof is finished.

Theorem 3.1. ***Let M be a 2m-dimensional almost hyperbolic Kählerian manifold of indefinite metric and signature $(2s,\ 2(m-s))$, $m > 1$. If for any point $p \in M$ there exists a constant $c(p)$, such that for any vector $x \in T_p(M)$ the holomorphic sectional curvature satisfies the inequality***

$$|H(x)| \leq c(p), \qquad \dots (5)$$

then M is a manifold of point-wise constant holomorphic sectional curvature.

Proof. Let $p \in M$, $\{x,a\}$ be an arbitrary antiholomorphic orthonormal pair of tangent vectors at the point p and $|t| < 1$. Then from (6) we have

$$\left| H\left(\frac{x + ta}{\sqrt{1 - t^2}} \right) \right| \leq c(p),$$

i.e.

$$\begin{aligned} |H(x) &+ 2t\{R(x,\ Jx,\ Jx,\ a) + R(x,\ Jx,\ Ja,\ x)\} \\ &+ t^2\{2R(x,\ Jx,\ Ja,\ a) + 2R(x,\ Ja,\ Jx,\ a)\} + K(x,\ Ja) + K(Ja,\ x) \qquad \dots (6) \\ &+ 2t^3\{R(a,\ Ja,\ Ja,\ x) + R(a,\ Ja,\ Jx,\ a)\} + t^4 H(a)| \leq c(p)\ (1 - t^2)^2. \end{aligned}$$

Because of the continuity of the function, we can take ($t = 1$, $t = -1$) and transform (6) into the form

$$\begin{aligned} |H(x) + t^2\{2R(x,\ Jx,\ Ja,\ a) + 2R(x,\ Ja,\ Jx,\ a)\} + K(x,\ Ja) + \\ K(Jx,\ a) + t^4 H(a)| \leq c(p)\ (1 - t^2)^2. \quad \dots (7) \end{aligned}$$

On the other hand, by the same reason ($t = \pm 1$), we get from (6)

$$\begin{aligned} H(x) &+ 2t\{R(x, Jx, Jx, a) + R(x, Jx, Ja, x)\} \\ &+ t^2\{2R(x, Jx, Ja, a) + 2R(x, Ja, Jx, a)\} + K(x, Ja) + K(Jx, a) \\ &+ 2t^3\{R(a, Ja, Ja, x) + R(a, Ja, Jx, a)\} + t^4 H(a) = 0 \end{aligned} \qquad \dots (8)$$

We obtain two equations from (8) for $t = 1$ and $t = -1$. Subtracting them, we get

$$R(x, Jx, Jx, a) + R(x, Jx, Ja, x) + R(a, Ja, Ja, x) + R(a, Ja, Jx, a) = 0. \qquad \dots (9)$$

Then we apply (9) to (6) and obtain

$$\begin{aligned} |H(x) &+ t^2\{2R(x, Jx, Ja, a) + 2R(x, Ja, Jx, a)\} + K(x, Ja) + K(Jx, a) \\ &+ 2t(1 - t^2)\{R(x, Jx, Jx, a) + R(x, Jx, Ja, x)\} \\ &+ t^4 H(a)| \leq c(p)\ (1 - t^2)^2. \end{aligned} \qquad \dots (10)$$

Both inequalities (10) and (7) give us

$$|t\{R(x, Jx, Jx, a) + R(x, Jx, Ja, x)\}| \leq c(p)\ (1 - t^2)$$

and due to Lemma 3.1., Theorem 3.1. is is proved. This theorem gives us a sufficient condition for an almost hyperbolic Kählerian manifold of indefinite metric to be a manifold of point-wise constant holomorphic sectional curvature.

REFERENCES

1. R. S. Kulkarni, The values of sectional curvature of indefinite metrics. ***Comment. Math. Helvetisi*** **54** (1979), 173-176.
2. M. Barros and A. Romero, Indefinite Kählerian manifolds. ***Math. Ann.*** **261** (1982), 55-62.
3. O. Kassabov and A. Borisov, On the sectional curvature of Kählerian manifolds of indefinite metrics. ***Pliska, Studia Mathematica Bulgarica*** **9** (1988), 66-70.
4. R. Ivanova, On the sectional curvature of the hyperbolic Kählerian manifolds of indefinite metric, ***J. Pure Math 16*** (1999), 109-115.
5. L. Graves and K. Nomizu, On sectional curvature of indefinite metrics, ***Math. Ann.*** **232** (1978), 267-276.
6. M. Dajczer and K. Nomizu, On the boundedness of Ricci curvature of an indefinite metric, ***Bol. Soc. Brasil Math.*** **11** (1980), 25-30.

7. G. Ganchev and A. Borisov, Isotrophic sections and curvature properties of Hyperbolic Kählerian manifolds, ***Publications de'l Institute Mathematique,*** N. S., **38(52)** (1985), 183-192.

8. A Borisov and O. Kassabov. On the boundedness of the sectional curvature of almost Hermitian manifolds, ***Math. and Education in Math.*** (1986), 179-185.

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CHARACTERIZATION OF "NEAR-RING GROUPS WITH IRREDUCIBLE SUBSTRUCTURES"

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ABSTRACT : Weakly n-injectivity and n-injectivity together with Beidleman's condition on a near-ring group E exhibit some interesting phenomena in case of those E which are not simple but its parts contain some irreducible substructures. In an m-simple weakly n-injective E, any maximal independent set of maximal strictly uniform ideals is complete in the sense that it contains an isomorphic copy of every maximal strictly uniform ideal of it. A quasi m-simple E is an essential extension of a direct sum of strictly uniform ideals and it is m-simple if and only if each strictly uniform ideal has a non-zero Socle 1. Moreover, coincidence of E with the n-injective hull of some of its uniform ideals is equivalent either to indecomposable or minimal n-injective character of E. Lastly, an n-injective E is with finite Goldie dimension if and only if it is a direct sum of finitely many non-zero strictly uniform n-injective ideals of it.

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Key words : Essential ideal, Goldie dimension, Direct sum, Injectivity, Quasi m-simple n-injective Hull, Socle, Uniform ideal.

1. INTRODUCTION

Oswald [6] and Chowdhury [2,3] have obtained some structure of near-ring with some sort of chain conditions. In this paper our attempt is on what may be termed as to study the near-ring groups satisfying the descending chain condition (DCC) on its ideals and with zero radical (Beidleman's condition) and the near-ring groups which are not necessarily simple but its substructures contain some irreducible substructures. How the n-injectivity and weakly n-injectivity character together with Beidleman's condition exhibit some interesting phernomena in case of such type of near-ring groups is our principal motivation behind this search.

Definition and Notations : All basic concepts used in this paper are available in Pilz [8]. Throughout the paper N will mean a zero symmetric near-ring with unity 1 and E an N-group. If H_1, H_2 are two N-subgroups of E such that $H_1 \subseteq H_2$ then H_1 is *strictly essential (essential)* in H_2 when any non-zero N-subgroup (ideal) C of E contained in H_2 has a nonzero intersection with H_1. In such cases H_2 is a *strictly essential (essential) extension* of H_1. If H_1 is strictly essential (essential) in E (when $H_2 = E$) then we say that H_1 is a *strictly essential (essential) N-subgroup* of E. Clearly a strictly essential N-subgroup is essential. It is easy to note that if A, B, C are three N-subgroups of E such that $A \subseteq B \subseteq C$ and A is strictly essential (essential) in C then B is strictly essential extension in E. An N-subgroup H of E is *irreducible (simple)* if it has no nonzero proper N-subgroup (ideal) of E.E. is *Irreducible (simple)* if it has no non-zero proper N-subgroups (ideal) of if [5]. An N-subgroup H of E is said to be *strictly uniform (uniform)* if for any two N-subgroups (ideals) A, B of E contained in H, $A \cap B = (0)$ implies $A = (0)$ or $B = (0)$ [9]. An irreducible (simple) N-subgroup and essential extension of an irreducible (simple) N-subgroup of an N-group are strictly uniform (uniform). Clearly, strictly uniform N-subgroup of E is uniform. A *complement ideal* is defined to be an ideal which is maximal with respect to having zero intersection with some non-zero ideal [5]. An ideal C of E is *indecomposable* if C cannot be written as $C = X \oplus Y$(direct sum) for any nonzero ideals X,Y of E. E is said to be *indecomposable* if E is indecomposable as an ideal of it. E is a *semi direct sum* of its N-subgroups A and B, written as $E = A + B$ if A is an ideal, $E = A + B$ and $A \cap B = (0)$. In such cases B is called a *semi-direct summand* of E and A is called a *complement* of the semi-direct summand of E [5]. An N-subgroup H of E is said to have the *strictly finitely Goldie dimension (finite Goldie dimension)* written as *strictly FGD (FGD)* if H does not have any infinite independent family of nonzero N-subgroups (ideals) of E. Clearly, strictly uniform (uniform) N-group has the strictly FGD (FGD). *Socle* of E, denoted by *Soc(E)*, is the sum of all minimal ideals of E([8]). In contrast with the definition of a Socle of E as in [8] we define *Socle-1* of an ideal H of E (denoted by $Soc_1(H)$) as follows :

$Soc_1(H)$ = Sum of all minimal ideals of E contained in H. Thus $Soc_1(E) = Soc(E)$. An ideal H of E is said to be *semisimple* provided $Soc_1(H) = H$. E is semisimple provided $Soc(E) = E$. An ideal I of E is said to be *m*-simple if every nonzero ideal I′ of E contained in I contains an irreducible ideal C of E such that $C \subseteq I'$ and hence E is an m-simple N-group if E is m-simple as an ideal of it. E is *quasi-m-simple* if every ideal I of E contains an ideal I′ of E with FGD. The *radical* $J_2(N)$ of N is defined to be the intersection of all the annihilators of irreducible N-groups and *J(N)* is the intersection of all the maximal left ideals of N which are also maximal as left N-subgroups [2,3,4]. If $1 \varepsilon N$, it is seen that $J_2(N) = J(N)$ (=J) [6].

E is with *Beidleman's condition* if its radical is zero and satisfies the descending chain condition (DCC) on its ideals. In E with Beidleman's condition every minimal ideal is irreducible [1]. Here we note that semisimple E with Beidleman's condition is evidently m-simple.

If A, B are ideals of E, then a homomorphism f : A → B is said to be a normal homomorphism if f(A) is an ideal of E contained in B. If E and E′ are two N-groups, then a homomorphism f : E → E′ is said to be a *normal homomorphism* if f(E) is an ideal of E′ ([5]).

E is *injective (normal injective)* if for every diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, "h"] \\ & E & \end{tikzcd}$$

of N-groups and N-group monomorphisms f and g in which f is normal there exists a homomorphism h : B → E such that hf = g. E is said to be *weakly n-injective* in case each normal homomorphism of an ideal N′ into E can be extended to a normal homomorphism of E into E. Simple N-group is an example of weakly n-injective N-group. Clearly, n-injective N-group is weakly n-injective.

2. EXAMPLES ON m-SIMPLE, UNIFORM, STRICTLY UNIFORM AND n-INJECTIVE N-GROUPS

In the near-ring (26) (0,5,4,0,2,1) as defined on Z_6 in p-341 [8] we see A = {0,3} and B = {0,2,4} are left N-subgroups of N. Here A = {0,3} is the only left ideal of N which is irreducible and hence N is m-simple. Also, since A ∩ B = (0), N is not strictly uniform. In the near-ring (46) (0,1,2,3,4,5,6,7) as defined on Z_8 (= N) in p-343 [8], {0,4} and {0,2,4,6} are left N-subgroups of N. Moreover these are left ideals of N. Here both the ideals {0,4} and {0,2,4,6} contain the irreducible ideal {0,4} and hence N is m–simple. Also, since {0,4} ∩ {0,2,4,6} ≠ (0), the N-subgroup {0,2,4,6} is strictly uniform and N is an example of strictly uniform N-group. The ideal {0,2,4,6} is strictly essential in N and hence essential.

The Dihedral group N(= D_8) = {0,a,2a,3a,b,a+b,2a+b,3a+b} is a near-ring w.r.t. the multiplication defined as in [8]. Here A = {0,b} and B = {0,2a,b,2a+b} are the only proper left N-subgroups of N with A ∩ B ≠ (0). Hence N is a strictly uniform N-group. Here, N-subgroup {0,2a,b,2a+b} is strictly essential in N(= D_8). The near-ring (13) (0,7,13,9) as defined on Klein's four group N = {0,a,b,c} [8] is such that A = {0,a}, B = {0,b},

$C = \{0,c\}$ are only left N-subgroups of N and these are also left ideals of N with $A+B+C = N$ and hence N is semisimple. Also, $\mathrm{Soc}_l(A) = A$, $\mathrm{Soc}_l(B) = B$ and $\mathrm{Soc}_l(C) = C$ and so A,B,C are semisimple N-subgroups of N, that is, in N every N-group is semisimple and so every N-group is n-injective [5]. Here since $A \cap B = B \cap C = C \cap A = (0)$, N-group N is not strictly uniform.

3. PRELIMINARIES

In our discussion any essential left N-subgroup would be strictly essential as followed by [7].

Before proving the main results let us prove the following

Lemma 3.1. If a nonzero ideal M of E has the FGD then it contains a uniform ideal of E.

Proof. Suppose M contains no uniform ideal of E. Then M is not uniform and so there exists two nonzero ideals M_1 and M_1' of E such that $M_1 \cap M_1' = (0)$ and $M_1 + M_1' \subseteq M$. By our supposition M_1' is not uniform and this implies that there exists two non-zero ideals M_2, M_2' of E such that $M_2 \cap M_2' = (0)$ and $M_2 + M_2' \subseteq M_1'$. If we continue this process we get two infinite sequences $\{M_i\}_1^\infty$, $\{M_i'\}_1^\infty$ of nonzero ideals of E contained in M such that $M_i \cap M_i' = (0)$ and $M_i + M_i' \subseteq M_{i-1}'$ for $i \geq 2$. Now, $\{M_i\}_1^\infty$ is an infinite independent family of nonzero ideals of contained in M – a contradiction. Hence M contains a uniform ideal of E.

Here we note that :

Note 1. If any essential N-subgroup is strictly essential then any nonzero ideal M of E with FGD contains a strictly uniform ideal of E ([9])

Using Zorn's Lemma we see that

Note 2. There exists a maximal independent set of maximal strictly uniform ideals of E

Lemma 3.2. If E is with Beidleman's condition and $(E_i)_{i\in I}$ is an independent family of ideals of E such that $E = \sum_{i\in I} E_i$ then $\mathrm{Soc}_l(E) = \sum_{i\in I} \mathrm{Soc}_l(E_i)$.

Prool. W.l.o.g. $\mathrm{Soc}_l(E) \neq (0)$. It is sufficient to show that $\mathrm{Soc}_l(E) \subseteq \sum_{i\in I} \mathrm{Soc}_l(E_i)$

Let L be any minimal ideal of E. Then L is irreducible since E is with Beidleman's condition and so $L = Na_1 \subseteq E$, $a_1 \in L$. Let $(0 \neq)\ a_1 = x_1 + x_2 + \ldots + x_k$ with $0 \neq x_k \in E_{i_k}$

$(k = 1,2,\ldots,n)$. Consider the homomorphism $\phi : Na_1 \to Nx_k$ given by $\phi(na_1) = nx_k$ $(n\varepsilon N)$. ϕ is clearly non-zero and onto and hence an isomorphism, since Na_1 is an irreducible ideal.

Now, $Nx_k \subseteq \mathrm{Soc}_1(E_{ik})$. Hence $a_1 = \sum_{k=1}^{n} x_k \varepsilon \sum_{k=1}^{n} \mathrm{Soc}_1(E_{ik}) \subseteq \sum_{i\varepsilon I} \mathrm{Soc}_1(E_i)$. This proves that $L \subseteq \sum_{i\varepsilon I} \mathrm{Soc}_1(E_i)$ and hence $\mathrm{Soc}_1(E) \subseteq \sum_{i\varepsilon I} \mathrm{Soc}_1(E_i)$.

Lemma 3.3. In a distributively generated near-ring (d.g.n.r.) N if E is weakly n-injective and A is a closed ideal of E then any normal homomorphism from an ideal K of E into A can be extended to a normal homomorphism of E into A.

Proof. By Zorn's lemma we take K such that a normal homomorphism f from K into A cannot be extended to a normal homomorphism from K_1 into A for any ideal K_1 of E which properly contains K. Since E is weakly n-injective, f can be extended to a normal homomorphism $g : E \to E$. Suppose $g(E) \not\subset A$ and L be the complement of A in E. Since A is closed, A is the complement of L. Thus $A \cap L = (0)$. Since $g(E) + A \supset A$, we see that $(g(E) + A) \cap L \neq (0)$. Let $(0 \neq)\ x = a + b\ \varepsilon\ (g(E) + A) \cap L, a\ \varepsilon\ g(E)$, be A. If $a\ \varepsilon A$ then $x\ \varepsilon\ A \cap L = (0)$, a contradiction. Therefore $a \not\varepsilon A$ and $a = x - b\ \varepsilon\ L \oplus A$. Now, $T = \{y\ \varepsilon\ E \mid g(y)\ \varepsilon\ L + A\}$ is an ideal of E containing K, since if $y\varepsilon\ E$ is such that $g(y) = a$ then $y\varepsilon\ T$; but $y \not\varepsilon\ K$ as $a \not\varepsilon\ A$. Let h denote the normal homomorphism of $L + A$ on A. Then hg is a normal homomorphism of T on A and $hg(y) = g(y) = f(y)$, $\forall\ y\varepsilon\ K$. Thus hg is a proper extension of f - a contradiction. Therefore $g(E) \subseteq A$ and g is the desired extension of.

Lemma 3.4. If E, E′ are N-groups and A is an ideal of E′ with A – essential in E′ and $f : A \to E$ is a monomorphism where E is n-injective then f extend to a homomorphism $E' \to E$.

Proof. Since E is n-injective, f must extend to a homomorphism $f' : E' \to E$. Now, let $x\ \varepsilon\ A \cap \mathrm{Ker}f'$. Then $x\ \varepsilon\ A$ and $x\ \varepsilon\ \mathrm{Ker}f'$ which gives $x\varepsilon\ A$ and $f'(x) = 0$. Thus $x\ \varepsilon A$ and $f(x) = 0$ and hence $x\ \varepsilon\ \mathrm{Ker}f = (0)$ and so $x = 0$. Thus, $A \cap \mathrm{Ker}f' = \mathrm{Ker}f = (0)$ and so $\mathrm{Ker}f' = (0)$. Hence f' is a monomorphism.

Before proving the next lemma we note the following

Note 3. If A is an N-subgroup of an n-injective E and A has no essential extension in E then A is n-injective [5].

Now, let A be an ideal of an n-injective E. Using Zorn's lemma we may pick an N-subgroup E′ of E that is maximal among those N-subgroups of E which are strictly essential extensions of A. E′ is clearly a maximal essential extension of A. Now, any essential extension of E′ in E is also an essential extension of A, but from the maximality of E′, it has no proper essential extension in E and hence E′ is n-injective (Note 3).

Here if f : A → E″ is a monomorphism where E″ is n-injective then, since A is essential in E′, f extends to a monomorphism E′ → E″ (Lemma 3.4).

Hence E′ is the minimal n-injective extension and maximal essential extension in E and such E′ is called the *n-injective hull* of A in E. In the n-injective N-group (13) (0,7,13,9) as defined on Klein's four group N = {0,a,b,c} [8], the only ideals are A = {0,a}, B = {0,b}, C = {0,c}. Here maximal strictly essential extension of A in N is A itself and hence A is an n-injective hull of A in N-group N. Similarly, B and C are n-injective hulls of B and C respectively in N-group N.

Lemma 3.5. If E′ and E″ are both n-injective hulls of an ideal A of E then the monomorphism on A extends to an isomorphism E onto E′.

Proof. By definition, the monomorphism A → E″ extends to a monomorphism f : E′ → E″ and hence A ≅ f(A) ⊆ f(E′) ⊆ E″ and since A is essential in E″, E′ ≅ f(E′) is essential in E″. But E′ has no proper essential extension and so f(E′) = E″. Thus f is an isomorphism.

Note 4. In this sense n-injective hull of an ideal of an N-group is unique.

Lemma 3.6. If M = A ⊕ B and A, B are n-injective ideals of E then M is n-injective

Proof. A diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "g"] \arrow[d, "f"'] & Y \\
 & M &
\end{tikzcd}$$

where g is

normal can be extended in a diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[dl, "h"] \\
 & M \arrow[d, shift right, "p"'] & \\
 & B \arrow[u, shift right, "i"'] &
\end{tikzcd}$$

where p and i are the homomorphisms associated with $M = A + B$. So, $pi = I_B$. Since B is n-injective there exists a homomorphism $h : Y \to B$ such that $hg = pf$ and so $ih = h : Y \to M$ is the required homomorphism.

Lemma 3.7. If an n-injective E is an essential extension of a direct sum of a family $(A_i)_{i\in I}$ of non-zero indecomposable n-injective ideals then any non-zero indecomposable n-injective ideals of E is isomorphic to some A_i

Proof. Let A be any non-zero indecomposable n-injective ideal of E. Since $\bigoplus_{i\in I} A_i$ is essential in E, $A \cap \sum_{i\in I} A_i \neq (0)$. Thus there exists $(0\neq)$ $a \varepsilon A \cap \sum_{i\in I} A_i$ and $a = a_1 + \ldots + a_n$ where $a_k \varepsilon A_{i_k}$ ($k = 1,2,\ldots n$). Let $A' = A \cap (A_{i1} + \ldots A_{in})$ then $A' \subseteq A$ and $A' \subseteq A_{i1} + \ldots + A_{in}$. Since $C = A_{i1} + \ldots A_{in}$ is n-injective (Lemma 3.6), it contains an n-injective hull $H(A')$. If an n-injective hull $H'(A') \subseteq A$ then $H(A') \cong H'(A')$. If $H'(A') = B$ then $A = B + C'$ for some ideal C' of E ([5]) and since A is an ideal of E, B must be an ideal of E. Also, A being indecomposable, we get $A = B$. Thus C contains an isomorphic copy B of A and B is a direct summand of C. If T is a complement of B in C then T is a direct sum of indecomposable n-injective ideals of E. Since in any two decomposition of an n-injective N-group into a direct sum of indecomposable n-injective indeals the summands are isomorphic, B is isomorphic to some A_{ij} $(I \leq j \leq n)$.

MAIN RESULTS

Now we prove our main results, the second and third being in some sense a characterization of N-groups with what termed as Beidleman's contition.

Theroem 4.1. If E is quasi-m-simple then E is an essential extension of a direct sum of strictly uniform ideals.

Proof. The set $\mathcal{A}$ of all strictly uniform ideals of E is non-empty (Note 1). Let $\mathcal{K}$ be a chain in $\mathcal{A}$. Then for nonzero N-subgroups A and B of $\cup\, \mathcal{K} = \cup\, \{K \mid K \in \mathcal{K}\}$ with $0 \neq a \in A$, $0 \neq b \in B$; there exists $K \varepsilon \mathcal{K}$ such that $0 \neq Na, Nb \subseteq K$ and $Na \cap Nb \subseteq A \cap B$ and since $Na \cap Nb \neq (0)$, $A \cap B \neq (0)$. Hence $\cup\, \mathcal{K} \in \mathcal{A}$ and therefore $\mathcal{A}$ has a maximal member which is also a maximal strictly uniform ideal of E. The set of maximal strictly uniform ideals of E being non empty contains a maximal independent subset $\mathcal{B}$ (Note 2). To prove $Y = \bigoplus_{X\varepsilon\mathcal{B}} X$ is essential in E, we assume the contrary. Then for some ideal A of E. $A \cap Y = (0)$. Since E is quasi-m-simple, A contains a non-zero ideal A' of E with FGD. Since $Y \cap A' = 0$ and A' contains a strictly uniform ideal C of E (Note 1), we have $Y \cap C = (0)$. Using Zorn's Lemma we get an ideal D of E that is maximal among those ideals of E which are strictly essential extensions of C. Again D is clearly mximal strictly uniform ideal of E. Now. $Y \cap D \cap C = Y \cap C = (0)$ and so $Y \cap D = (0)$. Hence $D \in \mathcal{B}$ and $\mathcal{B} \cup \{D\}$ is independent, contradicting the maximality of $\mathcal{B}$. Thus Y is essential in E.

Theorem 4.2. In $\mathcal{E}$ the following statements are equivalent

(i) E is m-simple

(ii) E is quasi-m-simple such that each strictly uniform ideal has a nonzero Socle-1.

(iii) Soc (E) is strictly essential in E.

Proof. (i) $\Rightarrow$ (ii) : E being m-simple each non-zero ideal I of E contains an irreducible ideal I of E with strictly FGD and hence I′ has the FGD and therefore E is quasi-m-simple. Moreover strictly uniform ideals being non-zero, have non-zero Socle – 1.

(ii) $\Rightarrow$ (iii) : E being with Beidleman's condition, if X is a minimal ideal of E then X is irreducible. And if $\{A_i\}_{i \in I}$ is a maximal independent family of strictly uniform ideals of E then $\sum_{i \in I} A_i$ is essential in E (Theorem 4.1) which gives $X \cap \sum_{i \in I} A_i \neq (0)$. Also, $X \cap \sum_{i \in I} A_i \subseteq X, \sum_{i \in I} A_i$ give $X \cap \sum_{i \in I} A_i = X \subseteq \sum_{i \in I} A_i$ (X being minimal). Hence $X \subseteq \mathrm{Soc}_1 (\sum_{i \in I} A_i) = \sum_{i \in I} \mathrm{Soc}_1 (A_i)$ (Lemma 3.2). Thus $\mathrm{Soc}_1 (E) \subseteq \sum_{i \in I} \mathrm{Soc}_1 (A_i)$. Also $\sum_{i \in I} A_i$ being an independent family of strictly uniform ideals of E, $\mathrm{Soc}_1 (\sum_{i \in I} A_i) \subseteq \mathrm{Soc}_1 (E)$ and so $\mathrm{Soc}_1(E) = \mathrm{Soc}_1 \sum A_i$. Now, A_i being strictly uniform, $\mathrm{Soc}_1(A_i)$ is strictly essential in $A_i, i \in I$. Thus $\mathrm{Soc}(E) = \mathrm{Soc}_1(E) = \sum_{i \in I} \mathrm{Soc}_1 (A_i) \subseteq \sum_{i \in I} A_i \subseteq E$ gives Soc (E) = $\mathrm{Soc}_1(E)$ is strictly essential in E [9].

(iii) $\Rightarrow$ (i) : Let A be a non-zero ideal of E and Soc (E) = $\bigoplus_{i=1}^{n} E_i$ where E_i's are some minimal ideals of E. Since Soc (E) is strictly essential in E, $A \cap \mathrm{Soc}(E) \neq (0)$. Now, $A \supseteq (0 \neq) A \cap \mathrm{Soc}(E) \supseteq A \cap E_i$ for some i, and $A \cap E_i \subseteq E_i$ and E_i being minimal $A \cap E_i = E_i$. Thus every ideal of E contains a minimal ideal which is irreducible as E is with Beidleman's condition. Thus E is m-simple. If $A \cap E_i = (0)$ the result is obvious.

That the non-zero Socle – 1 charaacter of strictly uniform ideals plays an important role for the inheritance of m-simple character of an quasi m-simple E is evident from the well known Z-group Z.

Corollary 1. If E is a direct sum of a family $(E_i)_{i \in I}$ of m-simple ideals then Soc (E) is strictly essential in E.

Proof. Here $E = \bigoplus_{i \in I} E_i$ gives Soc (E) $= \sum_{i \in I} \mathrm{Soc}_I\,(E_i)$ (Lemma 3.2). Since E_i's, $i \in I$, are m-simple ideals we get $\mathrm{Soc}_I\,(E_i)$ is strictly essential in E_i (Theorem 4.2) for each $i \in I$ and hence Soc (E) $= \sum_{i \in I} \mathrm{Soc}_I\,(E_i) \subseteq \sum_{i \in I} E_i = E$ gives Soc (E) is strictly essential in E. [9].

Corollary 2. An m-simple E is strictly uniform if and only if Soc (E) is an irreducible ideal.

Proof. Let E be strictly uniform. Then so is Soc (E) . Let Soc (E) $= \oplus A_i$, where A_i's, $i \in I$, are some minimal ideals of E. Since Soc (E) is strictly uniform. It contains only one minimal ideal and so Soc (E) $= A_i$ (for some i), which is minimal and hence Soc (E) is irreducible as E is with Beidleman's condition.

Conversely, let Soc (E) be an irreducible ideal. Then Soc (E) is strictly uniform. Let B and B′ be any two non-zero N-subgroups of E. Since Soc (E) is strictly essential in E, Soc (E) $\cap$ B $\neq$ (0) and Soc (E) $\cap$ B′ $\neq$ (0). Now, Soc (E) $\cap$ B $\subseteq$ Soc (E) and Soc (E) $\cap$ B′ $\subseteq$ Soc (E) and since Soc (E) is an irreducible ideal Soc (E) $\cap$ B = Soc (E) = Soc (E) $\cap$ B′. Thus (0 $\neq$) Soc (E) = Soc (E) $\cap$ B $\cap$ B′ and so B $\cap$ B′ $\neq$ (0). Hence E is strictly uniform.

Corollary 3. If (E_i) is a family of m-simple ideals of E with $E = \bigoplus_{i \in I} E_i$, then E is m-simple.

Proof. If $E = \bigoplus_{i \in I} E_i$ then Soc (E) $= \sum_{i \in I} \mathrm{Soc}_I\,(E_i)$ (Lemma 3.2). Since $\mathrm{Soc}_I\,(E_i)$ is strictly essential in E_p. for each i, and Soc (E) $= \sum_{i \in I} \mathrm{Soc}_I\,(E_i) \subseteq \sum_{i \in I} E_i = E$ we get Soc (E) is strictly essential in E and hence E is m-simple.

Before proving the next corollary let us note the following

Note 5. If A and B are two ideals of E and A is essential in B then Soc_I (A) $= \mathrm{Soc}_I$ (B).

Corollary 4. If E is m-simple and $(A_i)_{i \in I}$ is a maximal independent set of maximal strictly uniform ideals of E then Soc (E) $= \sum_{i \in I} \mathrm{Soc}_I\,(A_i)$

Proof. It is enough to show $\sum_{i \in I} A_i$ is essential in E. Assuming the contrary, we get a non-zero ideal X of E with $X \cap \sum_{i \in I} A_i = (0)$. Since E is m-simple, X contains an irreducible ideal of E and hence X contains a minimal ideal of E. Let Y be a minimal ideal of E contained in X. Then $Y \cap \sum_{i \in I} A_i = (0)$. Let T be a maximal strictly essential extension of Y in E so that T is a maximal strictly uniform ideal of E with $\mathrm{Soc}_1(T) = Y$. Now, $\mathrm{Soc}_1(T \cap \sum_{i \in I} A_i) \subseteq \mathrm{Soc}_1(T) \cap \mathrm{Soc}_1(\sum_{i \in I} A_i) = \mathrm{Soc}_1(T) \cap \sum_{i \in I} \mathrm{Soc}_1(A_i) = Y \cap \sum_{i \in I} \mathrm{Soc}_1(A_i) \subseteq Y \cap \sum_{i \in I} A_i = (0)$. Hence, $\mathrm{Soc}_1(T \cap \sum_{i \in I} A_i) = (0)$. It follows that $T \cap \sum_{i \in I} A_i = (0)$. Therefore, $\{T\} \cup \{A_i \mid i \varepsilon I\}$ is independent contradicting the maximality of $(A_i)_{i \varepsilon I}$. Hence, $\sum_{i \in I} A_i$ is essential in E and so $\mathrm{Soc}_1(\sum_{i \in I} A_i) = \mathrm{Soc}_1(E) = \mathrm{Soc}(E)$ (note 5).

Corollary 5. Any two maximal independent sets of maximal strictly uniform ideals of m-simple E has the same cardinality.

Proof. Lef $\mathcal{A} = \{A_i \mid i \varepsilon I\}$ be a maximal independent set of maximal strictly uniform ideals of E. Then $A_i \neq A_j \Leftrightarrow A_i \cap A_j = (0) \Leftrightarrow \mathrm{Soc}_1(A_i) \cap \mathrm{Soc}_1(A_j) = (0) \Leftrightarrow \mathrm{Soc}_1(A_i) \neq \mathrm{Soc}_1(A_j)$. Thus $\mathcal{A}$ and $\{\mathrm{Soc}_1(A_i) \mid i \varepsilon I\}$have the same cardinality. Now, $\mathrm{Soc}(E) = \mathrm{Soc}_1(E) = \sum_{i \in I} \mathrm{Soc}_1(A_i)$ (Lemma 3.2). Hence cardinality of $\mathcal{A}$ = Length of Soc (E) which is invariant.

Corollary 6. An m-simple E has the FGD if and only if Soc (E) is of finite length.

Proof. If E has the FGD then any maximal independent set of maximal strictly uniform ideals of E is of finite cardinality. Hence length of Soc (E) is finite.

Conversely, let $\mathrm{Soc}(E) = \sum X_i$ where $(X_i)_{i \varepsilon \{1,2,..,n\}}$ is an independent family of non-zero minimal ideals of E. Let $(Y_i)_{i \varepsilon I}$ be an independent family of non-zero ideals of E. Since E is m-simple, every non-zero ideal contains an irreducible ideal and hence minimal. Now, for each $i \varepsilon I$, pick a minimal ideal S_i of E contained in Y_i. Lef $N' = \oplus S_i$. Then length of $N' \leq$ length of Soc (E) = n. Hence I is of finite cardinality. Thus E has the FGD.

Theorem 4.3. If E is m-simple weakly n-injective and $\mathcal{B} = \{B_i \mid i \varepsilon I\}$ is any maximal independent set of maximal strictly uniform ideals of E then $\mathcal{B}$ contains an isomorphic copy of every maximal strictly uniform ideal of E.

Proof. Let A be any maximal strictly uniform ideal of E. Now, $\mathrm{Soc}(E) = \sum_{i \in I} \mathrm{Soc}_1(B_i)$ (Corollary 4). Hence for some $i \varepsilon I$, $\mathrm{Soc}_1(A) \cong \mathrm{Soc}_1(B_i)$. Let $f : \mathrm{Soc}_1(A) \rightarrow \mathrm{Soc}_1(B_i)$ be the isomorphism. Since E is weakly n-injective and $B_i \subseteq E$ is a closed ideal, the normal homomorphism $f : \mathrm{Soc}_1(A) \rightarrow B_i$ gets extended to a normal homomrphism

$f' : E \rightarrow B_i$ (Lemma 3.3). Let g be the restriction of f′ to A and assume $(0 \neq)\ x\ \varepsilon$ Ker g. Then $Nx \subseteq$ Ker g and so $\text{Soc}_1\ (Nx) \subseteq \text{Soc}_1$ (Kerg). Now, since A is strictly uniform, Ker g is strictly uniform and so if L is any minimal ideal of E contained in Kerg, then $Nx \cap L \neq 0$. Here $Nx \cap L \subseteq L$ and since L is irreducible (as E with Beidleman's condition) $Nx \cap L = L \subseteq Nx$ and hence $\text{Soc}_{1\,(\text{Ker } g)} \subseteq \text{Soc}_1\ (Nx)$. So $\text{Soc}_1\ (Nx) = \text{Soc}_1$ (Ker g) and thus $\text{Soc}_1\ (Nx) = \text{Soc}_1\ (\text{Ker } g) = \text{Soc}_1\ (A) \subseteq$ Kerg. It follows that $\text{Soc}_1\ (B_i) = f(s) = f'\ (s) = g(s) = 0,\ s\ \varepsilon\ \text{Soc}_1\ (A)$, a contradiction. Hence g is one to one and $A \cong g(A) \subseteq B_i$ together with the maximality of A as strictly uniform ideal implies $g\ (A) = B_i$. Hence g is an isomorphism.

Theorem 4.4. For a non-zero E the following statements are equivalent

(i) E is indecomposable n-injective

(ii) E is minimal n–injective

(iii) E is uniform and n–injective

(iv) E = H (A) for some uniform ideal A of E.

Proof. (i) ⇒ (ii) : Let E be indecomposable n-injective and $M \subseteq E$ be any non-zero n-injective ideal. Since an n-injective N-group is a direct summand of every N-group of which it is an ideal [5]. $E = M \oplus M'$ for some ideal M′ of E, which is a contradiction, since E is indecomposable. Therefore M = E and hence E is minimal n-injective.

(ii) ⇒ (iii) : Let E be minimal n-injective. Suppose there exists non-zero ideals A, B of E with $A \cap B = (0)$. Let E′ = H (A) in E. Since $E' \cap B \cap A = E' \cap (0)$ we have $E' \cap B = (0)$ (as A is strictly essential in E′) and hence $E' \subset E$, a contradiction. Hence E is uniform

(iii) ⇒ (iv) : It is obvious since E = H (E).

(iv) ⇒ (i) : It is also obvious, since E is an n-injective hull of A.

Theorem 4.5. If E is non-zero n-injective then E has the FGD if and only if E is a direct sum of finitely many non-zero strictly uniform n-injective ideals of E.

Proof. Necessary part of the proof is obvious. To prove the sufficient part, let $E = E_1 \oplus E_2 \oplus \ldots \oplus E_n$ where E_i's are non-zero strictly uniform n-injective ideals and hence indecomposable n-injective ideals. Thus for any two non-zero ideals A, B of E contained in E_i, $A \cap B \neq 0$. So, we do not get any independent family of non-zero ideals of E in E_i and hence E_i has the FGD. Thus every ideal of E contains an ideal of E with FGD and hence E is quasi m-simple. Consider any independent family $(X_i)_{i \in I}$ of non-zero ideals of E. Since E is quasi m-simple, each X_i conatains an ideal Y_i with FGD and so Yi contains a strictly uniform ideal (Note 1). Hence each X_i contains a strictly uniform ideal. For each i pick one strictly uniform ideal $A_i \subseteq X_i$. Let B_i be an N-subgroup of E which is maximal strictly essential extension of A_i in E. Then each B_i is strictly uniform n–

injective and $E = B_i + T$ for some ideal T of E ([5]) and hence B_i is an ideal of E. Also the family $(B_i)_{i \varepsilon I}$ is independent. The form $\{B_i \mid i \varepsilon I\}$ can be extended to a maximal independent set $\{B_j \mid j \varepsilon J\}$, where $J \supseteq I$, of strictly uniform n-injective ideals of E. Then $\sum_{j \varepsilon J} B_j$ is essential in E (Theorem 4.1) and each E_k is isomorphic to some B_{jk} ($k = 1,2,\ldots,n$) [Lemma 3.7).

Let $f_k : E_k \ldots\ldots B_{jk}$ be the isomorphism. Define $f : E \ldots\ldots E$ by $f(x = x_1 + x_2 + \ldots\ldots + x_n) = f_1(x_1) + \ldots + f_n(x_n)$ where $x_k \in E_k$. Then f is clearly an automorphism of E ([9]). Thus $E = f(E) = \Sigma f_k(E_k) = Bj_1 + \ldots + Bj_n$.

It follows that $\sum_{j \varepsilon J} B_j$ is essential in $E = \sum_{i=1}^{n} B_{ji}$ and hence $\sum_{j \varepsilon J} B_j = \sum_{i=1}^{n} B_{ji}$ and E being n-injective, cardinality of J is n. Hence I is a finite set. This completes the proof.

REFERENCES

1. J. Beidleman, A radical for near-ring modules, Mich. Math. J. 12 (1965), 377-383.
2. K. C. Chowdhury, A. Masum and H. K. Saikia, FSD N-group with acc on annihilators, Indian J. Pure and Appl. Math., 24 (2), (1993), 747-755.
3. ...., On left Goldie near-rings and its parts having minimum conditions, Indian J. Pure and Appl. Math., 25 (11) (1994), 1155-1162.
4. G. Mason and A. Oswald, Injective and projective near-ring modules, Tesside Polytechnic Mathematical Reports, 1981.
5. G. Mason, On strongly regular near-rings, Proc, Edinburgh Math. Soc., 23 (1980), 27-35.
6. A. Oswald, Near-rings in which each N-subgroup is principal, Proc. London Math. Soc., (3) 23 (1974), 67-88.
7. A. Oswald, Near-rings with chain conditions on right annihilators, Proc. Edinburgh Math. Soc., 23 (1980), 123-127.
8. G. Pilz, Near-rings, North Holland Publishing Company, Amsterdam, 1977.
9. B. Satyanarayana, Contributions to near-ring theory, Ph. D. Dissertation, Nagarjuna University, 1984.

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SOME RESULTS ON DIFFERENTIAL INEQUALITIES ASSOCIATED WITH ORDINARY DIFFERENTIAL EQUATION OF ORDER n

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ABSTRACT : In Chapter 1 of the book entitled 'Differential and Integral Inequalities' (vol 1) by V. Lakshmikanthan and S. Leela, the authors have proved two fundamental results on differential inequalities (vide theorem 1.2.1 and theorem 1.2.3 of the above book) involving Dini derivatives, over and under functions and first order differential equations with initial conditions. In this paper, these notions are generalised and modified proofs of the above results are given involving ordinary differential equations of order n with initial conditions.

Key words : Initial value problem, over function, under function.

1. INTRODUCTION

In Chapter 1 of [1] V. Lakshmikanthan and S. Leela have proved two fundamental results [1, Theorem 1.2.1 and Theorem 1.2.3.] involving Dini derivatives, over and under functions and first-order differential equations with initial conditions. In this paper, these notions are generalised and modified proofs of the above result are given involving ordinary differential equations of order n with initial conditions.

The real n-dimensional Euclidean space of elements

$\bar{u} = (u_1, \ldots, u_n)$ is denoted by R^n. The (n+1) tuple $(t, u_1', \ldots, u_n)$ denoted by $(t, \bar{u})$ is an element of R^{n+1}. Let E be an open $(t, \bar{u})$ set in R^{n+1}. The class of continuous mapping from E into R^n is denoted by $C\,[\,E, R^n]$.

The results proved by V. Lakshmikanthan and S. Leela are stated below :

Theorem 1′. Let E be an open (t, u) set in R^2 and $g \varepsilon C[E, R]$. Assume that $v, w \varepsilon C[[t_0, t_0 + a), R]$ and $(t, v(t)), (t, w(t)) \varepsilon E$, $t \varepsilon [t_0, t_0 + a)$. Suppose further that $v(t_0) < w(t_0)$ and for $t \varepsilon (t_0, t_0 + a)$, the inequalities

$D\!-\!v(t) \leq g(t, v(t))$

$D\!-\!w(t) > g(t, w(t))$ hold, where D–[.]. denotes a Dini derivative.

Then $v(t) < w(t)$, $t \varepsilon [t_0, t_0 + a)$.

The next theorem shows that any solution of the initial value problem, $u' = g(t, u)$, $u(t_0) = u_0$ ... (1.1)

can be bracketed between its under and over functions. For definitions of under and over functions sco [1, 1.2].

Theorem 2′. Let v (t), w (t) be under and over functions with respect to the initial value problem (1.1) on $[t_0, t_0 + a)$. If u (t) is any solution of (1.1) existing on $[t_0, t_0 + a)$ such that $v(t_0) = u_0 = w(t_0)$ then $v(t) < u(t) < w(t)$, $t \varepsilon [t_0\ t_0 + a)$.

In § 2, an initial value problem of order n, a modified version of Dini derivative, an n-over functions and an n-under function are defined.

The generalised theorem corresponding to theorem 1′ is then stated and proved in § 3. In § 4, the theorem corresponding to theorem 2′ is stated and proved.

2. SOME DEFINITIONS

Definition 1. The nth order differential equation with initial conditions is of the form

$$u^{(n)} = g(t, u, \ldots u^{(n-1)}),$$

$$\left.\begin{aligned} u(t_0) &= u_0^0 \\ u^1(t_0) &= u_0^1 \\ &\cdots\cdots\cdots\cdots \\ u^{(n-1)}(t_0) &= u_0^{(n-1)} \end{aligned}\right\} \quad \ldots (2.1)$$

where $u^1 = \dfrac{du}{dt}$, $u^{(2)} = \dfrac{d^2u}{dt^2}$, ..., $u^{(n)} = \dfrac{d^nu}{dt^n}$,

$g \varepsilon C[E, R]$, E is taken as an open $(t, u, \ldots, u^{(n+1)})$ set in R^{n+1}.

Definition 2. A real-valued function u (t) on some t–interval I containing t_0 is considered as a solution of (2.1) if it is n-times differentiable function such that

$u(t_0) = u_0^0,\ u^1(t_0) = u_0^1,\ \ldots\ u^{(n-1)}(t_0) = u_0^{(n-1)},$

$g(t, u(t), \ldots, u^{(n-1)}(t)) \varepsilon E$ and $u^{(n)}(t) = g(t, u(t), \ldots, u^{(n-1)}(t))$ ... (2.2)

for $t \varepsilon I$.

Similar to Dini derivatives we give the following definitions of generalised Dini derivative.

Definitions 3.

$$D^{+} u^{(j)}(t) = \underset{h \to 0^{+}}{Lim} \sup h^{-1} [u^{(j)}(t + h) - u^{(j)}(t)],$$

$$D_{+} u^{(j)}(t) = \underset{h \to 0^{+}}{Lim} \inf h^{-} [u^{(j)}(t + h) - u^{(j)}(t)],$$

$$D^{-} u^{(j)}(t) = \underset{h \to 0^{-}}{Lim} \sup h^{-1} [u^{(j)}(t + h) - u^{(j)}(t)],$$

$$D_{-} u^{(j)}(t) = \underset{h \to 0^{-}}{Lim} \inf h^{-1} [u^{(j)}(t + h) - u^{(j)}(t)],$$

where, $u^{(j)} \varepsilon C\, [(t_0, t_0 + a), R]$, $j = 1, 2, \ldots$

When $D^{+} u^{(j)}(t) = D_{+} u^{(j)}(t)$, then, the right derivative will be denoted by $u_{-}^{(j)}(t)$. Similarly $u_{-}^{(j)}(t)$ denotes the left derivative.

Definition 4. Let E be an open $(t, u, \ldots u^{(n-1)})$ set in R^{n+1} and $g \varepsilon C\, [E, R]$. Consider the differential equation $u^{(n)} = g(t, u, \ldots, u^{(n-1)})$,

where,

$$\left.\begin{aligned} u(t_0) &= u_0^0, \\ u'(t_0) &= u_0^1, \\ &\ldots\ldots\ldots\ldots\ldots \\ u^{(n-1)}(t_0) &= u_0^{(n-1)} \end{aligned}\right\} \qquad \ldots (2.3)$$

suppose, $v^{(j-1)} \varepsilon C\, [[t_0 + t_0 + a), R]$, $v_{+}^{(j)}(t)$ exist for $t \varepsilon [t_0, t_0 + a)$, $j = 1, 2 \ldots\ldots$, and $(t, v(t), \ldots, v^{(n-1)}(t)) \varepsilon E$, and $v^{(j)}(t_0) \leq u_0^{(j)}$

If v (t) satisfies the differential inequalities

$v_+^{(j)}(t) < g(t, v(t), \ldots, v^{(n-1)}(t))$, $j = 1, 2, \ldots, n$, $t \varepsilon [t_0, t_0 + a)$, it is said to be an *n-under function* with respect to the initial value problem (2.1). On the other hand, if

$v_+^{(j)}(t) > g(t, v(t) \ldots, v^{(n-1)}(t))$, $j = 1, 2, \ldots, n$,

$t \varepsilon [t_0, t_0 + a)$, $v(t)$ is said to be an *n-over function* with respect to the initial value problem (2.1).

3. PROOF OF THEOREM 1

Theorem 1. Let E be an open $(t, u, \ldots; u^{(n-1)})$ set in R^{n+1} and let $g \varepsilon C[E, R]$ satisfy the following conditions for each $t \varepsilon [t_0, t_0 + a)$,

(i) $g(t, p(t), \ldots, p^{(n-1)}(t)) \leq g(t, q(t), \ldots, q^{(n-1)}(t))$

whenever $p^{(i)}(t) <$ or, $= q^{(i)}(t)$ $\forall$ $i = 0, 1, \ldots, n-1$, ... (3.1)

where p, q have continuous derivaative upto (n-1) th order and $(t, p(t), \ldots p^{(n-1)}(t))$, $(t, q(t), \ldots q^{(n-1)}(t)) \varepsilon E$.

(ii) Assume that $v^{(i)}(t)$, $w^{(i)}(i)(t) \varepsilon C[[t_0, t_0 + a), R]$,

$i = 0, \ldots, n-1$, and $(t, v(t) \ldots, v^{(n-1)}(t))$;

$(t, w(t), \ldots, w^{(n-1)}(t)) \varepsilon E$, for $t \varepsilon [t_0, t_0 + a)$.

(iii) Suppose further that

$v^{(i)}(t_0) < w^{(i)}(t_0)$ for all $i = 0, \ldots, n-1$, (3.2)

(iv) and for $t \varepsilon (t_0, t_0 + a)$, the inequalities

$$\left.\begin{aligned} &D_-v^{(i)}(t) \leq g(t, v(t), \ldots, v^{(n-1)}(t)) \\ &D_-w^{(i)}(t) > g(t, w(t), \ldots, w^{(n-1)}(t)), \end{aligned}\right\} \quad i = 0, 1, \ldots, (n-1) \quad \ldots (3.3)$$

hold.

Then $v^{(i)}(t) < w^{(i)}(t)$ for $i = 0, \ldots n-1$, (3.8)

$t \varepsilon [t_0, t_0 + a)$.

Proof. We consider the set

$$\begin{aligned} Z = {} & [t \varepsilon [t_0, t_0 + a) : w(t) \leq v(t)] \\ & U [t \varepsilon [t_0, t_0 + a) : w'(t) \leq v'(t)] \\ & \ldots U [t \varepsilon [t_0, t_0 + a) : w^{(n-1)}(t) \leq v^{(n-1)}(t)] \end{aligned}$$

Then $Z = \bigcup_{i=0}^{n-1} Z_i$: Where $Z_i =[\, t \in [t_0, t_0 + a) : w^{(i)}(t) \leq v^{(i)}(t)]$.

If (3.4) is false, then the set Z is non-empty.

Let, $t_1 = \inf Z$.

We consider a sequence $\{t_1\}$ in Z_j, $(j = 0, 1, \ldots n-1)$, convenging to t_1

Then, $w^{(j)}(t_i) \leq v^{(j)}(t_1)$, $i = 1, 2. \ldots\infty$,

$\Rightarrow \lim\limits_{l\to\infty} w^{(j)}(t_i) \leq \lim v^{(j)}(t_1)$

$\Rightarrow w^{(j)}(\lim\limits_{l\to\infty} t_1) \leq v^{(j)}(\lim\limits_{l\to\infty} t_1)$, since, $w^{(j)}$ and $v^{(j)}$ are continuous. This implies that $\lim\limits_{l\to\infty} t_1 \in Z_j$.

Thus Z_j is closed, for all $j = 0, 1, \ldots n-1$.

Hence, $Z = \bigcup_{j=0}^{n-1} Z_j$ is closed.

Hence, $t_1 \in Z$.

From (3.2), we get $t_0 \notin Z$. Hence $t_0 < t_1$. Since, $t_1 \in Z$, for some arbitrary but fixed j, $t_1 \in Z_j$. Then we have $w^{(j)}(t_1) \leq v^{(j)}(t_1)$ ... (3.5)

Case 1. We assume that $t_1 \in Z_j$ for one j and $t_1 \notin Z_i$ for $i \neq j$.

Then we have, $v^{(i)}(t_1) < w^{(i)}(t_1)$ for $i \neq j$. ... (3.6)

Also we have,

$v^{(i)}(t) < w^{(i)}(t)$ for $t \in [\, [t_0, t_1), i = 0, 1, \ldots, n-1]$ ... (3.7)

from (3.5), (3.6) and (3.7)

$v^{(j)}(t_1) = w^{(j)}(t_1)$ ... (3.8)

Now for small $h < 0$, we get from (3.7)

$v^{(j)}(t_1 + h) < w^{(j)}(t_1 + h)$

$\Rightarrow v^{(j)}(t_1 + h) - v^{(j)}(t_1)$

$< w(t_1 + h) - w^{(j)}(t_1)$ by (3.8)

$$\Rightarrow \frac{v^{(j)}(t_1+h)-v^{(j)}(t_1)}{h} > \frac{w^{(j)}(t_1+h)-w^{(j)}(t_1)}{h} \text{ as } h < 0$$

$$\Rightarrow \lim_{h\to 0}\frac{v^{(j)}(t_1+h)-v^{(j)}(t_1)}{h} \geq \lim_{h\to 0}\frac{w^{(j)}(t_1+h)-w^{(j)}(t_1)}{h}$$

$\Rightarrow D_- v^{(j)} (t_1) \geq D_- w^{(j)} (t_1)$.

But

$D_- v^{(j)} (t_1) \leq g (t_1, v (t_1), \ldots, v^{(n-1)}(t_1))$ by (3.3)

$\leq g (t_1, w (t_1), \ldots, w^{(n-1)} (t_1))$ by (3.6), (3.8) and (3.1)

$< D_- w^{(j)} (t_1)$ by (3.3).

Thus we arrive at a contradiction.

Case II. Let $t_1 \varepsilon Z_j$ where j takes more than one value. Here, we also arrive at a contradiction. (the proof is similar to Case I). Thus Z_i is empty for all i = 0, 1, ..., n–1.

Hence, $v^{(i)} (t) < w^{(i)} (t)$, for i = 0,1, ..., n-1, $t \varepsilon [t_0, t_0 + a)$.

This completes the proof of theorem.

Remark. Theorem 1 also holds if the hypothesis (i) – (iii) of the theorem remain the same but (iv) is changed to

$$\text{(iv)} \qquad \left.\begin{array}{l} D_+ V^{(i)} (t) \leq g (t, v (t), \ldots, v^{(n-1)} (t)) \\ D_+ w^{(i)} (t) > g (t, w (t), \ldots, w^{(n-1)} (t)) \end{array}\right\} \qquad i = 0, \ldots, n\text{-}1.$$

4. PROOF OF THEOREM 2

Theorem 2. Let v (t), w (t) be n-under and n-over function respectively with respect to the initial value problem (2.1) on $[t_0, t_0 + a)$.

We consider all those $(t, u, \ldots, u^{(n-1)})$ in E which satisfy

$$\text{(i)} \int_{t_0}^{t} u^{(k)} (t)\, dt + u^{(k-1)} (t_0) \leq g (t, u (t), \ldots, u^{(n-1)} (t)), \; k = 1, \ldots, n \qquad \ldots (4.1)$$

(ii) Also g satisfies the condition,

$g (t, p (t), \ldots, p^{(n-1)} (t)) \leq g (t, q (t) \ldots, q^{(n-1)} (t))$

whenever $p^{(j)} (t) <$ or, $= q^{(j)} (t)$ for j = 0, ..., n-1 ... (4.2)

(iii) Let u(t) be any solution of (2.1) existing on $[t_0, t_0 + a)$

such that

$$v^{(j)}(t_0) = u_0^{(j)} = w^{(j)}(t_0),\ j = 0, \ldots, n\text{-}1 \qquad \ldots (4.3)$$

Then, $v^{(j)}(t)\,(t) < u^{(j)}(t) < w^{(j)}(t)$, $j = 0, 1, \ldots, n\text{-}1$ ... (4.4)

and $t \in [\, t_o, t_o + a)$.

Proof. We shall prove the right half of the inequality (4.4). Similar reasoning can be used for the left half.

Let w(t) be an n-over function and u(t) be a solution of (2.1).

Let, $m_j(t) = w^{(j)}(t) - u^{(j)}(t)$, $j = 0, 1, \ldots, n\text{-}1$.

From (4.1), we get

$$\int_{t_0}^{t} u^{(j+2)}(t)\,dt + u^{(j+1)}(t_0) \le g(t, u(t), \ldots, u^{(n-1)}(t)) \text{ for } j = 0, 1, \ldots, n\text{-}2$$

$\Rightarrow u^{(j+1)}(t_0) \le g(t_0, u(t_0) \ldots, u^{(n-1)}(t_0))$ for $j = 0, \ldots, n\text{-}2$.

For, $j = n-1$ we have,

$u^{(j+1)}(t_0) = u^{(n)}(t_0) = g(t_0, u(t_0) \ldots, u^{(n-1)}(t_0))$,

Since u(t) is a solution of (2.1).

$$\text{Thus, } m_{j+}^{(1)}(t_0) \ge w_+^{(j+1)}(t_0) - g(t_0, u(t_0) \ldots, u^{(n-1)}(t_0))$$

$$\ge w_+^{(j+1)}(t_0) - g(t_0, u_0^1, \ldots, u_0^n)$$

$$= w_+^{(j+1)}(t_0) - g(t_0, w(t_0), \ldots w^{(n-1)}(t_0))$$

> 0, (since w(t) is an n-over function)

for $j = 0, \ldots, n\text{-}1$

Thus, it follows that for any $j = 0, 1, \ldots, n-1$

$m_{j+}(t)$ is increasing to the right of t_0 in a sufficiently small interval $t_0 \le t \le t_0 + \varepsilon$ which implies that, $m_{j+}(t_0 + \varepsilon) - m_{j+}(t_0) > 0$

$\Rightarrow w_+^{(j)}(t_0 + \varepsilon) - u_+^{(j)}(t_0 + \varepsilon) - w_+^{(j)}(t_0) - u_+^{j}(t_0) > 0$

$\Rightarrow w_+^{(j)}(t_0 + \varepsilon) - u_+^{(j)}(t_0 + \varepsilon) > 0$, since $w_+^{(j)}(t_0) = u_+^{(j)}(t_0)$, by ... (4.3)

$\Rightarrow w^{(j)}(t_0 + \varepsilon) > u^{(j)}(t_0 + \varepsilon)$, for $j = 0, 1, \ldots, n-1$.

Further more, from (4.1)

$$u^{(j)}(t) = \int_{t_0}^{t} u^{(j+1)}(t)\,dt + u^{(j)}(t_0)$$

$\leq g(t, u(t), \ldots, u^{(n-1)}(t))$

and $w^{(j)}(t) > g(t, w(t), \ldots, w^{(n-1)}(t))$

for $j = 0, 1, \ldots n-1$.

Then by application of Theorem 1 (see statement after proof of theorem 1)

$u^{(j)}(t) < w^{(j)}(t)$, $j = 0, 1, \ldots, n-1$, for $t \varepsilon [t_0, t_0 + a)$.

Corollary 1. Let E be an open $(t, u, \ldots u^{(n-1)})$ – set in R^n, g_1, g_2 ε C [E, R] and $g_1(t, u, \ldots u^{(n-1)}) < g_2(t, u, \ldots u^{(n-1)})$, $(t, u, \ldots, u^{(n-1)}) \varepsilon E$.

Let $u_1(t)$ and $u_2(t)$ be two solutions of

$u^{(n)} = g_1(t, u, \ldots, u^{(n-1)})$

$u^{(n)} = g_2(t, u, \ldots, u^{(n-1)})$ respectively existing on $[t_0, t_0 + a)$

such that

$u_1^{(n-1)}(t_0) < u_2^{(n-1)}(t_0)$.

Then, $u_1^{(n-1)}(t) < u_2^{(n-1)}(t)$, $t \varepsilon [t_0, t_0 + a)$.

Proof. Let $m(t) = u_2^{(n-1)}(t) - u_1^{(n-1)}(t)$, $t\ [t_0, t_0 + a)$

$\Rightarrow m^1(t) = u_2^{(n)}(t) - u_1^{(n)}(t)$

$= g_2(t, u, \ldots, u^{(n-1)}), -g_1(t, u, \ldots u^{(n-1)}) > 0$.

Thus m (t) is strictly increasing function and it is given that

$m(t_0) = u_2^{(n-1)}(t_0) = u_1^{(n-1)}(t_0) > 0$.

From this we get,

$m(t) > 0$, $t \varepsilon [t_0, t_0 + a)$

$\Rightarrow u_2^{(n-1)}(t) > u_1^{(n-1)}(t)$, $t \varepsilon [t_0, t_0 + a)$.

REFERENCES

1. Lakshimikanthan, V. and Leela, S.—Differential and Integral inequalities VoL 1, Academic Press, 1969.

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A NEW TOPOLOGY ON FUNCTION SPACE

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1. INTRODUCTION

The object of the present paper is to introduce a type of set-set topology in Y^X, where X and Y are topological spaces. Our special interest would be on D (X, Y) which denote the class of all δ – continuous function from X to Y. The concept of δ – continuity was introduced by T. Noiri [6] and he showed that the concept is independent of continuity.

One of the original set-set topology is the compact—open topology C_0 which was introduced in 1945 by R. Fox [3] ; as expected from the name, for this topology, Ω is the collection of all open sets in Y & $\hat{C}$ is the collection of all compact sets in X and a topology in $F \subset Y^X$ is introduced via a sub-base $\mathcal{S}$ consisting of all sets $\{T(C,V) : C \in \hat{C}, V \in \Omega\}$ where $T(C,V) = \{f \in F : f(C) \subset V\}$.

The set-set topology to be discussed here shall utilize nearly – compact sets (NC sets in short) and open sets as the basic concepts.

§ 1.

Definition 1.1. [10]. A space (X, τ) is said to be nearly compact if for any open cover of X, there exists a finite subfamily, the interiors of the closures of which cover the space X.

Observation 1.2(a). A subset $A \subset (X, \tau)$ is nearly compact if it is nearly compact with respect to its subspace topology.

Definition 1.2(b). [2]. A subset $A \subset (X, \tau)$ is called an N-set in X if for any τ-open cover of A there exists a finite sub-collection, the interiors of the closures of which cover A ; interiors and closures are w.r.t. τ.

Definition 1.3. Let X & Y be topological spaces and let T(C,U) = $\{f \in Y^X : f(C) \subset U\}$; let N denote the NC set in X and let $\mathscr{R}$ denote the class of regular open sets in Y ; then $\{T(C,U) : C \in N, \& \ U \in \mathscr{R}\}$ is a sub-base for some topology on Y^X ; we call this topology the N-R topology on Y^X.

Observations 1.4(a). If Y is regular we can compare the compact-open topology C_0 and the N-R topology $N_{\mathscr{R}}$ in the perspective of C(X,Y) (the set of all continuous mappings from X to Y) in the following way : let $f \in T(C,V)$ where $T(C,V) \in C_0$ (having the usual meaning) ; then, f being continuous, $f(C) \subset V$, Y being regular and f(C) being compact, there exists a closed neighbourhood $\hat{V}$ of f(C) such that $f(C) \subset \hat{V} \subset V$; now $\text{int}_Y \hat{V}$ is regular open in Y and C is an NC set in X and so, $f \in (C, \text{int}_y \hat{V}) \subset T(C,V)$; thus $C_0 \subset N_{\mathscr{R}}$.

(b) In the same way as above it can be shown that if Y be regular and if P is the point-open topology on $Z \subset Y^X$, then $P \subset N_{\mathscr{R}}$.

Definition 1.5(a). [6]. A function f : X → Y where X & Y are topological spaces is said to be δ-continuous if, for each open neighbourhood V of f(x) in Y there is an open neighbourhood U of x in X such that f(int.cl.U) ⊂ int. (cl.V.)

Observations 1.5(b). It is obvious that if Y is regular then C(X,Y) = D(X,Y).

Observations 1.5(c). It follows from 1.4 & 1.6 that $N_{\mathscr{R}}$ and C_0 are comparable on D(X,Y) as well if Y is regular.

Definition 1.6. [9]. A space (X, τ) is said to be almost regular if any regularly closed set A and any singleton set $\{x\}$, disjoint from A can be strongly separated.

Lemma 1.7. [9] For a topological space (X, τ), the following conditions are equivalent :

(I) (X, τ) is almost regular.

(II) for each $x \in X$ and each regularly open set U containing x there exists a regular open set V such that $x \in V \subset \overline{V} \subset U$.

Lemma 1.8. [2]. A set is A an N-set in (X, τ) iff for any cover of A by regular open sets in (X, τ) there exists a finite subcover.

Theorem 1.9. Let $Z \subset Y^X$ be endowed with N-R topology $N_{\mathscr{R}}$; then Z is regular if the members of Z are δ-continuous and Y is almost regular.

Proof. Let the members of Z be δ-continuos ; let Y be almost regular. It must be shown that any $N_{\mathcal{R}}$-neighbourhood of each member f of Z contains a closed $N_{\mathcal{R}}$-neighbourhood ; however it suffices to show that each open neighbourhood of f which belongs to a subbase for $N_{\mathcal{R}}$ contains a closed $N_{\mathcal{R}}$-Neighbourhood of f.

Suppose f ∈ T(C,U) where C is an NC set in X and U is regularly open set in Y ; since f is δ-continuous, f(C) is an N-set in Y [6] and since Y is almost regular, for any point $y \in$ f(C) ∃ a regular open set $\hat{U}_y$, such that $y \in \hat{U}_y \subset \overline{\hat{U}_y} \subset U$ (lemma 1.7) ; thus $\{\hat{U}_y : y \in f(C)\}$ is a collection of regularly open sets in Y covering f (C) ; since f(C) is an N-set in Y, by theorem 1.8 there exists a finite subcollection $\hat{U}_{y_1}, \ldots\ldots \hat{U}_{y_n}$ covering f(C) ; then f(C) $\subset \bigcup_{i=1}^{n} \hat{U}_{y_i} \subset U$; hence ∃ a regualr closed neighbourhood L of f(C) such that f(C) $\subset L \subset U$, obviously fC) ⊂ int. L (= W(say)) $\subset L \subset U$, where int. L is a regular open set. Hence f ∈ $T(C,W) \subset T(C,L)$. Thus T(C,L) is a neighbourhood of f contained in T(C,U).

We demand that T(C,L) is a closed neighbourhood of f ; let g ∉ T(C,L) ; then g(C) ⊄ L i.e ∃ a g(x) (for some x ∈ C) such that g(x) ∉ L i.e g(x) ∈ Y \ L ; Y \ L is obviously regular open and as such g ∈ T ({x}, Y\L) and T ({x}, Y\L) ∈ $N_{\mathcal{R}}$ having null intersection with T(C,L) ; then T(C,L) is closed in (Z, $N_{\mathcal{R}}$).

Theorem 1.10. (Z, $N_{\mathcal{R}}$) is T_2 if Y is Urysohn and if Z ⊂ Y^{r}.

Proof. Let f ≠ g where f, g ∈ Z. Then f(x), g(x) ∈ Y and f(x) ≠ g(x) for some x ∈ X ; since Y is Urysohn there exists open sets U and V in Y such that f(x) ∈ U, g(x) ∈ V and $\overline{U} \cap \overline{V} = \phi$.

Obviously f ∈ T ({x}, int. $\overline{U}$) = W_1 and g ∈ T ({x}, int. $\overline{V}$) = W_2 ; int. $\overline{U}$ and int. $\overline{V}$ being regular open, W_1 and W_2 are both members of $N_{\mathcal{R}}$ and $W_1 \cap W_2 = \phi$ implying that (Z,$N_{\mathcal{R}}$) is T_2.

2. THE PROBLEM OF ADMISSIBILITY FOR $N_{\mathcal{R}}$ ON D(X,Y)

Arens [1] introduced the concept of admissibility for a topology T' on F ⊂ Y^X in the following manner :— A topology T′ on F ⊂ Y^X is called an admissible topology for F provided the evaluation mapping $P : (F,T') \times (X, \tau) \to (Y, \tau')$ defined by P(f,x) = f(x) is continuous. In the light of the above definition we shall introduce some other admissibility based on the following concepts :—

Definition 2.1. [11]. A function $f : (F,T') \to (Y, \sigma)$ is said to be almost continuous if for each x $\in$ X and each V $\in$ σ containing f(x), there exists U $\in$ τ containing x such that f(U) $\subset$ int. cl.(V)). We denote by A(X,Y) the set of all almost continuous functions from X to Y.

Definition 2.2. Let X,Y be spaces and Z $\subset$ Y^X. A topology for Z is said to be almost admissible (δ-admissible) iff the evaluation mapping $P : Z \times X \to Y$ defined by P(f,x) = f(x), where f $\in$ Z, x $\in$ X is almost continuous (δ-continuous).

Note 2.3. It is obvous that δ-continuity implies almost continuity ; thus δ-admissbility implies almost admissibility.

Definition 2.4. [2] A space (X, τ) is called locally nearly compact (l.n.c in short), if for each point x $\in$ X there exists an open neighbourhood U of x such that $cl_X\ (U)$ is an N-set in X.

Definition 2.5. [12]. A sub-set A of a space X is called an H-set iff every open cover of A admits a finite subcollection whose closures in X cover A.

Observation 2.6. Obviously every N-set in a topological space is a H-set.

Lemma 2.7. [13]. Let B be a regular closed subset of a T_2 space X. If A $\subset$ X is an H-set and B $\subset$ A then B is H-closed.

Observations 2.8. Obviously if B is a regular closed set as well as an H-set then it is an H-closed set in X.

Observation 2.9. It is easy to see that in an almost regular space an H-closed subset of a T_2 space X is a nearly compact subset of X.

Lemman 2.10. [2]. The following conditions are equivalent in a T_2 space X :

(1) X is locally nearly compact.

(2) For each N-set C in X and for each regular open neighbourhood U of x there exist an open set V such that $\overline{V}$ is an N-set and $C \subset V \subset \overline{V} \subset U$.

Observations 2.11. In 2.10 above $\overline{V}$ is an H-set as well since it is regular closed ; it is an H-closed set as well [2.9]. Further if X is almost regular also, then $\overline{V}$ is a nearly compact subspace.

Theorem 2.12. Let X be a l.n.c T_2 and almost regular space ; then $N_{\mathfrak{R}}$ is almost admissible for D(X,Y).

Proof. Let O be a regular open set in Y. Let $(f,x) \in P^{-1}(O)$; then $f(x) \in O$. Since f is δ-continuous there exists some regular open set U in X such that $x \in U$ and $f(U) \subset O$. Since X is l.n.c T_2, there exists an open neighbourhood U_1 of x such that $x \in U_1 \subset \overline{U}_1 \subset U$ with $\overline{U}_1$ as a nearly compact subset of X [2.11]. Hence $f \in T(\overline{U}_1, O)$, which is a neighbourhood of f in $(D(X,Y),N_{\mathcal{R}})$. Then $P(T(\overline{U}_1,O) \times U_1) \subset O$: infact, if $g \in T(\overline{U}_1,O)$, $g((\overline{U}_1) \subset O$ and thus $P(g,t) = g(t) \in O$ if $t \in U_1$. Thus $N_{\mathcal{R}}$ is almost admissible.

Observations 2.13. The result 2.12 can be further sharpened if we take Y to be almost regular. Infact, in that case by theorem 1.9 $(D(X,Y),N_{\mathcal{R}})$ is regular; so, for any $f \in T(\overline{U}_1,O)$, ∃ a closed neighbourhood L of f contained in $T(\overline{U}_1,O)$; but int. L is regular open and so is int. $L \times U_1$, since U_1 is regular open ; hence $P(\text{int. } L \times U_1) \subset O$ giving P to be δ-continuous.

Note 2.14. We now show by an example that theorem 2.12 need not hold if X is not a l.n.c. For working out the details of the example we need the follwing concepts :—

A point $x \in (X, \tau)$ is called θ—closure point of a subset $S \subset X$ if $S \cap \text{cl. } U = \phi$ for each open neighbourhood U of x. The set of all θ-closure point of S is called the θ-closure of S and is denoted by $cl_\theta(S)$. A subset S is said to be θ-closed if $cl_\theta(S) = S$. Then complement of a θ-closed set is called θ-open set ; the collection of all θ-open sets forms a topology for X and is denoted by τ_θ. Obviously $\tau_\theta \subset \tau$; further, if $D \subset X$ is dense in X then $(\tau_D)_\theta = (\tau_\theta)_D$ [4] ; further for an open set $U \subset X$, $cl_\theta U = \overline{U}$. If $A \subset B \subset X$, A is θ-closed in X and B is H-set in X, then A is H-set in X.

Theorem 2.15(a). [9]. Every dense subspace of an almost regular space is almost regular.

Theorem 2.15(b). [5]. For any subset $A \subset (X,\tau)$, $cl_\theta A$ is θ-closed if X is almost regular.

Theorem 2.15(c). [8]. In an almost regular space X if $A \subset X$ is an H-set, then it is an N-set.

Example 2.16. Let $A = \left\{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right\}$ and let τ_1 be the usual topology on R ; let $\tau = \{U \backslash \hat{A} : U \in \tau_1, \hat{A} \in P(A)\}$; then τ is a topology on R and $\tau \supset \tau_1$; for any $V = U \backslash \hat{A}$ obviously $cl_\tau (U \backslash \hat{A}) = cl_{\tau_1}(U) = \overline{U}$.

Obviously Q (the set of all rational numbers) is dense in (X, τ_1) and it is dense in (X, τ) as well. We demand that (Q, τ_Q) is not l.n.c ; if possible let $q \in Q$ have an

N-set neighbourhood L in Q ; and then L is an H-set and hence L is compact in $(Q,(\tau_Q)_\theta)$; but Q being dense in (X, τ), $(\tau_Q)_\theta = (\tau_\theta)_Q$; and thus L is compact in $(\tau_\theta)_Q$; but $\tau_\theta = \tau_1$ and then $(Q, (\tau_1)_Q)$ becomes locally compact which is a contradiction.

We consider the evaluation mapping $P : D(Q,I) \times Q \to I$. Let D(Q,I) be endowed with (N-R) topology $N_{\mathcal{R}}$. If possible let P be δ-continuous (or almost continuous) ; then P is continuous as well since I is regular ; obviously D(Q,I) = C(Q,I) = A(Q,I). Let $f_0 : Q \to I$ be the mapping such that $f_0(Q) = 0$ and let $q_0 \in Q$ be any point ; we prove P to be not continuous at (f_0, q_0) by showing that this point has no neighbourhood mapping into $W = \{t : t < 1\}$. Let U be any neighbourhood of q_0 in Q and let $V = \bigcap_{i=1}^{n} T(A_i, V_i)$ be any neighbourhood of f_0 in $(D(Q,I), N_{\mathcal{R}})$; since $cl_Q U$ is not an N-set it cannot be an H-set in (Q, τ_Q) as well since (Q, τ_Q) is almost regular ; now $cl_Q U$ is closed in (Q, τ_Q) ; but each A_i is an N-set and hence an H-set and thus $\bigcup_{1}^{n} A_i$ is an H-set in Q ; so $U \not\subset \bigcup_{1}^{n} A_i$; otherwise $cl_Q U$ would be an H-set in Q which is a contradiction ; so there is a $\tilde{q} \in U \setminus \bigcup_{1}^{n} A_i$. Now $(\tau_Q)_\theta = (\tau_\theta)_Q$ implies that $(Q, (\tau_Q)_\theta)$ is completely regular since $(Q, (\tau_\theta)_Q) = (Q,(\tau_1)_Q)$ is so.

Hence there is a continuous function $\tilde{f} : (Q,(\tau_\theta)_Q) \to I$ such that $\tilde{f}(q) = 1$ *and* $\tilde{f}(\bigcup_{1}^{n} A_i) = 0$ ($\bigcup_{1}^{n} A_i$ being an H-set is θ-closed in Q and hence is closed in $(Q(\tau_Q)_\theta)$ i.e $(Q,(\tau_\theta)_Q)$. We note that $o \in \bigcap_{i=1}^{n} V_i$; we conclude then $\tilde{f} \in V$, $\tilde{q} \in U$ and yet $P(\tilde{f},\tilde{q}) \notin W$; hence P is not continuous.

Example 2.17. Example of a δ-admissible topology on D(X,Y).

Let Z be endowed with discrete topology. Then the discrete topology is δ-admissible where Z = D(X,Y).

Proof. Let V be any regular open neighbourhood of f(x), $f \in Z$, $x \in X$. Since f is δ-continuous, there exists open neighbourhood U of x such that f (int. cl. U) $\subset$ int. cl. V = V. For such an open neighbourhood U of x, the subset $T = \{g \in Z : g(\text{int. } cl. U) \subset V\}$ is open and hence regular open since Z is endowed with discrete topology ; obviously $f \in T$.

Now T × U is an open neighbourhood of (f,x) in the product topology of Z × X and $P(\text{int}.Cl.(T \times U)) = P(\text{int. } cl.T \times \text{int. } cl.U) \subset V$: infact let $y \in$ P (int. cl. T × int. cl. U). Then $y = \text{P}(\phi, x_1)$ where $\phi \in$ int. cl. T = T as T is regular open and $x_1 \in$ int.cl. U. So $y = \phi(x_1) \in \phi\ (\text{int}.cl.U) \subset V$ *as* $\phi \in T$. Hence P is δ-continuous showing that the discrete topology is δ-admissible.

Note 2.11. Instead of D(X,Y), if we take A (X,Y), the class of almost continuous function from X to Y, then the discrete topology would be almost admissible.

§ 3.

Definition 3.1. A topology for a family $Z \subset Y^X$ of functions is δ-admissible (almost admissible) on a subset $A \subset X$ iff the mapping P restricted over $Z \times A$ is δ-continuous (almost continuous).

Definition 3.2. A topology for a family $Z \subset Y^X$ is δ-admissible (almost admissible) over near compacta iff it is δ-admissible (almost admissible) over nearly compact sets in X.

Theorem 3.3. If $Z \subset Y^X$ be endowed with δ-admissible (almost admissible) topology, then each member $f \in Z$ is δ-continuous (almost continuous).

Proof. Let τ be a δ-admissible topology on Z ; if U be an open neighbourhood of f(x) in Y then there exists an open neighbourhood T × V of (f,x) in Z × X such that P(int. cl. (T × V)) = P(int. cl. T × int. cl. V) $\subset$ int. cl. U. Now if $(f,x) \in T \times V \subset$ int. cl. (T × V) = int. cl. T × int. cl. V, we claim that f (int. cl. V) $\subset$ int. cl. U : infact if $y \in$ int. cl. U then $y = f(v)$ for some $v \in$ int. cl. V i.e $y = P(f,v) \in$ P(int.cl. T × int.cl.V) $\subset$ int. cl. U, i.e f is δ-continuous. The proof of the other part is similar.

Theorem 3.4. Let X, Y be spaces. Let $Z \subset Y^X$. Each topology for Z which is δ-admissible on near compacta is larger than the N-R topology N_R on Z.

Proof. Suppose that a topology σ for Z is δ-admissible on near compacta. Let U be a regular open subset of Y and let K be an NC set in X. We show that T(K,U) is σ-open. Let $f \in$ T(K,U)...(1). Now it is given that for $K \subset X$ (as in (1)) which is an NC set, $P : Z \times K \to Y$ such that P(f,x) = f(x) is δ-continuous ; let $(f,x) \in Z \times K$ where f is as in (1). By δ-continuity of P there exist open sets V_f^X & V_τ σ-open and open in K respectively such that $P(\text{int. cl. } V_f^X \times \text{int}_K\, cl_K\, V_x) \subset \text{int}_K\, \text{cl}_K$ U (where U is a neighbourhood of f(x) as in (1))...(2).

Now $\{V_x : x \in K\}$ is an open cover of K and K is an NC set in X ; hence there exists V_{x_i} : i = 1,2,...,n such that K $\subset \bigcup_{i=1}^{n} \text{int}_K\, cl_K\, V_{x_i}$; also f $\in \bigcap_{i=1}^{n} V_f^{x_i}$.

Now let g $\in \bigcap_{i=1}^{n} V_f^{x_i} \subset \bigcap_{i=1}^{n}$ int. $cl.$ $V_f^{x_i}$; for any x $\in$ K, x $\in \bigcup_{i=1}^{n}$ int. $cl.$ V_{x_i} as K $\subset \bigcup_{i=1}^{n}$ int. $cl.$ V_{x_i} ; hence for any x $\in$ K, x $\in \text{int}_K\, cl_K\, V_{x_{i_0}}$ for some i_0, $1 \le i_0 \le n$. Since g $\in$ int. $cl.$ $V_f^{x_{i_0}}$ and x $\in \text{int}_K\, cl_K\, V_{x_{i_0}}$; from (2) P(g,x) = g(x) $\in$ int. cl. U. Thus g(K) $\subset$ U and so g $\in$ T(K,U) for each g $\in \bigcap_{i=1}^{n} V_f^{x_i}$. For each f $\in$ T(K,U) then there exist a σ-open V = $\bigcap_{i=1}^{n} V_f^{x_i}$ containing f such that f $\in$ V $\subset$ T(K,U) on Z ; thus T(K,U) is σ-open and so σ is larger than $N_{\mathfrak{K}}$ on Z.

Theorem 3.5. Let X be a almost regular T_2 space, Y be a topological space and let Z $\subset Y^x$.

If each member of Z is δ-continuous on each N.C set in X, then $N_{\mathfrak{K}}$ is almost admissible on near compacta.

Proof. Let K be a nearly compact subset of X, f $\in$ Z & x $\in$ K. Let U be any open neighnourhood of f(x) in Y and let P(f,x) = f(x) We have to show that P is almost continuous.

Since f is δ-continuous on K there exists an open neighbourhood of x in K say V such that f($\text{int}_k\, cl_k V$) $\subset$ int. $cl.U$; now K being nearly compact T_2 is almost regular and so there exists a regular open neighbourhood M of x in K such that x $\in$ M $\subset$ $cl_K\, M \subset \text{int}_K\, cl_K\, V$.

Thus f($cl_K M$) $\subset$ U and $cl_K\, M$ is a regular closed set in K.

K is obviously H-closed T_2 and so, so is $cl_K\, M$ since $cl_k\, M$ is a regular closed in K. But K is almost regular as well and so $cl_K\, M$ is nearly compact. But if $cl_K\, M$ = L and int. cl. U = W then T(L, W) $\in N_{\mathfrak{K}}$ and is a neighbourhood of f.

Obviously P (T(L, W) × M) $\subset$ W giving the almost continuity of P on Z × K. Thus $N_{\mathfrak{K}}$ is almost admissible on near compacta.

Note. The result can be further sharpened if we take Y to be almost regular. Infact in that case (Z, $N_{\mathcal{R}}$) is regular and there exists a regular open set S in Z such that $f \in S \subset \bar{S} \subset T(L, W)$. Obviously $S \times M$ is a neighbourhood of (f,x) in $Z \times K$ and P (int. cl. $(S \times M)) = P$ (int. cl. $S \times \mathrm{int}_K\, \mathrm{cl}_K\, M) = P(S \times M) \subset W$. Thus P is δ-continuous on $Z \times K$.

Conclusion. The study of the $N_{\mathcal{R}}$ topology on D(X, Y) is to be continued and many of its implications are to be exposed in future.

REFERENCCS

1. R. Arens, Topology for homeomorphism groups, Amer. J. Math. 68 (1946), 593-610.

2. D. Charnahan, Locally nearly compact spaces, Boll. Un. Mat. Ital. (4), 6 (1972), 146-153.

3. R. Fox, On topologies for function spaces, Bull. Amer. Math. Soc. 51 (1945), 429-432.

4. S. Ganguly and S. sinharoy, More on θ-metrisable spaces (submitted).

5. T. Noiri, A charecterisation of almost regular spaces, Glasnik Matematicki. 13 (33) (1978), 335-338.

6. T. Noiri, On δ-continuous functions, J. Korean Math. Soc. 16 (1980), 161-166.

7. J. R. Porter & J. D. Thomas, On H-closed & minimal Hausdorff spaces, Trans. Amer. Math. soc. 138 (1969), 159-170.

8. T. G. Raghaban, θ-closed subsets of topological spaces. Journal of pure & Applied Mathematics. 15 (3) (1984), 233-240.

9. M. K. Singal & S. P. Arya, On almost regular spaces, Glasnik matematicki 4 (24) (1969), 89-99.

10. M. K. Sigal & A. Mathur, On nearly compact spaces, Boll. Un. Mat. Ital. 6 (4) (1972), 146-153.

11. M. K. Sigal and A. R. Sigal, Almost continuous mappings, Yokohama. Math. J. 16 (1968), 63-73.

12. N. V. Velicko, H-closed topological spaces, Amer. Math. Soc. Transl. 78 (2) (1968), 103-118.

13. J. Vermeer, Closed subspaces of H-closed spaces, Pacific. J. Maths. 118 (1) (1985), 229-247.

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REFERENCES

1. N. L. Alling, Foundations of Analysis on Surreal Number Fields, North-Holland Publishing Co., 1987.
2. E. Hewitt, Rings of continuous functions I, Trans. Amer. Math. Soc. 64(1948), 54-99.

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STRONGLY α-PREIRRESOLUTE FUNCTIONS AND STRONGLY β-PREIRRESOLUTE FUNCTIONS

YUSUF BECEREN AND TAKASHI NOIRI

ABSTRACT : The purpose of the present paper is to introduce two new classes of functions called strongly α-preirresolute functions in topological spaces. Some properties and several characterizations of these classes of functions are obtained. We investigate the relationships between these types of functions and other classes of non-continuous functions.

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Key words and phrases : Strongly α-preirresoluteness and strongly β-preirresoluteness.

1. INTRODUCTION

It is well-known that four classes of functions called strongly β-irresoluteness [15], β-irresoluteness [11], strongly M-precontinuity [4] and preirresoluteness [18]. Recently, Beceren [7] introduced notions of α-preirresoluteness and β-preirresoluteness.

The purpose of this paper is to introduce and investigate concepts of new classes of functions, namely strongly α-preirresolute functions and strongly β-preirresolute functions, and give several characterizations and their properties. Relations between these types of functions and other classes of functions are obtained. The new class of strongly α-preirresolute functions, which is stronger than α-preirresolute functions [7], is a generalization of strongly β-irresolute functions [15]. The new class of strongly β-irresolute functions, which is stronger than β-irresolute functions [11], is a generalization of strongly α-preirresolute functions.

2. PRELIMINARIES

Throughout this note, spaces always mean topological spaces and f : X→Y denotes a function of a space X into a space Y. Let S be a subset of a space X. The closure and the interior of S are denoted by Cl(S) and Int(S), respectively.

Definition 2.1. A subset S of a space X is said to be *α-open* [17] (resp. *preopen* [12], *β-open* [1]) if $S \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(S)))$ (resp. $S \subset \mathrm{Int}(\mathrm{Cl}(S))$, $S \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(s)))$).

The family of all α-open (resp. preopen, β-open) sets in a space X is denoted by τ^α(resp. PO(X), βO(X)). It is shown in [17] that τ^α is a topology for X. Moreover, $\tau \subset \tau^\alpha \subset PO(X)$

$\subset \beta O(X)$. The complement of an α-open (resp. preopen, β-open) set is said to be *α-closed* [14] (resp. *preclosed* [12], *β-closed* [1]). The intersection of all β-closed sets containing a subset S is called the *β-closure* [3] of S and is denoted by $\beta Cl(S)$; the union of all β-open sets contained in S is called the *β-interior* [3] of S and is denoted by $\beta Int(S)$.

Definition 2.2. A function $f : (X, \tau) \to (Y, \upsilon)$ is said to be *strongly M-precontinuous* [4] (resp.*α-preirresolute* [7], *preirresolute* [18], β-*preirresolute* [7]) if $f^{-1}(V)$ is open (resp. α-open, preopen, β-open) in X for every preopen set V of Y.

Definition 2.3. A function $f : (X, \tau) \to (Y, \upsilon)$ is said to be strongly *β*-irresolute [15] (resp. *β-irresolute* [11]) if $f^{-1}(V)$ is open (resp. β-open) in X for every β-open set V of Y.

Definition 2.4. A function $f : (X, \tau) \to (Y, \upsilon)$ is said to be *strongly α-preirresolute* (resp. *strongly β-preirresolute*) if $f^{-1}(V)$ is *α*-open (resp. preopen) in X for every *β*-open set V of Y.

From the definitions, we obtain the following diagram :

$$\begin{tikzcd}
\textit{Strongly } \beta\textit{-irresoluteness} \arrow[r] \arrow[d] & \textit{strongly } \alpha\textit{-preirresoluteness} \arrow[r] \arrow[d] & \textit{strongly } \beta\textit{-preirresoluteness} \arrow[r] \arrow[d] & \beta\textit{-irresoluteness} \arrow[d] \\
\textit{Strongly M-precontinuity} \arrow[r] & \alpha\textit{-preirresoluteness} \arrow[r] & \textit{preirresoluteness} \arrow[r] & \beta\textit{-preirresoluteness}
\end{tikzcd}$$

The following Remark enables us to realize that none of the above implications is reversible.

Remark. We have the following relationships :

(a) Strongly M-precontinuity does not imply β-irresoluteness ([5, Example 3.2]),

(b) β-irresoluteness does not imply preirresoluteness ([6, Example 2.2]),

(c) Strongly β-preirresoluteness does not imply α-preirresoluteness ([6, Example 2.1]),

(d) Strongly α-preirresoluteness dose not imply strongly M-precontinuity ([7, Example 2.1]).

3. STRONGLY α-PREIRRESOLUTE FUNCTIONS

Theorem 3.1 : *For a function $f : (X, \tau) \to (Y, \upsilon)$, the following are equivalent :*

(1) *f is strongly α-preirresolute ;*

(2) *$f : (X, \tau^{\alpha}) \to (Y, \upsilon)$ is strongly β-irresolute* ;

(3) *For each $x \in X$ and each β-open set V of Y containing $f(x)$, there exists an α-open set U of X containing x such that $f(U) \subset V$* ;

(4) *$f^{-1}(V) \subset Int(Cl(Int(f^{-1}(V))))$ for every β-open set V of Y* ;

(5) *$f^{-1}(F)$ is α-closed in X for every β-closed set F of Y;*

(6) *$Cl(Int(Cl(f^{-1}(B)))) \subset f^{-1}(\beta Cl)$ for every subset B of Y;*

(7) *$f(Int(Cl(Int(A)))) \subset \beta Cl(f(A))$ for every subset A of X.*

Proof. (1) $\Rightarrow$ (2). Let $x \in X$ and let V be any β-open set of Y containing f(x). By Definition 2.4, $f^{-1}(V)$ is α-open in X and contains x. Hence $f : (X, \tau^{\alpha}) \rightarrow (Y, \upsilon)$ is strongly β-irresolute.

(2) $\Rightarrow$ (3). Let $x \in X$ and let V be any β-open set of Y containing f(x). $U = f^{-1}(V)$, then by (2), U is an α-open subset of X containing x and $f(U) \subset V$.

(3) $\Rightarrow$ (4). Let V be any β-open set of Y and $x \in f^{-1}(V)$. By (3), there exists an α-open set U of X containing x such that $f(U) \subset V$. Thus, we have $x \in U \subset Intf^{-1}(Cl(Int(U))) \subset Int(Cl(Int(f^{-1}(V))))$ and hence $f^{-1}(V) \subset Int(Cl(Int(f^{-1}(V))))$.

(4) $\Rightarrow$ (5). Let F be any β-closed subset of Y. Set $V = Y-F$, then V is β-open in Y. By (4), we obtain $f^{-1}(V) \subset Int(Cl(Int(f^{-1}(V))))$ and hence $f^{-1}(F) = X-f^{-1}(Y-F) = X-f^{-1}(V)$ is α-closed in X.

(5) $\Rightarrow$ (6). Let B be any subset of Y. Since $\beta Cl(B)$ is a β-closed subset of Y, then $f^{-1}(\beta Cl(B))$ is α-closed in X and hence $Cl(Int(Cl(f^{-1}(\beta Cl(B))))) \subset f^{-1}(\beta cl(B))$. Therefore, we obtain $Cl(Int(Cl(f^{-1}(B)))) \subset f^{-1}(\beta Cl(B))$.

(6) $\Rightarrow$ (7). Let A be any subset of X. By (6), we have $Cl(Int(Cl(A))) \subset Cl(Int(Cl(f^{-1}(f(A))))) \subset f^{-1}(\beta Cl(f(A)))$ and hence $f(Cl(Int(Cl(A)))) \subset \beta Cl(f(A))$.

(7) $\Rightarrow$ (1). Let V be any β-open subset of Y. Since $f^{-1}(Y-V) = X-f^{-1}(V)$ is a subset of X and by (7), we obtain $f(Cl(Int(Cl(f^{-1}(Y-V))))) \subset \beta Cl(f(f^{-1}(Y-V))) \subset \beta Cl(Y-V) = Y-\beta Int(V) = Y-V$ and hence $X-Int(Cl(Int(f^{-1}(V)))) = Cl(Int(Cl(X-f^{-1}(V)))) = Cl(Int(Cl(f^{-1}(Y-V)))) \subset f^{-1}(f(Cl(Int(Cl(f^{-1}(Y-V)))))) \subset f^{-1}(Y-V) = X-f^{-1}(V)$. Therefore, we have $f^{-1}(V) \subset Int(Cl(Int(f^{-1}(V))))$ and hence $f^{-1}(V)$ is α-open in X. Thus, f is strongly α-preirresolute.

Lemma 3.1. *(Chae et al. [8], El-Deeb et al. [9] and Abd El-Monsef et al. [2]). Let $\{X_{\lambda} : \lambda \in \Lambda\}$ be a family of spaces and U_{λ_i} be a nonempty subset of X_{λ_i} for each i=1, 2, ……, n. Then $U = \Pi_{\lambda \neq \lambda_i} X_{\lambda} \times \prod_{i=1}^{n} U_{\lambda_i}$ is a nonempty α-open [8] (resp. preopen [9], β-open [2]) subset of ΠX_{λ} if and only if U_{λ_i} is α-open (resp. preopen, β-open) in X_{λ_i} for each i = 1, 2, ……, n.*

Theorem 3.2. *A function $f : X \rightarrow Y$ is strongly α-preirresolute if the graph function $g : X \rightarrow X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is strongly α-preirresolute.*

Proof. Let $x \in X$ and V be any β-open set of Y containing f(x). Then, by Lemma 3.1, $X \times V$ is a β-open set of $X \times Y$ containing g(x). Since g is strongly α-preirresolute, there exists an α-open set U of X containing x such that $g(U) \subset X \times V$ and hence $f(U) \subset V$. Thus, f is strongly α-preirresolute.

Theorem 3.3. *If a function* $f : X \to \prod Y_\lambda$ *is strongly* α*-preirresolute, then* $P_\lambda of : X \to Y_\lambda$ *is strongly* α*-preirresolute for each,* $\lambda \in \Lambda$, *where* P_λ *is the projection of* $\prod Y_\lambda$ *onto* Y_λ.

Proof. Let V_λ be any β-open set of Y_λ. Since P_λ is continuous and open, it is β-irresolute [1, Theorem 2.2], and hence $P_\lambda^{-1}(V_\lambda)$ is β-open in $\prod Y_\lambda$. Since f is strongly α-preirresolute, then $f^{-1}(P_\lambda^{-1}(V_\lambda)) = (P_\lambda of)^{-1}(V_\lambda)$ is α-open in X. Hence $P_\lambda of$ is strongly α-preirresolute for each $\lambda \in \Lambda$.

Theorem 3.4. *If the product function* $f : \prod X_\lambda \to \prod Y_\lambda$ *is strongly* α*-preirresolute, then* $f_\lambda : X_\lambda \to Y_\lambda$ *is strongly* α*-preirresolute for each* $\lambda \in \Lambda$.

Proof. Let $\lambda_0 \in \Lambda$ be an arbitrary fixed index and V_{λ_0} be any β-open set of Y_{λ_0}. Then, $\prod Y_\gamma \times V_{\lambda_0}$ is β-open in $\prod Y_\lambda$ by Lemma 3.1, where $\lambda_0 \neq \gamma \in \Lambda$. Since f is strongly α-preirresolute, then $f^{-1}(\prod Y_\gamma \times V_{\lambda_0}) = \prod X_\gamma \times f_{\lambda_0}^{-1}(V_{\lambda_0})$ is α-open in $\prod X_\lambda$ and hence, by Lemma 3.1, $f_{\lambda_0}^{-1}(V_{\lambda_0})$ is α-open in X_{λ_0}. This implies that f_{λ_0} is strongly α-preirresolute.

Theorem 3.5. *If* $f : (X, \tau) \to (Y, \upsilon)$ *is strongly* α*-preirresolute and A is a preopen subset of X, then the restriction f/A : A* $\to$ *Y is strongly* α*-preirresolute.*

Proof. Let V be any β-open set of Y. Since f is strongly α-preirresolute, then $f^{-1}(V)$ is α-open in X. Since A is preopen in X, $(f/A)^{-1}(V) = A \cap f^{-1}(V)$ is α-open in A [14, Lemma 1.1]. Hence f/A is strongly α-preirresolute.

Theorem 3.6. *Let* $f : (x, \tau) \to (Y, \upsilon)$ *be a function and* $\{ A_\lambda : \lambda \in \Lambda \}$ *be a cover of X by* α*-open sets of* (X, τ). *Then f is strongly* α*-preirresolute if* $f/A_\lambda : A_\lambda \to Y$ *is strongly* α*-preirresolute for each* $\lambda \in \Lambda$.

Proof : Let V be any β-open set of Y. Since f/A_λ is strongly α-preirresolute, $(f/A_\lambda)^{-1}(V)$ is α-open in A_λ. Since A_λ is α-open in X, then $(f/A_\lambda)^{-1}(V)$ is α-open in X for each $\lambda \in \Lambda$ [14, Lemma 1.2]. Therefore, $f^{-1}(V) = X \cap f^{-1}(V) = \cup \{A_\lambda \cap f^{-1}(V) : \lambda \in \Lambda\} = \cup \{(f/A_\lambda)^{-1}(V) : \lambda \in \Lambda\}$ is α-open in X because the union of α-open sets is an α-open set. Hence f is strongly α-preirresolute.

Theorem 3.7. *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be functions. Then the composition* $gof : X \to Z$ *is strongly* α*-preirresolute if f is strongly* α*-preirresolute and g is* β*-irresolute.*

Proof. Let W be any β-open subset of Z. Since g is β-irresolute, $g^{-1}(W)$ is β-open iin Y. Since f is strongly α-preirresolute, then $(gof)^{-1}(W) = f^{-1}(g^{-1}(W))$ is α-open in X and hence gof is strongly α-preirresolute.

4. STRONGLY β-PREIRRESOLUTE FUNCTIONS

The proofs of the following four theorems are similar to those in Section 3 and are thus omitted.

Theorem 4.1. *For a function* $f : (X, \tau) \to (Y, \upsilon)$, *the following are equivalent :*

(1) f *is strongly* β*-preirresolute ;*

(2) *For each* $x \in X$ *and each* β*-open set* V *of* Y *containing* $f(x)$, *there exists a preopen set* U *of* X *containing* x *such that* $f(U) \subset V$;

(3) $f^{-1}(V) \subset Int(Cl(f^{-1}(V)))$ *for every* β*-open set* V *of* Y;

(4) $f^{-1}(F)$ *is preclosed in* X *for every* β*-closed set* F *of* Y;

(5) $Cl(Int(f^{-1}(B))) \subset f^{-1}(\beta Cl(B))$ *for every subset* B *of* Y;

(6) $f(Cl(Int(A))) \subset \beta Cl(f(A))$ *for every subset* A *of* X.

Theorem 4.2. *A function* $f : X \to Y$ *is strongly* β*-preirresolute if the graph function* $g : X \to X \times Y$, *defined by* $g(x) = (x, f(x))$ *for each* $x \in X$, *is strongly* β*-preirresolute.*

Theorem 4.3. *If a function* $f : X \to \prod Y_{\lambda}$ *is strongly* β*-preirresolute, then* $P_{\lambda} o f : X \to Y_{\lambda}$ *is strongly* β*-preirresolute for each* $\lambda \in \Lambda$, *where* P_{λ} *is the projection of* $\prod Y_{\lambda}$ *onto* Y_{λ}.

Theorem 4.4. *If the product function* $f : \prod X_{\lambda} \to \prod Y_{\lambda}$ *is strongly* β*-preirresolute, then* $f_{\lambda} : X_{\lambda} \to Y_{\lambda}$ *is strongly* β*-preirresolute for each* $\lambda \in \Lambda$.

Theorem 4.5. *If* $f : (X, \tau) \to (Y, \nu)$ *is strongly* β*-preirresolute and* A *is a semi open subset of* X, *then the restriction* $f/A : A \to Y$ *is strongly* β*-preirresolute.*

Proof. Let V be any β-open set of Y. Since f is strongly β-preirresolute, then $f^{-1}(V)$ is preopen in X. Since A is semi open in X, $(f/A)^{-1}(V) = A \cap f^{-1}(V)$ is preopen in A [13, Lemma 2.1]. Hence f/A is strongly β-preirresolute.

Theorem 4.6. *Let* $f : (X, \tau) \to (Y, \nu)$ *be a function and* $\{A_{\lambda} : \lambda \in \Lambda\}$ *be a cover of* X *by preopen sets of* (X, τ). *Then* f *is strongly* β*-preirresolute if* $f/A_{\lambda} : A_{\lambda} \to Y$ *is strongly* β*-preirresolute for each* $\lambda \in \Lambda$.

Proof. Let V be any β-open set of Y. Since f/A_{λ} is strongly β-preirresolute, $(f/A_{\lambda})^{-1}(V)$ is preopen in A_{λ}. Since A_{λ} is preopen in X, then $(f/A_{\lambda})^{-1}(V)$ is preopen in X for each $\lambda \in \Lambda$ [13, Lemma 2.2]. Therefore, $f^{-1}(V) = X \cap f^{-1}(V) = \cup \{A_{\lambda} \cap f^{-1}(V) : \lambda \in \Lambda\} = \cup\{(f/A_{\lambda})^{-1}(V) : \lambda \in \Lambda\}$ is preopen in X because the union of preopen sets is a preopen set. Hence f is strongly β-preirresolute.

Theorem 4.7. *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be functions. Then the composition* $gof : X \to Z$ *is strongly* β*-preirresolute if* f *is strongly* β*-preirresolute and* g *is* β*-irresolute.*

Proof. The proof is similar to that of Theorem 3.7 and is thus omitted.

We recall that a space X is said to be *submaximal* if every dense subset of X is open in X and *extremally disconnected* if the closure of each open subset of X is open in X.

Theorem 4.8. Let (X, τ) be a submaximal and extremally disconnected space. Then the following are equivalent for a function $f : (X, \tau) \to (Y, \nu)$:

(1) *f is strongly β-irresolute ;*

(2) *f is strongly α-preirresolute ;*

(3) *f is strongly β-preirresolute ;*

(4) *f is β-irresolute ;*

Proof. This follows from the fact that if (X, τ) is a submaximal and extremally disconnected space, then $\tau = \tau^\alpha = PO(X) = \beta O(X)$ (Janković [10] and Nasef et al. [16]).

REFERENCES

1. M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, β-open sets and β-continuous mappings, *Bull. Fac. Sci. Assiut Univ.* A, 12 (1983), no. 1, 77-90.

2. M. E. Abd El-Monsef, A. N. Geaisa and R. A. Mahmoud, β-regular spaces, *Proc. Math. Phys. Soc. Egypt*, 60 (1985), 47-52.

3. M. E. Abd El-Monsef, R. A. Mahmoud and E. R. Lashin, β-closure and β-interior, *J. Fac. Ed. Ain Shams Univ.*, 10 (1986), 235-245.

4. M. E. Abd El-Monsef, R. A. Mahmoud and A. A. Nasef, A class of functions stronger than M-precontinuity, preirresolute and A-functions, *Qatar Univ. Sci. Bull.*, 10(1990), 41-48.

5. Y. Beceren, Almost α-irresolute functions, *Bull. Calcutta Math. Soc.,* 92 (2000), no 3, 213-218.

6. Y. Beceren and T. Noiri, On α-precontinuous functions, *Far East J. Math. Sci. (FJMS)* 2000, Special Volume, Part III, 295-303.

7. Y. Beceren, α-preirresolute functions and β-preirresolute functions, (submitted).

8. G. I. Chae, T. Noiri and D. W. Lee, On na-continuous functions, *Kyungpook Math* J., 26(1986), no. 1, 73-79.

9. N. El-Deeb, I. A. Hasanein, A. S. Mashhour and T. Noiri, On p-regular spaces, *Bul Math. Soc. Sci. Math. R. S. Roumanie* (N. S.), 27 (75) (1983), no. 4, 311-315.

10. D. S. Janković, On locally irreducible spaces, *Ann. Soc. Sci. Bruxelles Sér. I,* 97 (1983) no. 2, 59-72.

11. R. A. Mahmoud and M. E. Abd El-Monsef, β-irresolute and β-topological invariant, *Proc. Pakistan Acad. Sci.*, 27 (1990), no. 3, 285-296.
12. A. S. Mashhour, M. E. Abd El-Monsef and S. N. El–Deeb, On precontinuous and weak precontinuous mapping, *Proc. Math. Phys. Soc. Egypt,* 53 (1982), 47-53.
13. A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, A note on Semi-continuity and precontinuity, *Indian J. Pure Appl. Math.,* 13 (1982), no. 10, 1119-1123.
14. A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, α-continuous and α-open mappings, *Acta Math. Hungar.*, 41 (1983), no. 3-4, 213-218.
15. A. A. Nasef and T. Noiri, Strongly β-irresolute functions, *J. Natur. Sci. Math.*, 36 (1996), no. 2, 199-206.
16. A. A. Nasef and T. Noiri, Strong forms of faint continuity, *Mem. Fac. Sci. Kochi Univ. Ser. A Math.*, 19(1998), 21-28.
17. O. Njåstad, On some classes of nearly open sets, *Pacific J. Math.*, 15 (1965), 961-970.
18. I. L. Reilly and M. K. Vamanamurhy, On α-continuity in topological spaces, *Acta Math. Hungar.*, 45 (1985), no. 1-2, 27-32.

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E-LMC COMPACTIFICATION OF A SEMIGROUP

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ABSTRACT : The two different concepts, the concept of E-compactness for a topological space E and the concept Semigroup compactification are combined to introduce the notion E-LMC Compactification of a topologized semigroup X. We show that any Hausdorff semitopological semigroup X (i.e. one which is both left and right topological) has an E-compactification (e, $\delta_E X$) for a topological space E maximal with respect to the property that it is right topological and the requirement that $\lambda_{c(x)}$ be continuous for each $x \in X$.

1. INTRODUCTION

The notions of E-completely regular space and E-compact space for a topological space E originated in the 1958 paper by Engelking and Mrówka. The definitions generalize the characterization of a Tychonoff space as one that is homeomorphic to a subspace and a closed subspace respectively of a product of unit intervals.

It was shown in [1] that any Hausdorff semi-topological semigroup (i.e. one which is both left and right topological) has a compactification (e, δX) maximal with respect to the property that it is right topological and the requirement that $\lambda_{c(x)}$ be continuous for each $x \in X$.

The above two ideas are properly combined to prove the existence of the E-Compactification (e, $\delta_E X$) of a semitopological semigroup X and a properly chosen topological space E, maximal with respect to the above mentioned properties in Section 3.

2. PRELIMINARY CONCEPTS

2.1 Definition [3]. (1) A space X is E-completely regular for a topological space E, if X is homeomorphic to a subspace of a product of copies of E and X is called E-compact if X is homeomorphic to a closed subspace of product of copies of E.

(2) A subset U of X is called E-open if it is of the form $f^{-1}(V)$, where V is an open subset of some finite power E^n and $f \in C(X, E^n)$. A subset F of X is E-closed if and only if its complement is E-open.

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2.2 Theorem [3]. Let X and E be spaces. The following are equivalent.

(1) X is E-completely regular.

(2) For each closed subset A of X and each $p \in X \backslash A$, there is a positive integer n and $f \in C(X, E^n)$ such that $f(p) \notin cl_{E^n} f(A)$, i.e., $C_E(X) = \cup\{C(X, E^n) : n \in N\}$ separates points and closed sets of X.

(3) E-open subsets of X form a base for the open subsets of X. In general we cannot replace $\cup\{C(X, E^n) : n \in N\}$ by (X, E).

We have the theory of compact right topological semigroups. It is known that any Hausdorff Semitoplogical Semigroup S (i.e., one which is both left and right topological) has a compactification (e, X) maximal with respect to the property that it is right topological and the requirement that $\lambda_{c(s)}$ be continuous for each $s \in S$. (Here $\lambda_x(Y) = xy$).

Combining the above two concepts we show that any Hausdorff semitopological semigroup has an E-compactification $(e, \delta_E X)$ maximal with respect to the property that it is right topological and the requirement that $\lambda_{c(x)}$ be continuous for each $x \in X$.

We can study similar results with respect to the strong almost periodic and weak almost periodic E-compactifications of a topological semigroup S, for properly chosen space E.

3. THE E-LMC COMPACTIFICATION

Let X represent a Hausdorff semitopological semigroup. Let K represent the disjoint union set $\amalg E^n = \bigcup_{n \in N} (E^n \times \{n\})$.

Definition. If X is a topological semigroup, then an E-Bohr compactification of X is a pair (β_E, B) such that B is an E-compact semigroup. $\beta_E : X \rightarrow B$ is a continuous homomorphism and if $g : X \rightarrow T$ is a continuous homomorphism of X into an E-compact semigroup T, then there exists a unique continuous homomorphism $f : B \rightarrow T$ such that $f \cdot \beta_E = g$.

3.1 Result. Let X be a topological semigroup. Then there exists a collection $\{(f_\alpha, X_\alpha) : \alpha \in A\}$ such that X_α is an E-compact semigroup and $f_\alpha : X \rightarrow X_\alpha$ is a dense continuous homomorphism for each $\alpha \in A$ and if $g : X \rightarrow T$ is a dense continuous homomorphism of X into an E-compact semigroup T, then there exists $\delta \in A$ and a topological isomorphism $f : X \rightarrow T$ such that $f \cdot f_\delta = g$.

Proof. Let $\wedge = \{(K, m, \sigma) ; k \subseteq 2^{2^x}\}$, where K is an E-compact space, m is a continuous associative multiplication on K and $\sigma : X \rightarrow K$ is dense homomorphism. Here $\wedge \neq \phi$, since each singleton gives a member of $\wedge$.

Suppose that $g : X \to T$ is a dense continuous homomorphism of X into an E-compact semigroup T. [Suppose that T is a topological semigroup and $f : T \to K$ is a bijection of T onto K. We can give K the quotient topology τ induced by f. We can give K the multiplication m defined to be the composition $K \times K \xrightarrow{f^{-1}\times f^{-1}} T \times T \xrightarrow{p} T \xrightarrow{f} K$, where p is the multiplication on T. With the topology τ and multiplication m, K becomes a topological semigroup and $f : T \to K$ is a topological isomorphism. We call m the multiplication on K induced by f]. Since g(X) is a dense subset of the Hausdorff space T, Card $T \leq 2^{2^{g(X)}}$ (The function $f : T \to 2^{2^{g(X)}}$ defined by $f(x) : \{U : U = V \cap g(X)$ for some neighbourhood V of x in T$\}$ is an injection).

So there exists $K \subseteq 2^{2^{g(X)}}$ and a bijection $k : T \to K$. Let τ be the quotient topology on K induced by k. Define $\sigma : X \to K$ by $\sigma = k \cdot g$. Then $(K, m, \sigma) \in \wedge$ and $k^{-1} : K \to T$ is a topological isomorphism such that $k^{-1} \cdot \sigma = g$.

3.2. Result. If X is an E-completely regular semigroup, then there exists an E-Bohr compactification (β_E, B) of X.

Proof. Let $\mathcal{E} = \{(f_\alpha, X_\alpha) : \alpha \in A\}$ be the collection as above, $X_\alpha = E$ for all α. Define $\alpha : X \to \prod_{\alpha \in A} X_\alpha$ so that $\sigma(x)(\alpha) = \phi_\alpha(x)$ for each $\alpha \in A$ and $x \in S$. Then σ is a continuous homomorphism. Let $B = \overline{\sigma(X)}$ and define $\beta_E : X \to B$ so that $\beta_E(x) = \sigma(x)$ for each $x \in X$. Then (β_E, B) is E-Bohr compactification. Suppose $g : X \to T$ is a continuous homomorphism of X into an E-compact semigroup T. We can assume that g is dense. Then by the above result there exists $\delta \in A$ and a topological isomorphism $h : X \to T$ such that $h.\sigma = g$. Define $f : B \to T$ by $f = h \cdot \pi_\delta$. Then $f \cdot \beta_E = g$ and f is a continuous homomorphism. The uniqueness of f is a consequence of the fact that $\beta_E : X \to B$ is dense.

Remark. Let (β_E, B) be an E-Bohr compactification of X. Then β_E is a homeomorphism of X into B if and only if for every closed subset A of X an each $x \in X \backslash A$, there exists $f \in \mathcal{E}$ such that $f(A) = 0$ and $f(x) \neq 0$.

3.3. Definition. Let $C_E(X)^*$ denote the topological dual, i.e. all continuous functions from $C_E(X)$ to E^n, $n \in N$ and let $\sigma(C_E(X), C_E(X)^*)$ denote the weak topology induced on $C_E(X)$ by $C_E(X)^*$.

3.4. Definition. The function $f \in C_E(X)$ is defined to be in E-LMC if,

(1) $\{f \lambda_x : x \in X\} \subseteq C_E(X)$

(2) $\sigma(C_E(X), C_E(X)^*)$– closure of $\{f \cdot \rho_X : x \in X\} \subseteq C_E(X)$.

(3) For each $y \in X$, $\sigma(C_E(X), (C_E(X)^*)$ – closure of $\{ f \cdot \lambda_y \cdot \rho_x : x \in X\} \subseteq (C_E(X)$.

Put $\lambda_1 = \rho_1 = i$, the identify map. Then $f \in$ E-LMC if for each $y \in X \cup \{1\}$, $\sigma(C_E(X), C_E(X)^*)$ – closure of $\{f \cdot \lambda_y \cdot \rho_x : x \in X \cup \{1\}\} \subseteq C_E(X)$.

3.5 Result [4]. The following conditions are equivalent. Let X be E-completely regular. Then,

(a) X is E-Compact.

(b) Each E-Z-Ultrafilter in X converges.

(c) Each E-Z-Ultranet converges.

(d) Each E-Z-Filter has a cluster point.

(e) Each E-Z-Net has a cluster point.

(f) Each family of E-closed subsets of X with finite intersection property has non-empty intersection.

(g) Each E-Open cover of X has a finite subcover.

3.6. Result. Let T be an E-compact Hausdorff right topological semigroup and $\phi : X \to T$ be a continuous homomorphism such that $\lambda_{\phi(X)}$ is continuous for each $x \in X$. If $f \in C_E(T)$, then $f \cdot \phi \in$ E-LMC(X).

Proof. Let $t \in X \cup \{1\}$ and $g \in \sigma(C_E(X), C_E(X)^*)$ –closure of $\{(f \cdot \phi) \cdot \lambda_y \cdot \rho_x : x \in X \cup \{1\}\}$. It suffices to show that there is some $a \in T$ such that for each $x \in X$, $f \cdot \lambda_{\phi(tx)}(a) = g(x)$, for then, if $t = 1$, $g = f \cdot \lambda_{\phi(a)} = f \cdot \rho_a \cdot \phi$ and if $t \in X$, $g = f \cdot \lambda_{\phi(t)} \cdot \rho_a \cdot \phi$. In either case, g is the composition of continuous functions.

Let $(x_\alpha)_{\alpha \in I}$ be an E-Z-net such that $(f \cdot \phi \cdot \lambda_t \cdot \rho_{x_\alpha})_{\alpha \in I}$ converges to g. By taking an $E - Z$ subnet if necessary, we may assume $(\phi(x_\alpha))_{\alpha \in I}$ converges to some $a \in T$. Then for each $t \in X$, $\alpha \in I$,

$$f \cdot \lambda_{\phi(yt)}(\phi(x_\alpha)) = \Pi_t(f \cdot \phi \cdot \lambda_y \cdot \rho_{x_\alpha})$$

$$f \cdot \lambda_{\phi(yt)}(a) = g(t)$$

$$\therefore \quad f \cdot \lambda_{\phi(yt)}(a) = g(t) \text{ as required.}$$

3.7. Definition. (1) Define $e : X \to \prod_{f \in E-LMC} E_f$, where $E_f = E$ for every $f \in$ E-LMC, where $e(x)(f) = f(x)$.

(2) $\delta_E X = \text{cl } e[X]$

Then $\delta_E X$ is E-compact and Hausdorff and $e[X]$ is dense in $\delta_E X$.

3.8. Result. Let $f \in$ E-LMC, $x \in X$. Then $f \cdot \lambda_x \in$ E-LMC.

Proof. Let $t \in X \cup \{1\}$. Then

$\{(f \cdot \lambda_x) \cdot \lambda_t \cdot \rho_y : y \in X \cup \{1\}\} = \{f \cdot \lambda_{xt} \cdot \rho_y : y \in X \cup \{1\}\}$.

3.9. Definition. For $f \in$ E-LMC, $\upsilon \in \delta_E X$, define

$h_{\upsilon, f} : X \to K$ by $h_{\upsilon, f}(x) = \upsilon(f \cdot \lambda_x)$

By the above result, υ is defined at $f \cdot \lambda_x$.

3.10. Result. Let $f \in$ E-LMC, $\upsilon \in \delta_E X$. Then $h_{\upsilon,f} \in$ E-LMC.

Proof. Let $t \in X \cup \{1\}$. We show that for each $x \in X \cup \{1\}$,

$h_{\upsilon, f} \cdot \lambda_t \cdot \rho_x \in \mathrm{cl}\{f \cdot \lambda_t \cdot \rho_s : s \in X \cup \{1\}\}$ so that

$\mathrm{cl}\{h_{\upsilon, f} \cdot \lambda_t \cdot \rho_x : x \in X \cup \{1\}\} \subseteq \{\mathrm{cl}\, f \cdot \lambda_t \cdot \rho_s : s \in X \cup \{1\}\} \subseteq C_E(X)$

Let $x \in X \cup \{1\}$, F a finite subset of X, for each $y \in F$, U_y be a neighbourhood of $h_{\upsilon, f} \cdot \lambda_t \cdot \rho_x(y)$ in K^X.

Then, $U = \bigcap_{y \in F} \pi_y^{-1}(U_y)$ is a basic neighbourhood of

$h_{\upsilon, f} \cdot \lambda_t \cdot \rho_x$. (Here, $\prod_y$ map K^X to K). Let

$V = \bigcap_{y \in F} \prod_f^{-1} \lambda_{tyx}(U_y)$.

(Here $\prod_f \cdot \lambda_{tyx} : \prod_{g \in E-LMC} E_g \to K$, where $E_g = E$ for every $g \in$ E-LMC).

Given $y \in F$, $h_{\upsilon, f} \cdot \lambda_t \cdot \rho_x(y) = h_{\upsilon, f}(tyx) = \upsilon(f \cdot \lambda_{tyx})$ so that V is a neighbourhood of υ. Pick $s \in X$ such that $e(s) \in V$. Then given $y \in F$,

$f \cdot \lambda_t \cdot \rho_{xs}(y) = f(tyxs) = f \cdot \lambda_{tyx}(s) = e(s)\ (f \cdot \lambda_{tyx}) \in U_y$

Thus $f \cdot \lambda_t \cdot \rho_{xs} \in U$ as required.

3.11. Definition. For $\mu, \upsilon \in \delta_E X$, define $\mu\upsilon \in K^{E\text{-}LMC}$ by $\mu\upsilon(F) = \mu(h_{\upsilon,f})$.

3.12. Result. With the operation just defined, $\delta_E X$ is a right topological semigroup and for each $s \in X$, $\lambda_{e(s)}$ is continuous.

Proof. We first show that for $\mu, \upsilon \in \delta_E X$, $\mu\upsilon \in \delta_E X$.

Let F be a finite subset of E-LMC and for each $f \in F$, let U_f be a neighbourhood of $\mu\upsilon(f)$. Then

$U = \bigcap_{f \in F} \pi_f^{-1}(U_f)$ is a basic neighbourhood of $\mu\upsilon$.

Now, given $f \in F$,

$\mu(h_{\upsilon,f}) = \mu\upsilon(f) \in U_f$ so that $\bigcap_{f\in F} \Pi^{-1}_{h_{\upsilon,f}}(U_f)$ is a neighbourhood of μ. Pick $s \in X$ such that $e(s) \in \bigcap_{f\in F} \Pi^{-1}_{h_{\upsilon,f}}(U_f)$. Then given $f \in F$,

$\upsilon(f\cdot\lambda_s) = h_{\upsilon,f}(s) = e(s)\,(h_{\upsilon,f}) \in U_f$ so that $\bigcap_{f\in F} \Pi^{-1}_{f\cdot\lambda_s}(U_f)$ is a neighbourhood of υ. Pick $t \in X$ such that $e(t) \in \bigcap_{f\in F} \Pi^{-1}_{f.\lambda_s}(U_f)$. Then, given $f \in F$, $e(st)(f) = f(st) = (f.\lambda_s)(t) = e(t)(f.\lambda_s) \in U_f$ so that $e(st) \in U$ as required.

To see that the operation is associative, let $\mu, \upsilon, \eta \in \delta_E X$ and $f \in$ E-LMC. Then,

$(\mu\upsilon)\eta(f) = (\mu\upsilon)(h_{\eta,f}) = \mu(h_{\upsilon,h_{\eta,f}})$ and $\mu(\upsilon\eta)(f) = \mu(h_{\upsilon\eta,f})$,

so it suffices to show $h_{\upsilon,h_{\eta,f}} = h_{\upsilon\eta,\, f}$.

Let $s \in X$. Then $h_{\upsilon,h_{\eta,f}}(s) = (h_{\eta,f}\cdot\lambda_s)$ and

$h_{\upsilon\eta,f}(s) = \upsilon\eta(f\cdot\lambda_s) = \upsilon(h_{\eta,f}\cdot\lambda_s)$ so it suffices to show that $h_{\eta,f}\cdot\lambda_s = h_{\eta,f\cdot\lambda_s}$. So, let $t \in X$. Then,

$(h_{\eta,f\,\lambda_s})(t) = h_{\eta,f}(st) = \eta(f\cdot\lambda_{st}) = \eta(f\cdot\lambda_s\cdot\lambda_t) = (h_{\eta,f\cdot\lambda_s})(t)$.

To see that $\delta_E X$ is right topological, let $\upsilon \in \delta_E X$, $f \in$ E-LMC and U open in K. Then $\Pi^{-1}_f(U) \cap \delta_E X$ is a subbasic open set in $\delta_E X$.

$$\Pi^{-1}_{h_{\upsilon,f}}(U) \cap \delta_E X = \int^{-1}_{\upsilon}[\Pi^{-1}_f(U) \cap \delta_E X].$$

Let $s \in X$. Let $f \in$ E-LMC and U open in K. Then,

$\lambda^{-1}_{e(s)}[\Pi^{-1}_f(U) \cap \delta_E X] = \Pi^{-1}_{f\,\lambda_s}(U) \cap \delta_E X$.

Thus $\lambda_{c(s)}$ is continuous.

3.13 Result. The function $e : X \rightarrow \delta_E X$ is a continuous homomorphism.

Proof. Let $f \in$ E-LMC and U be open in K. Then, $e^{-1}(\Pi^{-1}_f(U) \cap \delta_E X) = f^{-1}(U)$. So e is continuous.

Let $s, t \in X$ and $f \in$ E-LMC. Then,

$e(s)\;e(t)(f) = e(s)\;(h_{c(t),f}) = h_{c(t),f}(s) = e(t)\;(f\cdot\lambda_s) = f(st) = e(st)(f)$.

3.14. Definition. Let (e, $\delta_E X$) be the E-compactification maximal with respect to the properties that $\delta_E X$ is right topological and $\lambda_{c(x)}$ is continuous for each $x \in X$. The functions, from X to E^n, $n \in N$ are called E-LMC-functions and $\delta_E X$ is called the E-LMC-compactification of X.

3.15. Theorem. Let $f : X \to K$. Then f extends continuously to $\delta_E X$ (i.e., there exists $g \in C_E(\delta_E X)$ with $g \cdot e = f$) if and only if $f \in$ E-LMC.

Proof. For the sufficiency, define

$g : \delta_E X \to K$ by $g(\upsilon) = \upsilon(f)$, given $x \in X$,

$g(e(x)) = e(x)(f) = f(x)$. Given U open in K,

$g^{-1}(U) = \Pi_f^{-1}(U) \cap \delta_E X$.

Necessity follows from result 3.6.

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REFERENCES

1. J. Berglund, H. Junghenn and P. Milness : Compact right topological semigroups and generalizations of almost periodicity, Lecture Notes in Math. 663 (1978).
2. S. Mrowka : Further results on E-compact spaces. I, Acta Math. 120 (1968), 161-185.
3. Porter J. R. and Woods R. G. : Extension and absolutes of Hausdorff Spaces, Springer—Verlag, Berlin, Heidelberg, New York (1987).
4. On maximal E-compactification : Mangalambal N. R. (to appear).

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ON α-S-CLOSED CRISP SUBSETS OF A FUZZY TOPOLOGICAL SPACE

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ABSTRACT : The present paper deals with a type of covering property viz. α-S-closedness of subsets of a space X, where X is endowed with a fuzzy toplogy. Basically, several characterizations of the α-S-closedness of such crisp subsets of a fuzzy space X are obtained, where among other things, ordinary nets and power-set filterbases are employed as supporting appliances.

1. INTRODUCTION

It is seen from literature that S-closedness for topological spaces, first introduced by Thompson [9], has so far got considerable attention from topologists. The generalizations of the concept to fuzzy sets and fuzzy space have also been carried out by many (e.g. see [5,7]). Gantner et al. [3] prescribed the notion of α-shading, which has paved a new direction towards the study of different covering properties for fuzzy spaces. Mashhour et et. [6] utilized the notion of α-shading to initiate the study of α-S-closed fuzzy topological spaces (henceforth abbreviated as fts).

In this paper, our aim is to study α-S-closedness of crisp subsets (i.e., ordinary subsets) of an fts. We obtain different characterizations of an α-S-closed crisp subset A of an fts, which also serve for the corresponding formulations of the α-S-closedness of X if one puts A = X. In the next section such characterizations are incorporated, while in the last section we search for the suitable class of crisp sets which inherit α-S-closedness of the whole space. A type of functions. keeping α-S-closedness invariant, is also investigated there.

In what follows, by (X, τ) or simply by X we mean an fts in the sense of Chang [2]. For a fuzzy set [10] A, the closure and interior of A in X will be denoted by clA and intA respectively. A fuzzy set A in X is called fuzzy regular open (semiopen) if A = int clA (resp. $U \leq A \leq$ clA, for some fuzzy open set U) [1]. The complement (1–A) of a fuzzy regular open (semiopen) set A in X is called fuzzy regular closed (resp. semiclosed). The semiclosure of a fuzzy set A in X, to be denoted by scl A, is the union of all those fuzzy points x_t (where x is the singleton support and t the value of the fuzzy point x_t, $0 < t \leq 1$) such that for any fuzzy semiopen set U with $U(x) + t > 1$, there exists $y \in X$ with $U(y) + A(y) > 1$ [4]. The semi-interior of a fuzzy set A, Written as sintA, is defined by sintA

$= 1-\mathrm{scl}(1\text{-}A)$ [4]. It is known [4] that a fuzzy set A in X is fuzzy semiclosed (semiopen) iff $A = \mathrm{scl}A$ (resp. $A = \mathrm{sint}A$).

2. α-S-CLOSEDNESS : CHARACTERIZATIONS

Definition 2.1 [3]. Let A be a crisp subset of an fts X. A collection $\mathcal{U}$ of fuzzy sets in X is called an α-shading (where $0 < \alpha < 1$) of A if for each $x \in A$, there is some $U_x \in \mathcal{U}$ such that $U_x(x) > \alpha$. If, in addition, the members are fuzzy open (semiopen) then $\mathcal{U}$ is called a fuzzy open (resp. semiopen) α-shading of A.

Definition 2.2. Let X be an fts and A be a crisp subset of X. A is said to be$\leq$ α-closed if for every semiopen α-shading ($0 < \alpha < 1$) $\mathcal{U}$ of A, there is a finite proximate α-subshading of A, i.e., there is a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{\mathrm{cl}U : U \in \mathcal{U}_0\}$ is again an α-shading of A. If $A = X$ in addition, then X is called an α-S-closed space, as proposed by Mashhour et al. [6].

Mashhour et al. [6] defined a fuzzy set A in an fts X to be fuzzy regular semiopen if there is a fuzzy regular open set U such that $U \leq A \leq \mathrm{cl}U$, and they proved that a fuzzy regular semiopen set is fuzzy semiopen but not conversely. The following result is a generalization of Theorem 5.7 of [6].

Theorem 2.3. A subset A of X is α-S-closed iff every α-shading of A by fuzzy regular semiopen sets in X has a finite proximate α-subshading.

Proof. The proof is immediate from the definition of α-S-closedness and the fact that whenever $\{U_i : i \in \Lambda\}$ is a fuzzy semiopen α-shading of A, then $\{(\mathrm{int\,cl}V_i) \cup V_i : i \in \Lambda\}$ is an α-shading of A by fuzzy regular semiopen sets.

Theorem 2.4. Let X be an fts. A crisp subset A of X is α-S-closed iff every family of fuzzy semiopen sets, the semi-interiors of whose closures form an α-shading of A, contains a finite subfamily, the closures of whose members form an α-shading of A.

Proof. It is sufficient to observe that for a fuzzy semiopen set U, $U \leq \mathrm{sint}(\mathrm{cl}U) \leq \mathrm{cl}(\mathrm{sint}(\mathrm{cl}U)) = \mathrm{cl}U$.

Theorem 2.5. A crisp subset A of an fts X is α-S-closed iff for every collection $\{F_i : i \in \Lambda\}$ of fuzzy open sets with the property that for each finite subset Λ_0 of Λ, there is $x \in A$ such that $\inf_{i \in \Lambda_0} F_i(x) \geq 1-\alpha$, one has $\inf_{i \in \Lambda} (\mathrm{scl}F_i)(y) \geq 1-\alpha$, for some $y \in A$.

Proof. Let A be α-S-closed, and if possible, let for a collection $\{F_i : i \in \Lambda\}$ of fuzzy open sets in X with the stated property, $(\bigcap_{i \in \Lambda} \mathrm{scl}\ F_i)(x) < 1-\alpha$, for each $x \in A$. Then $\alpha < (1 - \bigcap_{i \in \Lambda} \mathrm{scl}\ F_i)(x) = [\bigcup_{i \in \Lambda} (1-\mathrm{scl}F_i)](x)$, for each $x \in A$, so that $\{1-\mathrm{scl}F_i : i \in \Lambda\}$ is a semiopen α-shading of A. By α-S-closedness of A, there is a finite subset Λ_0 of Λ such that {cl(1−

$\mathrm{scl}F_i) : i\in\Lambda_0\}$ is an α-shading of A. Hence $\alpha < [\bigcup_{i\in\Lambda_0}(1-\mathrm{int\,scl}F_i)](x) = [1-\bigcap_{i\in\Lambda_0}\mathrm{int\,scl}F_i](x)$, for each $x\in A$. Then $(\bigcap_{i\in\Lambda_0}F_i)(x) \leq [\bigcap_{i\in\Lambda_0}\mathrm{int\,scl}F_i](x) < 1-\alpha$, for each $x\in A$, which contradicts the stated property of the collection $\{F_i : i\in\Lambda\}$.

Conversely, let under the given hypothesis, A be not α-S-closed. Then there is a semiopen α-shading $\mathcal{U}=\{U_i : i\in\Lambda\}$ of A such that for every finite subcollection Λ_0 of Λ, $\{\mathrm{cl}U_i : i\in\Lambda_0\}$ is not an α-shading of A, i.e., there exists $x\in A$ such that $\sup_{i\in\Lambda_0}(\mathrm{cl}U_i)(x) \leq \alpha$, i.e., $1-\sup_{i\in\Lambda_0}(\mathrm{cl}U_i)(x) \geq 1-\alpha$. Hence $\{1-\mathrm{cl}U_i : i\in\Lambda\}$ is a family of fuzzy open sets with the stated property. Consequently, there is some $y\in A$ such that $\inf_{i\in\Lambda}[\mathrm{scl}(1-\mathrm{cl}U_i)](y) \geq 1-\alpha$. Then $\sup_{i\in\Lambda}U_i(y) \leq \sup_{i\in\Lambda}(\mathrm{sint\,cl}U_i)(y) = 1-\inf_{i\in\Lambda}(1-\mathrm{sint\,cl}U_i)(y) = 1-\inf_{i\in\Lambda}[\mathrm{scl}(1-\mathrm{cl}U_i)](y) \leq \alpha$. This shows that $\{U_i : i\in\Lambda\}$ fails to be an α-shading of A, a contradiction.

Let us now introduce the following definition :

Definition 2.6. Let $\{S_n : n\in(D,\geq)\}$ (where $(D,\geq)$ is a directed set) be an ordinary net in X and $\mathcal{T}$ be a power-set filterbase on X, and $x\in X$ be any crisp point in X. Then x is called a semi-θ_α-adherent point of

(a) the net $\{S_n\}$ if for each fuzzy semiopen set U in X with $U(x) > \alpha$ and for each $m\in D$, there exists $k\in D$ such that $k\geq m$ in D and $(\mathrm{cl}U)(S_k) > \alpha$,

(b) the filterbase $\mathcal{T}$ if for each fuzzy semiopen set U with $U(x) > \alpha$ and for each $F\in\mathcal{T}$, there exists a crisp point x_F in F such that $(\mathrm{cl}U)(x_F) > \alpha$.

Theorem 2.7. A crisp subset A of an fts X is α-S-closed iff every net in A has a semi-θ_α-adherent point in A.

Proof. Suppose A be α-S-closed, but there is a net $\{S_n : n\in(D,\geq)\}$ in A ($(D,\geq)$ being a directed set, as usual) having no semi-θ_α-adherent point in A. Then for each $x\in A$, there is a fuzzy semiopen set U_x with $U_x(x) > \alpha$, and there is an $m_x\in D$ such that $(\mathrm{cl}U_x)(S_n) \leq \alpha$, for all $n\geq m_x$ $(n\in D)$. Now, $\mathcal{U} = \{1-\mathrm{cl}U_x : x\in X\}$ is a collection of fuzzy open sets such that for any finite subcollection $\{1-\mathrm{cl}\,U_{x_1}, \ldots, 1-\mathrm{cl}\,U_{x_k}\}$ (say) of $\mathcal{U}$ there exists $m\in D$ with $m\geq m_{x_1}, \ldots, m_{x_k}$ in D such that $(\bigcup_{i=1}^{k}\mathrm{cl}U_{x_i})(S_n) \leq \alpha$, for all $n\geq m$ $(n\in D)$, i.e., $\inf_{1\leq i\leq k}(1-\mathrm{cl}\,U_{x_i})(S_n) \geq 1-\alpha$, for all $n\geq m$. Hence by Theorem 2.5, there exists some $y\in A$ such that $\inf_{x\in A}[\mathrm{scl}(1-\mathrm{cl}U_x)(y)] \geq 1-\alpha$, i.e., $(\bigcup_{x\in A}U_x)(y) \leq [\bigcup_{x\in A}\mathrm{sint}(\mathrm{cl}U_x)](y) = 1-[1-(\bigcup_{x\in A}\mathrm{sint}$

$\mathrm{clU}_x)](y) = 1 - \inf_{x\in A}[\,\mathrm{scl}(1 - \mathrm{clU}_x)\,](y) \leq 1-1 + \alpha = \alpha$. We have, in particular, $U_y(y) \leq \alpha$, going against the definition of U_y. Hence the result is proved.

Conversely, let every net in A have semi-θ_α-adherent point in A and suppose $\{F_i : i \in \Lambda\}$ be an arbitrary collection of fuzzy open sets in X. Let Λ_f denote the collection of all subsets of Λ, then $(\wedge_f, \geq)$ is a directed set, where for $\mu, \lambda \in \wedge_f$, $\mu \geq \lambda$ iff $\mu \supseteq \lambda$. For each $\mu \in \wedge_f$, put $F_\mu = \cap\{F_i : i \in \mu\}$. Let for each $\mu \in \wedge_f$, there be a point $x_\mu \in A$ such that $\inf_{i\in\mu} F_i(x_\mu) \geq 1-\alpha$ ... (1).

Then by Theorem 2.5 it is enough to show that $\inf_{i\in\Lambda}(\mathrm{scl}\ F_i)(z) \geq 1-\alpha$ for some $z \in A$. If possible, let $\inf_{i\in\Lambda}(\mathrm{scl}\ F_i)(z) < 1-\alpha$, for each $z \in A$ ... (2).

Now, $S = \{x_\mu : \mu \in (\wedge_f, \geq)\}$ is clearly a net of points in A. By hypothesis, there is a semi-θ_α-adherent point z in A of this net. By (2), $\inf_{i\in\Lambda}(\mathrm{scl}\ F_i)(z) < 1-\alpha \Rightarrow$ there exists $i_0 \in \Lambda$ such that $(\mathrm{scl}\, F_{i_0})(z) < 1-\alpha$, i.e., $1 - \mathrm{scl}\ F_{i_0}(z) > \alpha$. Since z is a semi θ_α-adherent point of S, for the index $\{i_0\} \in \wedge_f$, there is $\mu_0 \in \Lambda_f$ with $\mu_0 \geq \{i_0\}$ (i.e., $i_0 \in \mu_0$) such that $\mathrm{cl}(1\text{-}\mathrm{scl}\, F_{i_0})(x_{\mu_0}) > \alpha$, i.e., $\mathrm{int\ scl}\, F_{i_0}(x_{\mu_0}) < 1-\alpha$. Since $i_0 \in \mu_0$, $\inf_{i\in\mu_0} F_i(x_{\mu_0}) \leq F_{i_0}(x_{\mu_0}) \leq \mathrm{int\ scl}\, F_{i_0}(x_{\mu_0}) < 1-\alpha$, which contradicts (1). This completes the proof.

Theorem 2.8. A crisp subset A of an fts X is α-S-closed iff every filterbase $\mathcal{G}$ on A has a semi-θ_α-adherent point in A.

Proof. Let A be α-S-closed and let there exist, if possible, a filterbase $\mathcal{F}$ on A having no semi-θ_α-adherent point in A. Then for each $x \in A$, there exists a fuzzy semiopen set U_x with $U_x(x) > \alpha$, and an $F_x \in \mathcal{F}$ such that $(\mathrm{clU}_x)(y) \leq \alpha$, for each $y \in F_x$. Then $\mathcal{U} = \{U_x : x \in A\}$ is a fuzzy semiopen α-shading of A. Thus there exist finitely many points $x_1, x_2, \ldots, x_n$ in A such that $F \leq F_{x_1} \cap \ldots \cap F_{x_n}$. Then $(\mathrm{cl}\,U_{x_i})(y) \leq \alpha$, for all $y \in F$ and for $i = 1, 2, \ldots, n$. Thus $\mathcal{U}_0$ fails to be an α-shading of A, a contradiction.

Conversely, let the condition hold and suppose, if possible, $\{y_n : n \in (D, \geq)\}$ be a net in A having no semi-θ_α-adherent point in A. Then for $x \in A$, there are a fuzzy semiopen set U_x with $U_x(x) > \alpha$ and an $m_x \in D$ such that $(\mathrm{clU}_x)(y_n) \leq \alpha$, for all $n \geq m_x$ ($n \in D$). Thus $\mathcal{B} = \{F_x : x \in A\}$, where $F_x = \{y_n : n \geq m_x\}$ generates a filterbase $\mathcal{F}$ on A. By hypothesis, $\mathcal{F}$ has a semi-θ_α-adherent point z (say) in A. But there are a fuzzy semiopen set U_z with $U_z(z) > \alpha$ and an $m_z \in D$ such that $(\mathrm{clU}_z)(y_n) \leq \alpha$, for all $n \geq m_z$, i.e., for all $p \in F_z$ ($\in \mathcal{F}$), $(\mathrm{clU}_z)(p) \leq \alpha$. Hence z cannot be a semi-θ_α-adherent point of the filterbase $\mathcal{F}$, a contradiction. Hence by Theorem 2.7, A is α-S-closed.

The next theorem gives slight variations of the results in the last two theorems. For this we need to introduce the following definition.

Definition 2.9. A crisp point x in an fts X is called an α-RC-adherent point of

(a) a net $\{x_n : n \in (D, \geq)\}$ of points in X if for each fuzzy regular closed set U with $U(x) > \alpha$ and for each $n \in D$, there exists $m \in D$ with $m \geq n$ such that $U(x_m) > \alpha$;

(b) a power-set filterbase $\mathcal{F}$ on X if for each fuzzy regular closed set U in X with $U(x) > \alpha$ and for each $F \in \mathcal{F}$, there exists $x_F \in F$ such that $U(x_F) > \alpha$.

Proceeding similarly as in the proofs of Theorems 2.7 and 2.8, we can establish the theorem below :

Theorem 2.10. For any crisp subset A of an fts X, the following are equivalent :

(a) A is α-S-closed.

(b) Each net in A has an α-RC-adherent point in A.

(c) Each filterbase on A has an α-RC-adherent point in A.

Definition 2.11. A family $\{F_i : i \in \Lambda\}$ of fuzzy sets is an fts X is said to have α-interiorly finite intersection property or simply α-IFIP (α-finite intersection property or simply α-FIP) in a subset A of X, if for each finite subset Λ_0 of Λ, there exists $x \in A$ such that $[\mathrm{int}(\bigcap_{i \in \Lambda_0} F_i)](x) \geq 1-\alpha$ (resp. $[\bigcap_{i \in \Lambda_0} F_i](x) \geq 1-\alpha$).

Theorem 2.12. A crisp subset A of an fts X is α-S-closed iff for every family $\mathcal{F} = \{F_i : i \in \Lambda\}$ of fuzzy semiclosed sets in X with α-IFIP in A, there exists $x \in A$ such that $\inf_{i \in \Lambda} F_i(x) \geq 1-\alpha$.

Proof. Assuming $A (\subseteq X)$ to be α-S-clsoed, let $\mathcal{F} = \{F_i : i \in \Lambda\}$ be a family of fuzzy semiclosed sets with α-IFIP in A. If possible, let for each $x \in A$, $\inf_{i \in \Lambda} F_i(x) < 1-\alpha$, i.e., $(\bigcap_{i \in \Lambda} (F_i)(x) < 1-\alpha$ and hence $[\bigcup_{i \in \Lambda} (1 - F_i)](x) > \alpha$. Thus $\mathcal{U} = \{1 - F_i : i \in \Lambda\}$ is a fuzzy semiopen α-shading of A. By α-S-closedness of A, there is a finite subset Λ_0 of Λ such that $[\bigcup_{i \in \Lambda_0} \mathrm{cl}(1 - F_i)](x) > \alpha$, i.e., $1 - (\bigcap_{i \in \Lambda_0} \mathrm{int} F_i)(x) > \alpha$, i.e., $(\mathrm{int} \bigcap_{i \in \Lambda_0} F_i)(x) < 1-\alpha$, for each $x \in A$, which shows that $\mathcal{F}$ does not have α-IFIP in A, a contradiction.

Conversely, let $\mathcal{U} = \{U_i : i \in \Lambda\}$ be a fuzzy semiopen α-shading of A. Then $\mathcal{F} = \{1 - U_i : i \in \Lambda\}$ is a family of fuzzy semiclosed sets with $\inf_{i \in \Lambda} (1 - U_i)(x) < 1-\alpha$, for each $x \in A$, so that $\mathcal{F}$ cannot have α-IFIP. Hence for some finite subset Λ_0 of Λ, we have for each $x \in A$,

$[\mathrm{int} \bigcap_{i\in\Lambda_0} (1 - U_i)](x) < 1-\alpha \Rightarrow 1-[\mathrm{cl}(\bigcup_{i\in\Lambda_0} U_i)](x) < 1-\alpha$, for each $x\in A \Rightarrow [\mathrm{cl}(\bigcup_{i\in\Lambda_0} U_i)](x) = [\bigcup_{i\in\Lambda_0} \mathrm{cl}U_i](x) > \alpha \Rightarrow A$ is α-S-closed.

Theorem 2.13. For a crisp subset A of an fts X, the following are equivalent :

(a) A is α-S-closed.

(b) Every α-shading of A by fuzzy regular closed sets in X has a finite α-subshading for A.

(c) For every family $\mathcal{F} = \{F_i : i\in\Lambda\}$ of fuzzy regular open sets in X with α-FIP in A, there exists $x\in A$ such that $\inf_{i\in\Lambda} F_i(x) \geq 1-\alpha$.

Proof. (a) $\Leftrightarrow$ (b) : Follows from the facts that fuzzy regular closed sets are fuzzy semiopen as well as fuzzy closed, and for any fuzzy semiopen set A in X, clA is a fuzzy regular closed set.

(a) $\Rightarrow$ (c) : We only note that a fuzzy regular open set in X is fuzzy semiclosed. The rest follows from Theorem 2.12.

(c) $\Rightarrow$ (a) : If possible, let $\mathcal{U} = \{U_i : i\in\Lambda\}$ be a fuzzy semiopen α-shading of A having no finite proximate α-subshading. Then $\{1-\mathrm{cl}U_i : i\in\Lambda\}$ is a family of fuzzy regular open sets possessing α-FIP. In fact, for any finite subcollection Λ_0 of Λ, $\bigcup_{i\in\Lambda_0} (\mathrm{cl}U_i)(x) \leq \alpha \Rightarrow \bigcap_{i\in\Lambda_0} (1-\mathrm{cl}U_i)(x) \geq 1-\alpha \Rightarrow \bigcap_{i\in\Lambda_0} \mathrm{int}(1-U_i)(x) \geq 1-\alpha \Rightarrow [\bigcap_{i\in\Lambda_0} (1-U_i)](x) \geq 1-\alpha$. Consequently by (c), there is $x\in A$ such that $\inf_{i\in\Lambda}\{(1-U_i)(x)\} \geq \inf_{i\in\Lambda}\{1-\mathrm{cl}U_i)(x)\} \geq 1-\alpha \Rightarrow 1-(\bigcup_{i\in\Lambda} U_i)(x) \geq 1-\alpha \Rightarrow (\bigcup_{i\in\Lambda} U_i)(x) \leq \alpha$, contradicting our assumption that $\mathcal{U}$ is an α-shading of A.

Note 2.14. The equivalence of (a) and (b) for A = X was proved by Mashhour et al. in [6].

It is known [8] that the collection of all fuzzy regular closed sets in an fts (X, τ) is same as that in its semiregularization space (X, τ_s), where τ_s is the fuzzy semiregularization topology on X generated by the class of all fuzzy regular open sets in (X, τ) as an open basis [8]. It then follows from Theorem 2.13 ((a) $\Leftrightarrow$ (b)) that

Theorem 2.15. A crisp subset A is α-S-closed in an fts (X, τ) iff it is so in (X, τ_s).

Putting A = X in the characterization theorems so far, of α-S-closed crisp subset A, we obtain as follows.

Theorem 2.16. For an fts (X, τ) the following are equivalent :

(a) X is α-S-closed.

(b) Every α-shading of X by fuzzy regular semiopen sets has a finite proximate α-subshading.

(c) Every family of fuzzy semiopen sets, the semi-interiors of whose closures form an α-shading of X, contains a finite subfamily, the closures of whose members form an α-shading of X.

(d) For every collection $\{F_i : i\in\Lambda\}$ of fuzzy open sets with the property that for each finite subset Λ_0 of Λ, there is $x\in X$ such that $\inf_{i\in\Lambda_0} F_i(x) \geq 1-\alpha$, one has $\inf_{i\in\Lambda} (\mathrm{scl}F_i)(y) \geq 1-\alpha$, for some $y\in X$.

(e) Every net in X has a semi-$\theta_\alpha$adherent point in X.

(f) Every filterbase on X has a semi-θ_α-adherent point in X.

(g) Every net in X has an α-RC-adherent point in X.

(h) Each filterbase on X has an α-RC-adherent point in X.

(i) For every family $\mathcal{F} = \{F_i : i\in\Lambda\}$ of fuzzy semiclosed sets in X with α-IFIP in X, there exists $x\in X$ such that $\inf_{i\in\Lambda} F_i(x) \geq 1-\alpha$.

(j) Every α-shading of X by fuzzy regular closed sets has a finite α-subshading.

(k) For every family $\mathcal{F} = \{F_i : i\in\Lambda\}$ of fuzzy regular open sets in X with α-FIP, there exists $x\in X$ such that $\inf_{i\in\Lambda} F_i(x) \geq 1-\alpha$.

(l) (X, τ_s) is α-S-closed.

3. θ_s^α-CLOSED SET, θ_s^α-CONTINUITY AND α-S-CLOSEDNESS

In this section, our intention is to define a class of crisp subsets of an fts X such that α-S-closedness of X is inherited by each member of the class, and to find a class of fts's where the above crisp subsets form precisely the class of those subsets which inherit the α-S-closedness. We also investigate for a kind of functions under which α-S-closedness remains invariant.

Definition 3.1. Let (X, τ) be an fts and $A\subseteq X$. A point $x\in X$ is said to be a θ_s^α-limit point of A if for every fuzzy semiopen set U in X with $U(x) > \alpha$, there exists $y\in A\setminus\{x\}$ such that $(\mathrm{cl}U)(y) > \alpha$. The set of all θ_s^α-limit points of A will be denoted by $[A]_s^\alpha$.

The θ_s^α-closure of A, to be denoted by θ_s^α-clA, is defined by $\theta_s^\alpha\text{-cl}A = A \cup [A]_s^\alpha$.

Definition 3.2. A crisp subset A of an fts X is said to be θ_s^α-closed if it contains all its θ_s^α-limit points. Any subset B of X is called θ_s^α-open if $X \setminus B$ is θ_s^α-closed.

Remark 3.3. For any $A \subseteq X$ it is clear that $A \subseteq \theta_s^\alpha$-clA, and θ_s^α-clA = A iff $[A]_s^\alpha \subseteq A$. Then in view of Definition 3.1, it follows that A is θ_s^α-closed iff θ_s^α-cl A = A. It is also clear that $A \subseteq B \subseteq X \Rightarrow [A]_s^\alpha \subseteq [B]_s^\alpha$.

Theorem 3.4. A θ_s^α-closed subset A of an α-S-closed space X is α-S-closed.

Proof. Let A ($\subseteq X$) be θ_s^α-closed in X. Then for any $x \notin A$, there is a fuzzy semiopen set U_x such that $U_x(x) > \alpha$, and $(\mathrm{cl}U_x)(y) \leq \alpha$ for every $y \in A$. Consider the collection $\mathcal{U} = \{U_x : x \notin A\}$. Now to prove that A is α-S-closed, consider a fuzzy semiopen α-shading $\mathcal{V}$ of A. Clearly $\mathcal{U} \cup \mathcal{V}$ is a fuzzy semiopen α-shading of X. Since X is α-S-closed, there exists a finite subcollection $\{V_1, \ldots, V_n\}$ of $\mathcal{U} \cup \mathcal{V}$ such that for every $t \in X$, there exists V_i $(1 \leq i \leq n)$ with $\mathrm{cl}V_i(t) > \alpha$. For every member U_x of $\mathcal{U}$, $\mathrm{cl}U_x(y) \leq \alpha$ for every $y \in A$. So if this subcollection contains any member of $\mathcal{U}$, we omit it and hence we get the result.

Definition 3.5. An fts (X, τ) is said to be α-Urysohn if for any two distinct points x, y of X, there exist U, $V \in \tau$ with $U(x) > \alpha$, $V(y) > \alpha$ and $\min(\mathrm{cl}U(z), \mathrm{cl}V(z)) \leq \alpha$ for each $z \in X$.

Theorem 3.6. An α-S-closed set in an α-Urysohn space X is θ_s^α-closed.

Proof. Let $x \in X \setminus A$. For each $y \in A$, we have $x \neq y$ and hence by α-Urysohn property of X, there exist fuzzy open sets U_y and V_y such that $U_y(x) > \alpha$, $V_y(y) > \alpha$ and $\min((\mathrm{cl}U_y)(z), (\mathrm{cl}V_y)(z)) \leq \alpha$, for all $z \in X$ ... (1).

Then $\mathcal{U} = \{V_y : y \in A\}$ is a fuzzy open and hence fuzzy semiopen α-shading of A. By α-S-closedness of A, there are finitely many points $y_1, \ldots, y_n$ in A such that $\mathcal{U}_o = \{\mathrm{cl}\, V_{y_1}, \ldots, \mathrm{cl}\, V_{y_n}\}$ is again an α-shading of A. Now, $U = U_{y_1} \cap \ldots \cap U_{y_n}$ is a fuzzy open set and hence a fuzzy semiopen set such that $U(x) > \alpha$. In order to show A to be θ_s^α-closed, it now suffices to show that $(\mathrm{cl}U)(y) \leq \alpha$ for each $y \in A$. In fact, if for some $z \in A$, we assume $(\mathrm{cl}U)(z) > \alpha$ then as $z \in A$, we have $(\mathrm{cl}\, V_{y_k})(z) > \alpha$ for some k $(1 \leq k \leq n)$. Also, $(\mathrm{cl}\, U_{y_k})(z) > \alpha$. Hence $\min[(\mathrm{cl}\, U_{y_k})(z), (\mathrm{cl}\, V_{y_k})(z)] > \alpha$, contradicting (1).

Corollary 3.7. In an α-S-closed, α-Urysohn space X, a subset A of X is α-S-closed iff it is θ_s^α-closed.

Theorem 3.8. In an α-S-closed space X, every cover of X by θ_s^α-open sets has a finite subcover.

Proof. Let $\mathcal{U} = \{U_i : i \in \Lambda\}$ be a cover of X by θ_s^α-open sets. Then for each $x \in X$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Now, as $X \setminus U_x$ is θ_s^α-closed, there exists a fuzzy semiopen set V_x in X such that $V_x(x) > \alpha$, and $(\mathrm{cl}V_x)(y) \leq \alpha$ for each $y \in X \setminus U_x$ ... (1).

Then $\{V_x : x \in X\}$ forms a fuzzy semiopen α-shading of the α-S-closed space X. Thus for a finite subset $\{x_1, \ldots, x_n\}$ of X, $\{\mathrm{cl}\, V_{y_i} : i = 1, 2, \ldots, n.\}$ is an α-shading of X ... (2)

We claim that $\{U_{x_1}, \ldots, U_{x_n}\}$ is a finite subcover of $\mathcal{U}$. If not, then there exists $y \in X \setminus \bigcup_{i=1}^{n} U_{x_i} = \bigcap_{i=1}^{n} (X \setminus U_{x_i})$. Then by (1), $\mathrm{cl}\, V_{x_i}(y) \leq \alpha$ for $i = 1, 2, \ldots, n$. Therefore, $(\bigcup_{i=1}^{n} \mathrm{cl}\, V_{x_i})(y) \leq \alpha$, contradicting (2).

Theorem 3.9. Let (X, τ) be an fts. If X is α-S-closed then every collection of θ_s^α-closed sets in X with finite intersection property has nonempty intersection.

Proof. Follows immediately from Theorem 3.8.

Let us now introduce a class of functions as follows.

Definition 3.10. Let X, Y be fts's. A function $f : X \to Y$ is said to be fuzzy θ_s^α-continuous if for each point $x \in X$ and each fuzzy semiopen set V in Y with $V(f(x)) > \alpha$, there is a fuzzy semiopen set U in X with $U(x) > \alpha$ such that $\mathrm{cl}U \leq f^{-1}(\mathrm{cl}V)$.

Theorem 3.11. If $f : X \to Y$ is fuzzy θ_s^α-continuous (where X, Y are, as usual, fts's), then the following are true :

(a) $f([A]_s^\alpha) \subseteq [f(A)]_s^\alpha$, for every $A \subseteq X$.

(b) $[f^{-1}(A)]_s^\alpha \subseteq f^{-1}([A]_s^\alpha)$, for every $A \subseteq Y$.

(c) For each θ_s^α-closed set A in Y, $f^{-1}(A)$ is θ_s^α-closed in X.

(d) For each θ_s^α-open set A in Y, $f^{-1}(A)$ in θ_s^α-open in X.

Proof. (a) Let $x \in [A]_s^\alpha$ and U be any fuzzy semiopen set in Y such that $U(f(x)) > \alpha$. Then there exists a fuzzy semiopen set V in X such that $V(x) > \alpha$ and $(\mathrm{cl}V) \leq f^{-1}(\mathrm{cl}\, U)$. Now, $x \in [A]_s^\alpha$ and V is fuzzy semiopen in X with $V(x) > \alpha \Rightarrow \mathrm{cl}V(x_0) > \alpha$ for some $x_0 \in A \setminus \{x\} \Rightarrow \alpha < (\mathrm{cl}V)(x_0) \leq (f^{-1}(\mathrm{cl}U))(x_0) = (\mathrm{cl}U)(f(x_0))$, where $f(x_0) \in f(A) \setminus \{f(x)\} \Rightarrow f(x) \in [f(A)]_s^\alpha$. Thus $f([A]_s^\alpha) \subseteq [f(A)]_s^\alpha$.

(b) By (a), $f\left(\left[f^{-1}(A)\right]_s^\alpha\right) \subseteq \left[ff^{-1}(A)\right]_s^\alpha \subseteq [A]_s^\alpha \Rightarrow \left[f^{-1}(A)\right]_s^\alpha \subseteq f^{-1}([A]_s^\alpha)$.

(c) We have, $[A]_s^\alpha = A$. By (b), $\left[f^{-1}(A)\right]_s^\alpha \subseteq f^{-1}([A]_s^\alpha) = f^{-1}(A) \Rightarrow \left[f^{-1}(A)\right]_s^\alpha = f^{-1}(A) \Rightarrow f^{-1}(A)$ is θ_s^α-closed in X.

(d) Follows from (c).

Theorem 3.12. Let X, Y be fts's and f : X → Y be fuzzy θ_s^α-continuous. If A ($\subseteq$ X) is α-S-closed then so is f(A) in Y.

Proof. Let $\mathcal{V} = \{V_i : i \in \Lambda\}$ be a fuzzy semiopen α-shading of f(A), where A is α-S-closed in X. For each $x \in A$, $f(x) \in f(A)$ and hence there exists $V_x \in \mathcal{V}$ such that $V_x(f(x)) > \alpha$. By fuzzy θ_s^α-continuity of f, there exists a fuzzy semiopen set U_x in X such that $U_x(x) > \alpha$ and $f(\mathrm{cl}U_x) < \mathrm{cl}V_x$. Then $\{U_x : x \in X\}$ is a fuzzy semiopen α-shading of A. By α-S-closedness of A, there exist finitely many points $a_1, \ldots, a_n$ in A such that $\{\mathrm{cl}\,U_{a_i} : i = 1, 2, \ldots, n\}$ is again an α-shading of A. We claim that $\{\mathrm{cl}\,V_{a_i} : i = 1, 2, \ldots, n\}$ is an α-shading of f(A). In fact, $y \in f(A) \Rightarrow$ there exists $x \in A$ such that $y = f(x)$. Now, there is a U_{a_i} (for some j, $1 \le j \le n$) such that $(\mathrm{cl}\,U_{a_j})(x) > \alpha$ and hence $\mathrm{cl}\,V_{a_j}(y) \geq [f(\mathrm{cl}\,U_{a_j})](y) \geq \mathrm{cl}\,U_{a_j}(x) > \alpha$.

REFERENCES

1. Azad, K. K. ; On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl. 82 (1981), 14-32.
2. Chang, C. L. ; Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968), 182-190.
3. Gantner, T. E., Steinlage, R. C. and Warren, R. H. ; Compactness in fuzzy topological spaces, J. Math. Anal. Appl. 62 (1978), 547-562.
4. Ghosh, B. ; Semi-continuous and semiclosed mappings and semi-connectedness in fuzzy setting, Fuzzy Sets and Systems 35 (1990), 345-355.
5. Malakar, S. and Mukherjee, M. N. ; Some characterizations of S-closed fuzzy topological spaces ; J. Pure Math. 10 (1993), 1-6.
6. Mashhour, A. S., Ghanim, M. H. and Fath Alla, M. A. ; α-separation axioms and α-compactness in fuzzy topological spaces ; Rocky Mountain J. Math. 16 (3) (1986), 561-600.
7. Mukherjee, M. N. and Ghosh, B. ; On fuzzy S-closed spaces and FSC-sets, Bull. Malaysian Math. Soc. (Second Series) 12 (1989), 1-14.

8. Mukherjee, M. N. and Ghosh, B. ; Fuzzy semiregularization topologies and fuzzy submaximal spaces, Fuzzy Sets and Systems 44 (1991), 283-294.

9. Thompson, T. ; S-closed spaces, proc. Amer. Math. Soc. 60 (1976), 335-338.

10. Zadeh, L. A. ; Fuzzy Sets, Inform. Control 8 (1965), 338-353.

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ON QUASI EINSTEIN PSEUDO RICCI SYMMETRIC MANIFOLDS

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1. INTRODUCTION

The notion of a Pseudo Ricci Symmetric Manifold was introduced by the first author (Chaki) in 1988 [1]. According to him a non-flat Riemannian Manifold (M^n, g) $(n>2)$ is called Pseudo Ricci Symmetric if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the condition

(1) $(\nabla_x S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X)$

where A is a non-zero 1-form such that

(2) $g(X,U) = A(X)$ for all vector fields X and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.

In such a case A is called the associated 1-form and U is called the basic vector field corresponding to the 1-form A. An n-dimensional manifold of this kind was denoted by the symbol $(PRS)_n$.

The object of this paper is to study a type of $(PRS)_n$ which is Quasi Einstein. The notion of a quasi Einstein manifold was introduced in a recent paper [2] by the first author and R. K. Maity who defined such a manifold as follows :

A non-flat Riemannian manifold $(M^n, g)(n>2)$ is said to be quasi Einstein if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition.

(3) $S(X,Y) = ag(X,Y) + bB(X)B(Y)$ where a, b are scalars of which $b \neq 0$ and B is a non-zero 1-form such that

(4) $g(X,V) = B(X)\ \forall\ X$

V being a unit vector field.

In such a case, a, b are called the associated scalars, B is called the associated 1-form and V is called the generator of the manifold. A Riemannian manifold (M^n, g)

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($n > 2$) satisfying (1), (2), (3) and (4) shall be called a Quasi Einstein Pseudo Ricci symmetric manifold and such an n-dimensional manifold shall be denoted by the symbol $\chi(PRS)_n$. Thus in a $\chi(PRS)_n$ there are two associated scalars a, b, two associated 1-forms A, B and two vector fields U and V. In such a case, a, b shall be called the associated scalars, A and B shall be called the main and auxiliary 1-forms and U, V shall be called the basic vector field and the generator respectively.

In this paper it is shown that in case of a $\chi(PRS)_n$, the sum of the associated scalars is zero, the auxiliary 1-form is a scalar multiple of the main 1-form and the generator is a scalar multiple of the basic vector field. Since the sum of the associated scalars is zero, only one of them, say a deserves consideration. It is to be noted that a cannot be zero because $b \neq 0$. Also a cannot be a non-zero constant for otherwise the scalar curvature will be zero and therefore a will be zero. The significance of a is pointed out by showing that a is an eigenvalue of the Ricci tensor S having multiplicity n–1. The nature of the generator is determined by showing that it is a proper concircular vector field [3]. A Riemannian manifold is said to be conformally conservative [4] if the divergence of its conformal curvature tensor C is zero. An interesting property of a $\chi(PRS)_n$ is obtained by showing that a $\chi(PRS)_n$ is conformally conservative. Finally, the meaningfulness of the study of $\chi(PRS)_n$ is pointed out by proving that a conformally flat $(PRS)_n$ is a $\chi(PRS)_n$.

Thus the results of a $\chi(PRS)_n$ hold in a conformally flat $(PRS)_n$ but the converse is not, in general, true. For example, it is known [5] that the relation $R(X,Y) \cdot S = 0$ holds in a conformally flat $(PRS)_n$, but this relation does not, in general hold in a $\chi(PRS)_n$.

1. PRELIMINARIES

Let a, b be the associated scalars, A, B be the main and auxiliary 1-forms and U, V be the basic vector field and the generator respectively of a $\chi(PRS)_n$ ($n>3$). Then (1), (2), (3) and (4) will hold.

Contracting (3) over X and Y we get.

$$r = na + b \quad [\because g(V,V) = 1] \tag{1.1}$$

where r is the scalar curvature.

Next, contracting (1) over Y and Z we get

$$dr(X) = 2rA(X) + 4S(X,U) \tag{1.2}$$

Again from (1) we have

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = A(X)S(Y,Z) - A(Z)S(Y,X) \tag{1.3}$$

Now, contracting (1.3) over Y and Z we get

$$dr(X) = 2rA(X) - 2S(X,U) \tag{1.4}$$

From (1.2) and (1.4) it follows that

(1.5) $S(X,U) = 0$

Therefore from (1.4) we get

(1.6) $dr(X) = 2rA(X)$

Since $ddr = 0$, it follows from (1.6) that

(1.7) $rdA(X,Y) = 0$

These formulas will be used in the sequel.

2. RELATIONS BETWEEN THE ASSOCIATED SCALARS, BETWEEN THE MAIN AND AUXILIARY 1-FORMS AND BETWEEN THE BASIC VECTOR FIELD AND THE GENERATOR OF A $\chi(PRS)_n$.

From (3) it follows that

$a\, g(X, U) + b\, B(X)B(U) = S(X,U)$

or $a\, A(X) + b\, B(X)B(U) = 0$

Hence $B(U) \neq 0$, because $A(X) \neq 0$, $a \neq 0$

Therefore,

(2.1) $B(X) = -\frac{a}{b}\frac{A(X)}{B(U)}$ $[\because b \neq 0,\ B(U) \neq 0]$

From (2.1) we get $B(V) = -\frac{a}{b}\frac{A(V)}{B(U)}$

or, (2.2) $-\frac{a}{b}\frac{A(V)}{B(U)}=1$ $[\because B(V) =1]$

But $B(U) = g(U,V) = A(V)$

Hence from (2.2) we get $-\frac{a}{b} = 1$

Therefore,

(2.3) $a + b = 0$

In virtue of (2.3) it follows from (2.1) that

(2.4) $B(X) = \frac{1}{A(V)}A(X)$

We can express (2.4) as follows

$$g(X,V) = g\left(\frac{1}{A(V)}U,X\right) \forall \ X$$

Hence

$$(2.5) \qquad V = \frac{1}{A(V)}U$$

In virtue of (2.3), it follows from (1.1) that

$$(2.6) \qquad r = (n-1)a.$$

If a is a non-zero constant, then from (2.6) it follows that r is a non-zero constant. Hence from (1.6) it follows that $r = 0$. Therefore from (2.6) we get $a = 0$ which is inadmissible. Therefore a can neither be zero nor can be a non-zero constant.

Summing up we can state as follows :—

Theorem 1. In a $\chi(PRS)_n$ the sum of the associated scalars is zero, the auxiliary 1-form is a scalar multiple of the main 1-form and the generator is a scalar multiple of the basic vector field. Further, none of the associated scalars can be zero or a non-zero constant.

3. SIGNIFICANCE OF THE ASSOCIATED SCALARS

From (3) we get

$$\begin{aligned} S(X,V) &= a\ g(X,V) + b\ B(X)B(V) \\ &= a\ B(X) + b\ B(X) \qquad [\because B(V) = 1] \\ &= (a+b)\ B(X) \end{aligned}$$

Hence

$$(3.1) \qquad S(X,V) = 0 \qquad [\because a+b = 0]$$

From (3.1) it follows that 0 is an eigenvalue of the Ricci tensor S and V is an eigenvector corresponding to this eigenvalue.

Let W be any other vector orthogonal to V

Then $g(W,V) = 0$

or

$$(3.2) \qquad B(W) = 0$$

From (3) we get

$$S(X,W) = a\ g(X,W) + b\ B(X)B(W)$$

Hence

$$(3.3) \qquad S(X,W) = a\, g(X,W) \qquad [\text{by } (3.2)]$$

From (3.3) it follows that a is an eigenvalue of the Ricci tensor and W is an eigenvector corresponding to this eigenvalue. Since $a \neq 0$ the eigenvalues 0 and a of the Ricci tensor are different. In virtue of the fact that the manifold under consideration is n-dimensional and W is any vector orthogonal to V, it follows from a well known result of linear algebra [6] that the eigenvalue a is of multiplicity n-1. Hence the multiplicity of the eigenvalue 0 must be 1. This leads to the following theorem :

Theorem 2. In a $\chi(PRS)_n$ the Ricci tensor S has only two distinct eigenvalues, namely a and 0 of which the former is of multiplicity n–1 and the latter is simple.

4. PROPERTIES OF THE GENERATOR OF A $\chi(PRS)_n$

From (3) we get

$$(4.1) \qquad (\nabla_X S)(Y,Z) = da(X)\,[g(Y,Z) - B(Y)B(Z)] - a[(\nabla_X B)(Y)\, B(Z) + (\nabla_X B)(Z)\, B(Y)]$$

Again from (1) we get in virtue of (3) and (2.4)

$$(4.2) \qquad (\nabla_X S)(Y,Z) = 2a\, A(V)\, B(X)\, [\, g(Y,Z) - B(Y)B(Z)\,] + aA(V)B(Y)\,[g(X,Z) - B(X)\, B(Z)\,] + aA(V)B(Z)\,[\, g(Y,X) - B(Y)B(X)\,]$$

From (4.1) and (4.2) it follows that

$$(4.3) \qquad 2aA(V)B(X)\,[\, g(Y,Z) - B(Y)\, B(Z)\,] + aA(V)B(Y)\,[g(X,Z) - B(X)B(Z)] + aA(V)\, B(Z)\,[\, g(Y,X) - B(Y)\, B(X)\,]$$

$$= da(X)\,[\, g(Y,Z) - B(Y)B(Z)\,] - a[\,(\nabla_X B)(Y)B(Z) + (\nabla_X B)(Z)B(Y)\,]$$

Putting Z = V in (4.3) we get

$$(4.4) \qquad (\nabla_X B)(Y) = -A(V)g(X,Y) + A(X)B(Y) \qquad [\text{by } (2.4)]$$

We can express (4.4) as follows :

$$g(\nabla_X V, Y) = g\,[-A(V)X + A(X)V,\ Y\,]$$

Hence

$$(4.5) \qquad \nabla_X V = -A(V)X + A(X)V$$

Since $r \neq 0$ it follows from (1.7) that the 1-form A is closed. Also $A(V) \neq 0$. Hence from (4.5) we conclude that the vector field V is a proper concircular vector field. From (4.4) we get.

$$(4.6) \qquad (\nabla_X B)(Y) = (\nabla_Y B)(X)$$

We can write (4.6) as follows :

(4.7) $g(\nabla_X V, Y) = g(\nabla_Y V, X)$

Putting Y = V in (4.7) we get

$g(\nabla_X V, V) = g(\nabla_V V, X) \; \forall \; X$

Hence

(4.8) $g(\nabla_V V, X) = 0 \;\; \forall \; X \qquad [\because g(\nabla_X V, V) = 0]$

From (4.8) it follows that

(4.9) $\nabla_V V = 0$

This implies that the integral curves of V are geodesics.

Summing up we can state the following properties of the generator.

Theorem 3. In a $\chi(PRS)_n$ the generator is a proper concircular vector field whose integral curves are geodesics.

5. NATURE OF THE CONFORMAL CURVATURE TENSOR IN A $\chi(PRS)_n$

From (2.6) we get

(5.1) $dr(X) = (n-1)da(X)$

Hence $2rA(X) = (n-1)da(X)$ [by (1.6)]

Therefore

(5.2) $2aA(X) = da(X)$ [by (2.6)]

Let

(5.3) $$H(X,Y,Z) = (\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) - \frac{1}{2(n-1)}[\, dr(X)g(Y,Z) - dr(Z)g(Y,X\,]$$

It is known [7] that in a Riemannian manifold $(M^n, g)(n>3)$ the conformal curvature tensor C is conformally conservative i.e. C is of zero divergence if and only if $H(X,Y,Z) = 0$

From (4.2) we have

(5.4) $(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = a\,[\, A(X)g(Y,Z) — A(Z)g(Y,X)\,]$ [by (2.4)]

Again

(5.5) $$\frac{1}{2(n-1)}[\, dr(X)g(Y,Z) - dr(Z)g(Y,X\,]$$

$$= a\,[\, A(X)g(Y,Z) — A(Z)g(Y,X)\,] \qquad [\text{ by } (5.2)\,]$$

From (5.4) and (5.5) we get

(5.6) $H(X,Y,Z) = 0$

This leads to the following theorem :—

Theorem 4. A $\chi(PRS)_n$ (n>3) is conformally conservative.

6. CONFORMALLY FLAT $(PRS)_n$ (n>3) AS $\chi(PRS)_n$ (n>3)

In this section it is shown that a conformally flat $(PRS)_n$ may be regarded as a $\chi(PRS)_n$.

Let A and U be the associated 1-form and the basic vector field of a conformally flat $(PRS)_n$ (n>3). Then it is known [1] that its curvature tensor 'R of type (0.4) is of the following form :

(6.1) $$'R(X,Y,Z,W) = \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + T(X)T(Z)g(Y,W) - T(Y)T(Z)g(X,W) + T(Y)T(W)g(X,Z) - T(X)T(W)g(Y,Z)]$$

where

(6.2) $$T(X) = \frac{A(X)}{\sqrt{A(U)}}$$

Let

(6.3) $g(X,P) = T(X) \ \forall \ X$

Then $g(X,P) = g\left(\frac{1}{\sqrt{A(U)}}U, X\right)$

Hence (6.4) $$P = \frac{1}{\sqrt{A(U)}}U$$

Therefore $g(P,P) = g(\frac{1}{\sqrt{A(U)}}U, \frac{1}{\sqrt{A(U)}}U) = \frac{1}{A(U)}g(U,U) = 1$

Hence P is a unit vector field.

Now contracting (6.1) over X and W we get

(6.5) $S(Y,Z) = c\, g(Y,Z) + d\, T(Y)T(Z)$

where

(6.6) $$c = \frac{r}{n-1},\ d = -\frac{r}{n-1},\ T(X) = g(X,P)$$

P being a unit vector field.

From (6.5) and (6.6) it follows that the manifold under consideration is quasi Einstein.

We can therefore state the following theorem :

Theorem 5. A conformally flat $(PRS)_n$ $(n>3)$ is a Quasi Einstein $(PRS)_n$.

This theorem shows that a result holding in a $\chi(PRS)_n$ holds also in a conformally flat $(PRS)_n$, but the converse is not, in general true. For example the relation $R(X,Y)\bullet S = 0$ holds in a conformally flat $(PRS)_n$ [5] but this relation does not, in general, hold in a $\chi(PRS)_n$.

In conclusion, we state that the theorem 5 shows why the study of Quasi Einstein $(PRS)_n$ is meaningful.

REFERENCES

1. M. C. Chaki, On Pseudo Ricci symmetric manifolds, Bulg, Journ, Phys, 15 (1988) No. 6, 526-531.
2. M. C. Chaki and R. K. Maity, On quasi Einstein manifolds, Publ. Math. Debrecen Vol. 57 (2000) 297-306.
3. J. A. Schouten, Ricci calculus, Springer Verlag Berlin, 1954, 389.
4. N. J. Hicks, Notes on Differential Geometry, Affiliated East-West Press Pvt. Ltd., 1969, 95.
5. M. C. Chaki and M. Tarafdar, On conformally flat Pseudo Ricci symmetric Manifolds, Periodica Math. Hungar 19 (1988) 209-215.
6. P. Shirokov, Tensor Calculus (Russian) Mathematics Literature 1934, 161-165.
7. L. P. Eisenhart, Riemannian Geometry, Princeton Univ. 1926, 91, (28.16).

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ON REGULAR BANDS*

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1. INTRODUCTION

Zhang L., Shum K. P. and Zhang R. H. introduced refined semilattice and described regular bands by using it (see [4]). Is refined semilattice decomposition of a regular band unique? We give an affirmative answer to this problem in this paper. Also, the homomorphism problem is investigated. As a special case, we point out when two regular bands are isomorphic. This leads to a way of producing all regular bands, up to isomorphism, with some fixed family of pairwise disjoint rectangular bands indexed by a semilattice. We illustrate by an example how to find, up to isomorphism, all regular bands based on certain family of rectangular bands.

1. PRELIMINARIES

In what follows, we use notations and terminologies of [1,2]. We begin by recalling some concepts.

Let S be a semigroup. If σ is a congruence on S, then We call the partition induced by σ a *congruence partition* on S, denoted by $\{S_d \mid d \in D\}$, or S_D, where each S_d is a σ-class and D is an index set. Combining the definition of refined semilattice in [5], we rewrite the definition of it in [4] as follows.

Let $\{S_\alpha \mid \alpha \in Y\}$ be a family of pairwise disjoint semigroups of type T indexed by a semilattice Y and $S = \bigcup_{\alpha \in Y} S_\alpha$. For each pair α, $\beta \in Y$ with $\alpha \geq \beta$, let $D(\alpha, \beta)$ be an index set and $\{S_{d(\alpha,\beta)} \mid d(\alpha,\beta) \in D(\alpha,\beta)\}$ a congruence partition on S_β. Let $\{S_{D(\alpha,\beta)} \mid \beta \in Y, \alpha \geq \beta\}$ be a family of congruence partitions such that for all $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$ and any $d(\alpha, \gamma) \in D(\alpha, \gamma)$, there exists $d(\beta,\gamma) \in D(\beta,\gamma)$ satisfying $S_{d(\alpha,\gamma)} \subseteq S_{d(\beta,\gamma)}$. Moreover, for any $d(\alpha,\beta) \in D(\alpha,\beta)$, let $\Phi_{d(\alpha,\beta)}$ be a homomorphism of S_α into $S_{d(\alpha,\beta)}$. Suppose that the following conditions are satisfied :

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(a) $\Phi_{d(\alpha,\alpha)}$ is the identity mapping of S_α for each $\alpha \in Y$, where $d(\alpha, \beta)$ is the unique element of $D(\alpha,\beta)$.

(b) For all $d(\alpha,\beta) \in D(\alpha,\beta)$ and all $d(\beta,\gamma) \in D(\beta,\gamma)$, there exists $d(\alpha,\gamma) \in D(\alpha,\gamma)$ such that $\Phi_{d(\alpha,\beta)}\Phi_{d(\beta,\gamma)} = \Phi_{d(\alpha,\gamma)}$ and $S_{d(\alpha,\beta)}\Phi_{d(\beta,\gamma)} \subseteq S_{d(\alpha,\gamma)}$.

(c) For each $d(\alpha\beta,\gamma) \in D(\alpha\beta, \gamma)$ and each $a_\alpha \in S_\alpha$, there exists $\bar{d}(\beta,\gamma) \in D(\beta,\gamma)$ such that $\{a_\alpha\Phi_{d(\alpha,\gamma)} \mid d(\alpha,\gamma) \in D(\alpha,\gamma)\} \cap S_{d(\alpha\beta,\gamma)} \subseteq S_{\bar{d}(\beta,\gamma)}$.

(d) For any $d(\alpha,\beta), d'(\alpha,\beta) \in D(\alpha,\beta)$, $a_\alpha \in S_\alpha$ and $b_\beta \in S_\beta$,

$$(a_\alpha\Phi_{d'(\alpha,\beta)})b_\beta \in S_{d(\alpha,\beta)} \Rightarrow (a_\alpha\Phi_{d'(\alpha,\beta)})b_\beta = (a_\alpha\Phi_{d(\alpha,\beta)})b_\beta$$

and

$$b_\beta(a_\alpha\Phi_{d'(\alpha,\beta)}) \in S_{d(\alpha,\beta)} \Rightarrow b_\beta(a_\alpha\Phi_{d'(\alpha,\beta)}) = b_\beta(a_\alpha\Phi_{d(\alpha,\beta)}).$$

For any $a_\alpha \in S_\alpha$ and $b_\beta \in S_\beta$, pick $d(\alpha,\alpha\beta) \in D(\alpha,\alpha\beta)$ and $d(\beta,\alpha\beta) \in D(\beta,\alpha\beta)$, and define

$$a_\alpha \, o \, b_\beta = (a_\alpha\Phi_{\bar{d}(\alpha,\alpha\beta)})(b\,\Phi_{\bar{d}(\beta,\alpha\beta)}),$$

Where $a_\alpha\Phi_{d(\alpha,\alpha\beta)} \in S_{\bar{d}(\beta,\alpha\beta)}$ and $b_\beta\Phi_{d(\beta,\alpha\beta)} \in S_{\bar{d}(\alpha,\alpha\beta)}$. With respect to the multiplication "o", S is a semigroup (see [4]). This semigroup is said to be a *refined semilattice* of S_α, Written as

$$[\,Y; S_\alpha; D(\alpha,\beta),\ S_{d(\alpha,\beta)}, \Phi_{d(\alpha,\beta)}].$$

It is well known that every band is a semilattice of rectangular bands. A band B is called a regular band if for all $e, f, g \in B$, we always have $efge = efege$. Zhang L. Shum K. P. and Zhang R. H. concretely described regular bands by applying refined semilattice (see [4]) as follows.

Lemma 1.1. *A semigroup B is a regular band if and only if B is a refined semilattice of rectangular bands.* □

We refer to a band B as a *normal band* if $e,f,g \in B$ implies that $efge = egfe$. We quote an alternative description for normal bands from [1].

Lemma 1.2. *A semigroup B is a normal band if and only if it is a strong semilattice of rectangular bands.* □

Just as every normal band is also a regular one, strong semilattice is a special case of refined semilattice in which each $D(\alpha,\beta)$ is singleton.

2. THE UNIQUENESS PROBLEM OF REFINED SEMILATTICE DECOMPOSITION

In this section, we show that every regular band has a unique refined semilattice decomposition. This also leads to the fact that all normal bands have unique strong semilattice decomposition. First we rewrite some results in [4] as follows.

Let $I = \bigcup_{\alpha \in Y} I_\alpha$ be a regular band, where Y is a semilattice and $\{I_\alpha \mid \alpha \in \mathrm{Y}\}$ is a family of rectangular bands. For any $\alpha, \beta \in Y$ with $\alpha \geq \beta$, some fixed $i_\alpha \in I_\alpha$ and $j_\beta \in I_\beta$, let $I(i_\alpha, j_\beta) = \{i \in I_\beta \mid i_\alpha i i_\alpha = i_\alpha j_\beta i_\alpha\}$, then we have

Lemma 2.1. *$\{I(i_\alpha, j_\beta) \mid j_\beta \in I_\beta\}$ is a congruence partition on I_β.* □

Denote the above congruence partition by $\{I_{d(\alpha\beta)} \mid d(\alpha,\beta) \in D(\alpha,\beta)\}$for any $\alpha, \beta \in Y$ with $\alpha \geq \beta$, where $D(\alpha,\beta)$ is an index set. Define $\Phi_{d(\alpha,\beta)} : I_\alpha \rightarrow I_{d(\alpha,\beta)}$ by $i_\alpha \Phi_{d(\alpha,\beta)} = i_c I_{d(\alpha,\beta)} i_\alpha$. Thus with these hypotheses, we obtain

Lemma 2.2. *$\Phi_{d(\alpha,\beta)}$ is a homomorphism of I_α into $I_{d(\alpha,\beta)}$.* □

Lemma 2.3. *$[\, Y ; I_\alpha ; D(\alpha,\beta), I_{d(\alpha,\beta)}, \Phi_{d(\alpha,\beta)}]$ is a refined semilattice decomposition of I.* □

Using the above lemmas, we have

Theorem 2.4. *A regular band can be uniquely expressed as a refined semilattice of rectangular bands.*

Proof. Let $I = [\, Y ; I_\alpha ; D(\alpha,\beta), I_{d(\alpha,\beta)}, \Phi_{d(\alpha,\beta)}]$. To prove the uniqueness of the decomposition, we need to show the refined semilattice of I coincides with that described in Lemma 2.3. That is to show.

(1) The congruence partitions coincide with that described in Lemma 2.1. That is, for any $d(\alpha,\beta) \in D(\alpha,\beta)$, there exist some $i_\alpha \in I_\alpha$ and $I_\gamma \in I_\beta$ such that $I_{d(\alpha,\beta)} = \{i \in I_\beta \mid i_\alpha i i_\alpha = i_\alpha j_\beta i_\alpha\}$. In fact, if $i_\beta \in I_{d(\alpha,\beta)}$, then for any $i_\alpha \in I_\alpha$, $i_\alpha i_\beta i_\alpha = i_\alpha \Phi_{d(\alpha,\beta)} = i_\alpha (i_\alpha \Phi_{d(\alpha,\beta)}) i_\alpha$, in other words, $i_\beta \in \{i \in I_\beta \mid i_\alpha i i_\alpha = i_\alpha (i_\alpha \Phi_{d(\alpha,\beta)}) i_\alpha\}$. Furthermore, if $i_\beta \in \{i \in I_\beta \mid i_\alpha i i_\alpha = i_\alpha (i_\alpha \Phi_{d(\alpha,\beta)}) i_\alpha\}$, then we have

$$i_\beta i_\alpha = i_\beta i_\alpha i_\beta i_\alpha = i_\beta (i_\alpha \Phi_{d(\alpha,\beta)}) \text{ and } i_\alpha i_\beta = i_\alpha i_\beta i_\alpha i_\beta = (i_\alpha \Phi_{d(\alpha,\beta)}) i_\beta.$$

Also, if $i_\beta \in I_{d'(\alpha,\beta)}$, then $i_\beta i_\alpha = i_\beta (i_\alpha \Phi_{d'(\alpha,\beta)})$ and $i_\alpha i_\beta = (i_\alpha \Phi_{d'(\alpha,\beta)}) i_\beta$ so that $i_\alpha i_\beta i_\alpha = (i_\alpha i_\beta)(i_\beta i_\alpha) = i_\alpha \Phi_{d'(\alpha,\beta)}$ which yields $d'(\alpha,\beta) = d(\alpha,\beta)$. Thus, from the multiplication defined on a refined semilattice we know that $i_\beta \in I_{d(\alpha,\beta)}$.

(2) For any $d(\alpha,\beta) \in D(\alpha,\beta)$, the homomorphism $\Phi_{d(\alpha,\beta)}$ of I_α into $I_{d(\alpha,\beta)}$, agrees with that in Lemma 2.2. Indeed, for any $i_\alpha \in I_\alpha$ and $i_\beta \in I_{d(\alpha,\beta)}$, we have

$$i_\alpha \Phi_{d(\alpha,\beta)} = (i_\alpha \Phi_{d(\alpha,\beta)}) i_\beta (i_\alpha \Phi_{d(\alpha,\beta)}) = i_\alpha i_\beta i_\alpha,$$

So that $i_\alpha \Phi_{d(\alpha,\beta)} = i_\alpha I_{d(\alpha,\beta)} i_\alpha$. This implies the uniqueness as required. □

The result below can be immediately obtained from Lemma 1.2 and Theorem 2.2.

Corollary 2.5. *A normal band can be uniquely expressed as a strong semilattice of rectangular bands.* □

3. HOMOMORPHISMS BETWEEN REGULAR BANDS

Hall solved the isomorphism problem for orthodox semigroups in [3]. In this section, the same problem for regular bands is investigated as a special case of homomorphism problem. This also leads to another way to describe homomorphisms between a special type of orthodox semigroups. The solution of this problem is again based on the refined semilattice of a regular band. Throughout the section, we fix two regular bands

$$I = [\,Y\,; I_\alpha\,; D(\alpha,\beta),\ I_{d(\alpha,\beta)}, \Phi_{d(\alpha,\beta)}]$$

and

$$J = [\,Y'\ ;\ J_{\alpha'}\ ; E(\alpha',\beta'),\ J_{e(\alpha',\beta')}, \psi_{e(\alpha',\beta')}\,].$$

Definition 3.1. Let $\phi : Y \to Y'$ and $\xi_\alpha\ : I_\alpha \to J_{\alpha\phi}$ for any $\alpha \in Y$ be homomorphisms and $\psi_{\alpha,\beta} : D(\alpha,\beta) \to E(\alpha\phi,\ \beta\phi)$ for any $\alpha,\beta \in Y$ with $\alpha \geq \beta$ a mapping such that for any $d(\alpha,\beta) \in D(\alpha,\beta)$,

$I_{d(\alpha,\beta)}\xi_\beta \subseteq J_{d\psi(\alpha\phi,\beta\phi)}$ and $\Phi_{d(\alpha,\beta)}\xi_\beta = \xi_\alpha\psi_{d\psi(\alpha\phi,\beta\phi)}$,

where $d\psi(\alpha\phi,\ \beta\phi)$ stands for $(d(\alpha,\beta))\psi_{\alpha,\beta}$. Then we call $(\phi\ ;\ \{\xi_\alpha\}\ ;\ \{\psi_{\alpha,\beta}\})$ an *rh-system* for I and J. For any $i \in I$, define a mapping $\xi = \xi\ (\phi\ ;\{\xi_\alpha\}\ ;\ \{\psi_{\alpha,\beta}\})$ by

$$i\xi = i\xi_\alpha \text{ if } i \in I_\alpha.$$

Particularly, if both $D(\alpha,\beta)$ and $E\ (\alpha',\beta')$ are singleton for I and J, I and J turn to be normal bands. Denote such I and J respectively by $I_N = \{Y\,;\ I_\alpha,\ \Phi_{\alpha,\beta}\ \}$ and $J_N = \{\,Y'\,;\ I_{\alpha'}, \Psi_{\alpha',\beta'}\,\}$.

Definition 3.2. Let $\phi : Y \to Y'$ and $\xi_\alpha : I_\alpha \to J_{\alpha\phi}$ for any $\alpha \in Y$ be homomorphisms such that $\Phi_{\alpha,\beta}\xi_\beta = \xi_\alpha\Psi_{\alpha\phi,\beta\phi}$ for any $\alpha,\ \beta \in Y$ with $\alpha \geq \beta$. Then we call $(\phi\,;\ \{\xi_\alpha\})$ an *nh-system* for I_N an J_N. Define a mapping $\zeta = \zeta(\phi\,;\ \{\xi_\alpha\})$ of I_N by

$$i\zeta = i\xi_\alpha \text{ if } i \in I_\alpha.$$

Now we proceed to present our results, first we point out that each corollary below is immediate as we have Lemma 1.2.

Theorem 3.3. *$\xi = \xi(\phi\,;\ \{\xi_\alpha\}\ ;\ \{\psi_{\alpha,\beta}\})$ is a homomorphism of I into J for any rh-system $(\,\phi\,;\ \{\xi_\alpha\}\ ;\ \{\psi_{\alpha,\beta}\})$ for I and J. Conversely, if ξ is a homomorphism of I into J, then there exists a unique rh-system $(\,\phi\,;\ \{\xi_\alpha\}\ ;\ \{\psi_{\alpha,\beta}\})$ such that $\xi = \xi(\ \phi\,;\{\xi_\alpha\}\ ;\{\psi_{\alpha,\beta}\})$.*

Proof. (1) *Direct part.* Suppose that $(\phi\,;\{\xi_\alpha\}\ ;\ \{\psi_{\alpha,\beta}\})$ is an rh-system for I and J. For any $i_\alpha \in I_\alpha$ and $i_\beta \in I_\beta$, if $i_\alpha\Phi_{d(\alpha,\alpha\beta)} \in I_{\bar{d}(\beta,\alpha\beta)}$, then $i_\alpha\Phi_{d(\alpha,\alpha\beta)}\xi_{\alpha\beta} \in J_{\bar{d}\psi(\beta\phi,(\alpha\beta)\phi)}$ so that $i_\alpha\xi_\alpha\psi_{d(\alpha,\alpha\beta)} \in J_{\bar{d}\psi(\beta\phi,(\alpha\beta)\phi)}$. Similarly $i_\beta\Phi_{d(\beta,\alpha\beta)} \in I_{\bar{d}(\alpha,\alpha\beta)}$ implies that $i_\beta\xi_\beta\psi_{d(\beta,\alpha\beta)} \in J_{\bar{d}\psi(\alpha\phi,(\alpha\beta)\phi)}$. Then by applying (c) in the definition of refined semilattice, for any $i_\alpha \in I_\alpha$ and $i_\beta \in I_\beta$, we have

$$\begin{aligned}(i_\alpha i_\beta)\xi &= (i_\alpha \Phi_{\bar{d}(\alpha,\alpha\beta)} i_\beta \Phi_{\bar{d}(\beta,\alpha\beta)})\xi_{\alpha\beta} \\ &= (i_\alpha \Phi_{\bar{d}(\alpha,\alpha\beta)} \xi_{\alpha\beta})(i_\beta \Phi_{\bar{d}(\beta,\alpha\beta)} \xi_{\alpha\beta}) \\ &= (i_\alpha \xi_\alpha \Psi_{\bar{d}\psi(\alpha\phi,(\alpha\beta)\phi)})(i_\beta \xi_\beta \Psi_{\bar{d}\psi(\beta\phi,(\alpha\beta)\phi)}) \\ &= (i_\alpha \xi_\alpha)(i_\beta \xi_\beta) \\ &= (i_\alpha \xi)(i_\beta \xi),\end{aligned}$$

so that ξ is a homomorphism.

(2) *Converse*. Take any homomorphism $\xi : I \to J$. Define ϕ by $\alpha' = \alpha\phi$ if and only if $j_{\alpha'} = i_\alpha \xi$ for any $\alpha \in Y$ and $i_\alpha \in I_\alpha$. To see that ϕ is a mapping of Y into Y', we need to show that ϕ is independent on the choice of i_α. Suppose that $i'_\alpha \in I_\alpha$ and $i'_\alpha \xi = j'_{\beta'}$, then

$$j_{\alpha'} = i_\alpha \xi = (i_\alpha i'_\alpha i_\alpha)\xi = (i_\alpha \xi)(i'_\alpha \xi)(i_\alpha \xi) = j_{\alpha'} j'_{\beta'} j_{\alpha'}$$

so that $\alpha' \leq \beta'$. By symmetry we deduce that $\beta' \leq \alpha'$, so $\alpha' = \beta'$. Furthermore, if $i_\alpha \xi = j_{\alpha'}$ and $i_\beta \xi = j_{\beta'}$ for some $i_\alpha \in I_\alpha$, and $i_\beta \in I_\beta$, then $i_\alpha i_\beta \in I_{\alpha\beta}$ and $(i_\alpha i_\beta)\xi = (i_\alpha \xi)(i_\beta \xi) = j_{\alpha'} j_{\beta'} \in J_{\alpha'\beta'}$. So we have $(\alpha\beta)\phi = (\alpha\phi)(\beta\phi)$, in other words, ϕ is an homomorphism.

Put $\xi_\alpha = \xi | I_\alpha$. Combining the above argument we can see that ξ_α is an homomorphism of I_α into $J_{\alpha\phi}$.

From Lemma 2.1–2 and the proof of Theorem 2.4 we know that for any $d(\alpha,\beta) \in D(\alpha,\beta)$, $I_{d(\alpha,\beta)} = \{\mathrm{i} \in I_\beta | \, i_\alpha i i_\alpha = i_\alpha \Phi_{d(\alpha,\beta)}\}$ for any $i_\alpha \in I_\alpha$, and that $\{j \in J_{\beta\phi} | \, (i_\alpha \xi) j (i_\alpha \xi) = i_\alpha \Phi_{d(\alpha,\beta)} \xi\}$ is a congruence partition block of $J_{\beta\phi}$, written as $J_{e(\alpha\phi,\beta\phi)}$. It is obvious that $\psi_{\alpha,\beta} : D(\alpha,\beta) \to E(\alpha\phi,\beta\phi)$ defined by $e(\alpha\phi,\beta\phi) = (d(\alpha,\beta))\psi_{\alpha,\beta}$ is a mapping, and we denote $(d(\alpha,\beta))\psi_{\alpha,\beta}$ by $d\psi(\alpha\phi,\beta\phi)$. It is also evident that $I_{d(\alpha,\beta)}\xi = I_{d(\alpha,\beta)}\xi_\beta \subseteq J_{d\psi(\alpha\phi,\beta\phi)}$.

To complete the proof of this part, we also need to show that for any $d(\alpha,\beta) \in D(\alpha,\beta)$, $\Phi_{d(\alpha,\beta)}\xi_\beta = \xi_\alpha \Psi_{d\psi(\alpha\phi,\beta\phi)}$. In fact, we can see this from the following argument. For each $i_\alpha \in I_\alpha$ and $i_\beta \in I_{d(\alpha,\beta)}$,

$$i_\alpha \Phi_{d(\alpha,\beta)} \xi_\beta = (i_\alpha i_\beta i_\alpha)\xi = (i_\alpha \xi_\alpha)(i_\beta \xi_\beta)(i_\alpha \xi_\alpha) = i_\alpha \xi_\alpha \Psi_{d\psi(\alpha\phi,\beta\phi)}.$$

Also, $\Phi_{d(\alpha,\beta)}\xi_\beta$ and $\xi_\alpha \Psi_{d\psi(\alpha\phi,\beta\phi)}$ have the same domain so that $\Phi_{d(\alpha,\gamma)}\xi_\beta = \xi_\gamma \Psi_{d\psi(\alpha\phi,\beta\phi)}$. In addition, from the above, we can see that ϕ, $\{\xi_\alpha\}$ and $\{\psi_{\alpha,\beta}\}$ are uniquely determined by ξ. □

Corollary 3.4. *If* $(\phi\,;\{\xi_\alpha\})$ *is an nh-system for* I_N *and* $\mathrm{J}_{\mathrm{N}'}$, *then* $\zeta = \zeta(\phi\,;\{\xi_\alpha\})$ *is a homomorphism of* I_N *into* $\mathrm{J}_{\mathrm{N}'}$. *Conversely, if* ζ *is homomorphism of* I_N *into* $\mathrm{J}_{\mathrm{N}'}$, *then there exists a unique nh-system* $(\phi\,;\{\xi_\alpha\})$ *such that* $\zeta = \zeta(\phi\,;\{\xi_\alpha\})$. □

Theorem 3.5. *The homomorphism* $\xi = \xi(\phi; \{\xi_\alpha\}; \{\psi_{\alpha,\beta}\})$ *of* I *into* J *is injective if and only if so are* ϕ, ξ_α *for any* $\alpha \in Y$ *and* $\phi_{\alpha,\beta}$ *for any* $\alpha, \beta \in Y$ *with* $\alpha \geq \beta$.

Proof. (1) *Necessity.* Suppose that $\xi = \xi(\phi; \{\xi_\alpha\}; \{\psi_{\alpha,\beta}\})$ is monomorphism of I into J. Then for any $\alpha \in Y$, $\xi\big|_{I_\alpha}$ is injective, that is, ξ_α is injective.

Pick $\alpha, \beta, \in Y$ and assume that $\alpha\phi = \beta\phi$. If $\alpha \neq \beta$, then both α and β are not equal to $\alpha\beta$. Without loss of generality, suppose that $\alpha \neq \alpha\beta$, then for any $i_1 \in I_\alpha$ and $i_2 \in I_\beta$, we have $(i_1i_2i_1)\xi = (i_1\xi)(i_2\xi)(i_1\xi) = i_1\xi$ since $i_1\xi, i_2\xi \in J_{\alpha\phi} = J_{\beta\phi}$. However, $i_1i_2i_1 \in I_{\alpha\beta}$ and $i_1 \in i_\alpha$ so that $i_1 \neq i_1i_2i_1$ which yields that ξ maps at least two distinct elements to $i_1\xi$. This is a contradiction. It follows that $\alpha = \beta$, so ϕ is also injective.

For any $\alpha, \beta, \in Y$ with $\alpha \geq \beta$, and $d(\alpha,\beta), d'(\alpha,\beta) \in D(\alpha,\beta)$, suppose that $d(\alpha,\beta) \neq d'(\alpha,\beta)$. Thus, for any $i_\alpha \in I_\alpha$ and $i_\beta \in I_{d(\alpha,\beta)}$, we have $i_\alpha i_\beta i_\alpha = i_\alpha \Phi_{d(\alpha,\beta)} \neq i_\alpha \Phi_{d'(\alpha,\beta)}$ so that $(i_\alpha\xi)(i_\beta\xi)(i_\alpha\xi) = i_\alpha\Phi_{d(\alpha,\beta)}\xi \neq i_\alpha\Phi_{d'(\alpha,\beta)}\xi$ since ξ is injective. It follows from the proof of Theorem 3.3 that $J_{d\psi(\alpha\phi,\beta\phi)} \neq J_{d'\psi(\alpha\phi,\beta\phi)}$, so we obtain that $d\psi(\alpha\phi,\beta\phi) \neq d'\psi(\alpha\phi,\beta\phi)$. Therefore, each $\psi_{\alpha,\beta}$ is injective.

(2) *Sufficiency.* That ξ is injective follows directly from the fact that ξ_α is injective for each $\alpha \in Y$. □

Corollary 3.6. *The homomorphism* $\zeta = \zeta(\phi; \{\xi_\alpha\})$ *of* I_N *into* J_N *is injective if and only if so are* ϕ *and* ξ_α *for any* $\alpha \in Y$.

Theorem 3.7. *If the homomorphism* $\zeta = \xi(\phi; \{\xi_\alpha\}; \{\psi_{\alpha,\beta}\})$ *is surjective, then so is* ϕ. *However,* ξ_α *with* $\alpha \in Y$ *and* $\psi_{\alpha,\beta}$ *for* $\alpha,\beta \in Y$ *with* $\alpha \geq \beta$ *are not necessarily surjective.*

Proof. For any $\alpha' \in Y'$, and $j \in J_{\alpha'}$, there exists $i \in I_\alpha$ for some $\alpha \in Y$ such that $i\xi = j$. It follows from the proof of Theorem 3.3 that $\alpha\phi = \alpha'$. Therefore, ϕ is surjective.

The last assertion of the result can be seen from the following example. □

Example 1. Let $I_1 = \{$a, b, c, d$\}$ and $I_2 = \{e, f, g\}$ are two disjoint semigroups with multiplications defined respectively by the following Cayley tables.

	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	c
d	d	d	d	d

	e	f	g
e	e	f	g
f	f	f	f
g	g	g	g

Then one can see that both I_l and I_2 are regular bands. Let $I_\alpha = \{a\}$, $I_\beta = \{b\}$ and $I_\gamma = \{c,d\}$ with $\alpha \geq \beta \geq \gamma$, then from the table we see that $I_\alpha \cup I_\beta \cup I_\gamma$ is the semilattice decomposition of I_l. Similarly, let $I_{\alpha'} = \{e\}$ and $I_{\beta'} = \{f,g\}$ with $\alpha' \geq \beta'$, we obtain that $I_{\alpha'} \cup I_{\beta'}$ is the semilattice decomposition of I_2. Define a surjection ξ of I_l into I_2 by

$$a \mapsto \mathrm{e},\ \mathrm{b} \mapsto \mathrm{f},\ \mathrm{c} \mapsto \mathrm{g},\ \mathrm{d} \mapsto \mathrm{f}.$$

We can see by routine calculation that ξ is a homomorphism. However, $\xi\big|_{I_\beta}$ is not a surjection of I_β into $I_{\beta'}$. Moreover, for the refined semilattice decomposition of I_l and I_2, We can obtain that $\psi_{\alpha,\beta}$ described as in Definition 3.1 and Theorem 3.3 is not surjective.

Corollary 3.8. *If the homomorphism $\zeta = \zeta(\phi\,;\{\xi_\alpha\})$ is surjective, then so is ϕ. However, ξ_α with $\alpha \in Y$ is not necessarily surjective.* □

To this corollary, we only verify the last assertion by the following example.

Example 2. Let $I_1 = \{a,b,c\}$ be a semigroup with the multiplication defined by the Cayley table as follows.

	a	*b*	*c*
a	*a*	*c*	*c*
b	*b*	*b*	*b*
c	*c*	*c*	*c*

Then it is easily seen that I_1 is a normal band. Let $I_\alpha = \{a\}$ and $I_\beta = \{b,c\}$ with $\alpha \geq \beta$, then $I_\alpha \cup I_\beta$ is the semilattice decompositon of I_1. Let $I_2 = \{d,e\}$ be a left zero semigroup such that $I_1 \cap I_2 = \phi$. Define a mapping ζ of I_1 into I_2 by

$$a \mapsto \mathrm{d},\ \mathrm{b} \mapsto \mathrm{e},\ \mathrm{c} \mapsto \mathrm{d}.$$

It is obvious that ζ is a homomorphism. However, $\zeta\big|_{I_\alpha}$ is not a surjection of I_α into I_2.

Theorem 3.9. *The homomorphism $\xi = \xi(\phi\,;\{\xi_\alpha\}\,;\{\psi_{\alpha,\beta}\})$ of I into J is bijective if and only if so are ϕ, ξ_α for any $\alpha \in Y$ and $\phi_{\alpha,\beta}$ for any $\alpha,\beta \in Y$ with $\alpha \geq \beta$.*

Proof. (1) *Necessity.* To complete the proof of this part, we only need to show that ξ_α with $\alpha \in Y$ and $\psi_{\alpha,\beta}$ for $\alpha,\beta \in Y$ with $\alpha \geq \beta$ are surjective. It suffices to show that $I_{d(\alpha,\beta)}\xi_\beta = J_{d\psi(\alpha\phi,\beta\phi)}$ for any $d(\alpha,\beta) \in D(\alpha,\beta)$.

Suppose that $j \in J_{d\psi(\alpha\phi,\beta\phi)}$, then there exists $i \in I_\beta$ such that $i\xi = j$. So $(\mathrm{i}_x\xi)(\mathrm{i}\xi)(\mathrm{i}_\alpha\xi) = i_\alpha\Phi_{d(\alpha,\beta)}\xi$, that is, $(i_\alpha i i_\alpha)\xi = i_\alpha\Phi_{d(\alpha,\beta)}\xi$. This implies that $i_\alpha i i_\alpha = i_\alpha\Phi_{d(\alpha,\beta)}$ since ξ is injective. Therefore we have obtained that $i \in I_{d(\alpha,\beta)}$ and $j = i\xi = i\xi_\beta$. so that $J_{d\psi(\alpha\phi,\beta\phi)} \subseteq I_{d(\alpha,\beta)}\xi_\beta$. It follows from Definition 3.1 that $J_{d\psi(\alpha\phi,\beta\phi)} = I_{d(\alpha,\beta)}\xi_\beta$.

(2) *Sufficiency.* This is obvious. □

Corollary 3.10. *The homomorphism* $\zeta = \zeta\,(\phi\,;\{\xi_\alpha\})$ *of* I_N *into* J_N *is bijective if and only if so are* ϕ *and* ξ_α *for any* $\alpha \in Y$. □

4. A CONSTRUCTED EXAMPLE

In general, we have not been able to obtain a universal way to find, up to isomorphism, all regular bands with a family of pairwise disjoint rectangular bands indexed by a semilattice. Here, we just illustrate by an example about this problem.

Let $Y = \{\alpha,\beta\}$ with $\alpha \geq \beta$ be a semilattice. $I_\alpha = \{a,b\}$ and $I_\beta = \{c, d, e\}$ are two disjoint left zero semigroups. We know that every mapping of a left semigroup into another is a homomorphism, and it is easy to see that all partitions on a left zero semigroup are congruence partition. Also, since Y is a 2-element semilattice, it is evident that for any partition $\{I_{d(\alpha,\beta)} \mid d(\alpha,\beta) \in D(\alpha,\beta)\}$ and any corresponding homomorphism family $\{(\Phi_{d(\alpha,\beta)} : I_\alpha \to I_{d(\alpha,\beta)}) \mid d(\alpha,\beta) \in D(\alpha,\beta)\}$, we can always establish a refined semilattice of $\{I_\alpha, I_\beta\}$.

Now we proceed to find all distinct regular bands, up to isomorphism, established on $\{I_\alpha, I_\beta\}$.

Case 1. $D(\alpha,\beta) = \{d_1(\alpha,\beta), d_2(\alpha,\beta), d_3(\alpha,\beta)\}$. Without loss of generality, suppose that $I_{d_1(\alpha,\beta)} = \{c\}$, $I_{d_2(\alpha,\beta)} = \{d\}$ and $I_{d_3(\alpha,\beta)} = \{e\}$. Then we have the following unique homomorphism family :

$$\Phi_{d_1(\alpha,\beta)} : a \mapsto \mathrm{c},\ \mathrm{b} \mapsto \mathrm{c}\,;$$

$$\Phi_{d_2(\alpha,\beta)} : a \mapsto \mathrm{d},\ \mathrm{b} \mapsto \mathrm{d}\,;$$

$$\Phi_{d_3(\alpha,\beta)} : a \mapsto \mathrm{e},\ \mathrm{b} \mapsto \mathrm{e}.$$

Thus we obtain a regular band with Cayley table as follows :

	a	*b*	*c*	*d*	*e*
a	*a*	*a*	*c*	*d*	*e*
b	*b*	*b*	*c*	*d*	*e*
c	*c*	*c*	*c*	*c*	*c*
d	*d*	*d*	*d*	*d*	*d*
e	*e*	*e*	*e*	*e*	*e*

It is the unique regular band (in fact, left regular band) in this case.

Case 2. $D(\alpha,\beta) = \{d_1(\alpha,\beta), d_2(\alpha,\beta)\}$. If $I_{d_1(\alpha,\beta)} = \{c,d\}$, and $I_{d_2(\alpha,\beta)} = \{e\}$, we also have two cases.

(1) The homomorphism of I_α into $I_{d_1(\alpha,\beta)}$ is not onto. Define the homomorphism family as follows :

$$\Phi_{d_1(\alpha,\beta)} : a \mapsto \mathrm{c},\ \mathrm{b} \mapsto \mathrm{c};$$

$$\Phi_{d_2(\alpha,\beta)} : a \mapsto \mathrm{e},\ \mathrm{b} \mapsto \mathrm{e}.$$

Then we get a regular band with the following Cayley table :

.	*a*	*b*	*c*	*d*	*e*
a	*a*	*a*	*c*	*c*	*e*
b	*b*	*b*	*c*	*c*	*e*
c	*c*	*c*	*c*	*c*	*c*
d	*d*	*d*	*d*	*d*	*d*
e	*e*	*e*	*e*	*e*	*e*

Suppose that the homomorphism family is as follows :

$$\Psi_{d_1(\alpha,\beta)} : a \mapsto \mathrm{d},\ \mathrm{b} \mapsto \mathrm{d};$$

$$\Psi_{d_2(\alpha,\beta)} : a \mapsto \mathrm{e},\ \mathrm{b} \mapsto \mathrm{e}.$$

We know that the following mapping of I_β into itself is an automorphism of I_β :

$$\phi_\beta : \mathrm{c} \mapsto \mathrm{d},\ \mathrm{d} \mapsto \mathrm{c},\ \mathrm{e} \mapsto \mathrm{e}.$$

Moreover, we have $\Phi_{d_i(\alpha,\beta)}\Phi_\beta = \psi_{d_i(\alpha,\beta)}$ for i = 1, 2. So $[Y; I_\alpha; D(\alpha,\beta), I_{d(\alpha,\beta)}, \Phi_{d(\alpha,\beta)}]$ is isomorphic to $[Y; I_\alpha; D(\alpha,\beta), I_{d(\alpha,\beta)}, \Psi_{d(\alpha,\beta)}]$ by Theorem 3.9. Therefore, we can only find one regular band in this case up to isomorphism.

(2) The homomorphism of I_α into $I_{d_1(\alpha,\beta)}$ is onto. We also can show that there is only one regular band up to isomorphism in this case which has the following Cayley table :

	a	*b*	*c*	*d*	*e*
a	*a*	*a*	*c*	*c*	*e*
b	*b*	*b*	*d*	*d*	*e*
c	*c*	*c*	*c*	*c*	*c*
d	*d*	*d*	*d*	*d*	*d*
e	*e*	*e*	*e*	*e*	*e*

By applying Theorem 3.9, we can similarly deduce that there exist precisely the above two regular bands up to isomorphism in Case 2.

Case 3. $D(\alpha,\beta) = \{d(\alpha,\beta)\}$. This is the strong semilattice case. Here we omit the details while we find all distinct normal bands with the following Cayley tables :

	a	b	c	d	e
a	a	a	c	c	e
b	b	b	d	d	e
c	c	c	c	c	c
d	d	d	d	d	d
e	e	e	e	e	e

	a	b	c	d	e
a	a	a	c	c	e
b	b	b	d	d	e
c	c	c	c	c	c
d	d	d	d	d	d
e	e	e	e	e	e

Thus we have found all distinct regular bands, up to isomorphism, based on $\{I_\alpha, I_\beta\}$.

REFERENCES

1. Howie J. M., *An Introduction to Semigroup Theory*, Academic Press, London, 1976.
2. Petrich M., *Lectures in semigroups*, Akad. Berlag. Berlin, 1977.
3. Hall T. E., The isomorphism problem for orthodox semigroups, *Pacefic* J. Math., 136 (1989), 123-128.
4. Zhang L., Shum K. P. and Zhang R. H., On the refined semilattice of semigroups, *Algebra Colloquium* 8 : 1 (2001) 93-108.
5. Zhang R. H., On the refined semilattice decomposition structure of left *C-rpp* semigroups, *Southeast Asian Bulletin of Mathematics*, (2000) 24 : 137-145.

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ON REARRANGEMENTS OF DOUBLE SERIES

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ABSTRACT : Some properties of the rearrangements of double series are studied from point of view of the topology in the X of all permutations of the set of all positive integers endowed with the Frechet metric.

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Key words : Rearrangement, Residual G_δ set, First category.

1. INTRODUCTION

Let X denote the set of all permutations of the set of positive integers endowed with the Frechet metric $\rho(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1+|x_k - y_k|}$ where $x = \{x_k\}_{k=1}^{\infty}$ and $y = \{y_k\}_{k=1}^{\infty}$ are points in X. (X,ρ) is a metric space of second category. But the space is not complete. The convergence in this space is pointwise convergence.

Several papers [1], [3], [4] are devoted to study on the rearrangement of single series of real terms. Let us consider a double series $\sum_{m,n=1}^{\infty} a_{mn}$. We now determine the position of a term a_{mn} in the double series $\sum_{m,n=1}^{\infty} a_{mn}$ accordingly to the following plan.

We write the double series like a single series as

(1) $a_{11} + a_{21} + a_{22} + a_{12} + \ldots\ldots + a_{n1} + a_{n2} + \ldots\ldots + a_{(n-1)n} + \ldots\ldots + a_{1n} + \ldots\ldots$

If a_{mn} occupies the r-th position in this single series then the position of the term in the double series will also be counted as r-th position. So $a_{mn}(r)$ denotes the term in r-th position. To each $x = \{x_k\}_{k=1}^{\infty} \in X$ there corresponds a rearrangemnet $\sum_{k=1}^{\infty} a_{mn}(x_k)$ of the series

(I), that is, a rearrangement of the double series $\sum_{m,n} a_{mn}$. We denote this rearrangement $\sum_{k=1}^{\infty} a_{mn}(x_k)$ corresponding to the element $x = \{x_k\}_{k=1}^{\infty} \in X$ by $\sum_{m,n} a_{mn}(x)$.

Thus to each point $x = \{x_k\}_{k=1}^{\infty} \in X$ there corresponds a rearrangement $\sum_{m,n} a_{mn}(x)$. Therefore to each $x \in X$ there exists a rearrangement of the above form and also for each rearrangement of the double series there exists a point $x \in X$ corresponding to it.

A double series $\sum_{m,n} a_{mn}$ is said to be of type (α) if the series formed by its non-negative terms and the series formed by the absolute value of the negative terms of the double series $\sum a_{mn}$ both separately diverges at $+\infty$.

Theorem 1. If $\sum_{m,n} a_{mn}$ be a double series of real numbers of type (α) such that $a_{mn} \to 0$ as m $\to \infty$, n $\to \infty$ then the set of all those $x = \{x_k\}_{k=1}^{\infty} \in X$ for which $\limsup_{N\to\infty} \sum_{m,n=1}^{N} a_{mn}(x) = +\infty$, $\liminf_{N\to\infty} \sum_{m,n=1}^{N} a_{mn}(x) = -\infty$

is a residual G_δ set.

Proof. For each $x = \{x_k\}_{k=1}^{\infty} \in X$, let $S_r = \sum_{m,n=1}^{r} a_{mn}(x)$ be the partial square sums of the rearrangement $\sum_{m,n} a_{mn}(x)$. Suppose $\lambda_N = max\{S_1, S_2, \ldots, S_N)$ and $\mu_N = min \{S_1, S_2, \ldots, S_N\}$. Then clearly the sequence $\{\lambda_N\}$ is monotonic increasing and $\{\mu_N\}$ is monotonic decreasing. Therefore $\lim_{N\to\infty} \lambda_N$ and $\lim_{N\to\infty} \mu_N$ both exist.

Let $A_n = \{x \in X : \lim_{k\to\infty} \lambda_k > n\}$, where $n = 1,2, \ldots$. We first show that A_n is open for each n. Let $\{x_r\}_{r=1}^{\infty} \in A_n$. Let $\varepsilon > 0$ and l be the smallest positive integer for which $\sum_{t=l+1}^{\infty} 2^{-t} < \varepsilon$.

Consider an open sphere $S(x,\varepsilon)$ and $y = \{y_i\}_{i=1}^{\infty} \in S(x,\varepsilon)$; then $x_i = y_i$, $i = 1,2, \ldots, l$. Therefore $\sum_{m,n=1}^{l} a_{mn}(y) = \sum_{m,n=1}^{l} a_{mn}(x)$ and so $y \in A_n$. Hence $S(x,\varepsilon) \subset A_n$. This shows that A_n is open. Therefore $A = \bigcap_{n=1}^{\infty} A_n$ is a G_δ set in X. Clearly A is the set of all those points $x \in X$ for which $\limsup_{N\to\infty} \sum_{m,n=1}^{\infty} a_{mn}(x) = +\infty$.

Now we shall show that A is dense in X. Let z = $\{z_i\}_{i=1}^{\infty} \in X$ *and* $\varepsilon > 0$. Let m be the smallest positive integer for which $\sum_{i=m+1}^{\infty} 2^{-i} < \varepsilon$. We take a point x = $\{x_i\}_{i=1}^{\infty} \in X$ Such that $x_i = z_i$; i =1,2, ..., m. Let $B = \{\, k \in N : a_{mn}(k) < 0\}$. So $B \setminus \{z_1, z_2, \ldots, z_m\} = \{q_1, q_2, \ldots\}$ where $q_1 < q_2 < \ldots$. Since the series is of type (α) it is possible to determine a positive integer m_1 and pairwise distinct positive integers $x_{m+1}, x_{m+2}, \ldots, x_{m+m_1}$ all different form x_i ($i = 1,2, \ldots, m$) in such a way that

$$\{x_i\}_{i=1}^{m+m_1} \subset \{i\}_{i=1}^{q_1-1} \text{ and } \sum_{i=1}^{m+m_1} a_{rs}(x_i) > 1 - a_{rs}(q_1)$$

$$\text{i.e., } \sum_{i=1}^{m+m_1+1} a_{rs}(x_i) > 1 \text{ where } x_{m+m_1+1} = q_1.$$

Similarly, we can find a positive integer $m_2 > m_1 + 1$ and positive distinct positive integers $x_{m+m_1+2}, \ldots, x_{m+m_2}$ all different from x_i(i=1,2,...,m+m_1+1) such that

$$\{x_i\}_{i=m+m_1+2}^{m+m_2} \subset \{i\}_{i=q_1+1}^{q_2-1} \text{ and } \sum_{i=1}^{m+m_1} a_{rs}(x_i) + \sum_{i=m+m_1+2}^{m+m_2} a_{rs}(x_i) > 2 - a_{rs}(q_1) - a_{rs}(q_2)$$

$$\text{i.e., } \sum_{i=1}^{m+m_2+1} a_{rs}(x_i) > 2 \text{ where } x_{m+m_2+1} = q_2.$$

If this process is continued, we obtain a point $x \in X$ such that $\sum_{r,s} a_{rs}(x)$ diverges to $+\infty$ and hence $x \in A$. But $\rho(x,z) < \varepsilon$ and so $x \in S(Z,\varepsilon)$. Hence $A \cap S(z,\varepsilon) \neq \phi$. Therefore,

A is dense in X. Thus A is a residual G_δ set. Similarly C = $\left\{x \in X : \lim_{N\to\infty} \inf \sum_{m,n=1}^{N} a_{mn}(x) = -\infty\right\}$ is also a residual G_δ set. Hence $A \cap C$, the set of all those $x \in X$ for which $\limsup_{N\to\infty} \sum_{m,n=1}^{N} a_{mn}(x) = +\infty$, $\liminf_{N\to\infty} \sum_{m,n=1}^{N} a_{mn}(x) = -\infty$ is a residual G_δ set.

Corollary. Let $\sum_{m,n=1}^{\infty} a_{mn}$ be a double series of type (α) with $a_{mn} \to 0$. Then the set of all $x \in X$ for which the series $\sum_{m,n=1}^{\infty} a_{mn}(x)$ is convergent is a set of first category.

Theorem 2. Let $\sum_{m,n} a_{mn}$ be a double series with $a_{mn} \to 0$. Then $\sum_{m,n} a_{mn}$ is absolutely convergent if and only if there exists a second categorey set $S \subset X$ such that $\sum_{m,n} a_{mn}(x)$ is convergent for each $x \in S$.

Proof. Suppose $\sum_{m,n} a_{mn}$ is absolutely convergent ; then $\sum a_{mn}(x)$ is convergent for all $x \in X$ and X is a second category set.

Conversely, let there exists a second category set $S \in X$ such that $\sum a_{mn}(x) < +\infty$ for each $x \in S$. If $\sum a_{mn}$ is not absolutely convergent then the set of all those $x \in S \subset X$ for which $\limsup_{N\to\infty} \sum_{m,n=1}^{N} a_{mn}(x) = +\infty$; $\liminf_{N\to\infty} \sum_{m,n=1}^{N} a_{mn}(x) = -\infty$ is a residual G_δ set. Therefore the set of all those points for which $\sum_{m,n} a_{mn}(x)$ is convergent is of first category, a contradiction. Hence the theorem.

Theorem 3. Let $\sum a_{mn}$ be a double series of type (α) and $\lim_{m,n\to\infty} a_{mn} = 0$. If for each $x \in X$ except those belonging to a set of first category, the sequence $\{s_{mn}(x)\}$ is compact then the set of all limit points of the sequence is precisely the interval $\langle -\infty, \infty \rangle$.

The proof of the theorem is same as the Theorem 3 of paper [2].

REFERENCES

1. R. P. Agnew, On rearrangment of series. Bull. of Amer. Math. Soc. 46 (1940).
2. D. K. Ganguly, Chandana Dutta, Some results on subseries of Divergent Series. Journal of Pure Mathematics. Vol. 12, 1995, pp.22-26.
3. T. Salat, On subseries of divergent series. Mat. Cas. Sav. 18. 1968, pp. 312-318.
4. H. M. Sengupta, Rearrangement of Series. Proc. Amer. Math. Soc. 1950, pp. 71-75.

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ENTIRE FUNCTIONS OF L-BOUNDED INDEX

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ABSTRACT : In this paper we have established a necessary and sufficient condition for an entire function to be of L-bounded index. We have also proved that if an entire function $f(z)$ is of L-bounded index, then $f(az + b)$ is also of L-bounded index for a, $b \in C$. Finally, we have shown that the product of two functions of L-bounded index is again a function of L-bounded index.

1. INTRODUCTION

The concept of 'entire functions of bounded index' was first introduced by Lepson [1]. According to him :

Definition 1. An entire function f is said to be of **bounded index** if there exists an integer N, independent of Z, such that

$$\frac{|f^{(n)}(z)|}{n!} \le \max_{0 \le j \le N} \left\{ \frac{|f^{(j)}(z)|}{j!} \right\} \text{ for all } z \in C \text{ and } n \ge 0.$$

At present the theory of entire functions of bounded index is well-developed mainly due to the efforts of the mathematicians like F. Gross, S. M. Shah, G. H. Fricke etc. Later on T. V. Lakshminarasimhan [5] introduced the concept of entire functions of L-bounded index. In this paper we present a few properties of entire functions of L-bounded index. Throughout this paper the set of all entire functions of L-bounded index will be denoted by LBI. We first consider the following.

Definition 2. [4] A positive continuous function $L(r)$ is said to be a 'slowly changing function' if for any $\varepsilon > 0$ there exists an $r_0(\varepsilon) = r_0 > 0$ such that for all $r > r_0$.

$$(1.1) \qquad \frac{1}{k^{\varepsilon}} \le \frac{L(kr)}{L(r)} \le k^{\varepsilon} \text{ holds uniformly for } k \ge 1.$$

It is known that [4] if $L(r)$ is differentiable, then the condition (1.1) is equivalent to the condition :

$$(1.2) \qquad \lim_{r \to \infty} \frac{rL^{(l)}r}{L(r)} = 0.$$

· In this paper Singh and Barker proved the following theorem.

Theorem 1.1. Let $L(r)$ be a slowly changing function. Then

(i) $L(kr) \sim L(r)$ as $r \to \infty$ uniformaly for $0 < a \leq k \leq b < \infty$

(ii) for every $\beta\,(>0)$ and $r \geq r_0(\beta)$, $L(r)r^{\beta}$ is strictly increasing and $\lim_{r\to\infty} r^{\beta}L(r) = \infty$

(iii) for every $\beta\,(>0)$ and $r \geq r_0(\beta)$, $r^{-\beta}L(r)r^{\beta}$ is strictly decreasing and $\lim_{r\to\infty} r^{-\beta}L(r) = 0$

Definition 3. Let $L(r)$ be a slowly changing function. An entire function f is said to be of *L-bounded index* at a point $z_0 \in C$ if there exits a non-negative integer N_{z_0} such that

$$(1.3) \qquad L(j+2)\frac{|f^{(j)}(z_0)|}{j!} \leq \max_{0\leq i\leq N_{z_0}} \left\{ L(i+2)\frac{|f^{(i)}(z_0)|}{i!} \right\} \text{ for j= 0, 1, 2, ...}$$

The least integer N_{z_0} for which (1.3) holds is called the *L-index* of f at the point z_0.

Definition 4. [5] Let f be an entire function and N_z be the L-index of f at the point z. Let $A = \{N_z : z \in C\}$. Then f is said to be of **L-bounded index** if the set A is bounded and the least upper bound of A is called the **L-index** of f and is denoted by N_L. It follows that f will be of L-bounded index with index N_L if and only if

$$(1.4) \qquad L(j+2)\frac{|f^{(j)}(z_0)|}{j!} \leq \max_{0\leq i\leq N_L} \left\{ L(i+2)\frac{|f^{(i)}(z_0)|}{i!} \right\} \text{ for all } j \geq 0 \text{ and for all } z \in C.$$

We observe that when $L(r) \equiv 1$, the L-index of f becomes the index of f in the sense of Lepson. So, the class of entire functions of bounded index is a subclass of LBI.

§ **2.** Let $L(r)$ be a slowly changing function. Also, let $r > 0$ and $z_0 \in C$ and h be a positive integer. For an entire function f and $n = 0, 1, \ldots, h$, we define $R_n(r,h,z_0)$ and $M_l(f,z,r)$ by

$$(2.1) \qquad R_n(r,h,z_0) = \text{Max } \left\{L(l+2)\frac{|f^{(l)}(z)|}{l!} \Big/ |z-z_0| \leq \frac{nr}{h} \text{ and } l = 0, 1, 2, \ldots \right\}$$

$$(2.2) \qquad M_l(f,z,r) = \max_{|\xi - z| = r} \left\{ \left| f^{(l)}(\xi) \right| \right\}$$

We now prove the following lemma.

Lemma 2.1. If f is an entire function of L-bounded index with index N_L and $\forall$ r, h are such that

$$\text{(2.3)} \qquad \frac{r}{h} \leq \frac{1}{4(N_L+1)(N_L+3)}, \text{ then}$$

(i) $R_t(r,h,z_0) \leq 2R_{t-1}(r,h,z_0)$ for all $z_0 \in C$ and $1 \leq t \leq h$, and

$$\text{(ii)} \qquad \max_{\substack{l \leq N_L \\ |z-z_0|=r}} \left\{ L(l+2)\frac{|f^{(l)}(z)|}{l!} \right\} \leq 2 \max_{l \leq N_L} \left\{ L(l+2)\frac{|f^{(l)}(z_0)|}{l!} \right\}$$

Proof. (i) If possible, suppose there exists an integer t with $1 \leq t \leq h$ and $z_0 \in C$ such that

$$\text{(2.4)} \qquad R_t(r,h,z_0) > 2R_{t-1}(r,h,z_0).$$

Since f is of L-bounded index with index N_L, there is an integer l_t with $0 \leq l_t \leq N_L$ such that

$$\text{(2.5)} \qquad R_t(r,h,\ z_0) = L(l_t + 2)\frac{|f^{(l_t)}(z_t)|}{l_t!} \text{ for some } z_t \in C \text{ with } |z_t - z_0| = \frac{tr}{h}.$$

We choose $z_t' = z_0 + \frac{t-1}{t}(z_t - z_0)$. Then $|z_t' - z_0| = (t-1)\frac{r}{h}$.

Hence

$$\text{(2.6)} \qquad L(l_t+2)\frac{\left|f^{(l_t)}(z_t')\right|}{l_t!} \leq R_{t-1}(r,h,z_0)$$

From (2.5) and (2.6) it follows that

$$\text{(2.7)} \qquad L(l_t+2)\frac{\left|f^{(l_t)}(z_t) - f^{(l_t)}(z_t')\right|}{l_t!} \geq L(l_t+2)\frac{\left|f^{(l_t)}(z_t)\right| - \left|f^{(l_t)}(z_t')\right|}{l_t!}$$

$$\geq R_t(r,h,z_0) - R_{t-1}(r,h,z_0)$$

By Lagrange's mean value theorem there exists a point $\xi = z_t' + k(z_t - z_t')$ for some $k \in (0,1)$, such that $\left|f^{(l_t+1)}(\xi)\right| = \dfrac{\left|f^{(l_t)}(z_t) - f^{(l_t)}(z_t')\right|}{|z_t - z_t'|}$

This implies, $L(l_t+3)\dfrac{\left|f^{(l_t+1)}(\xi)\right|}{(l_t+1)!} = \dfrac{L(l_t+3)}{(l_t+1)!}\,\dfrac{l_t!}{|z_t-z_t'|}\,\dfrac{\left|f^{(l_t)}(z_t)-f^{(l_t)}(z_t')\right|}{l_t!}$

$$= \frac{(l_t+3)L(l_t+3)}{(l_t+3)(l_t+1)}\,\frac{1}{|z_t-z_t'|}\,\frac{\left|f^{(l_t)}(z_t)-f^{(l_t)}(z_t')\right|}{l_t!}$$

$$\geq \frac{(l_t+2)L(l_t+2)}{(l_t+3)(l_t+1)}\,\frac{1}{r/h}\,\frac{\left|f^{(l_t)}(z_t)-f^{(l_t)}(z_t')\right|}{l_t!}$$

[since $rL(r)$ is a monotonic increasing function and $|z_t-z_t'|=\dfrac{r}{h}$]

$> \dfrac{1}{(N_L+3)(N_L+1)}\,\dfrac{\frac{1}{2}R_t(r,h,z_0)}{r/h} \geq 2R_t(r,h,z_0)$ (using (2.3)) ; that is,

$$(2.8) \qquad L(l_t+3)\frac{\left|f^{(l_t+1)}(\xi)\right|}{(l_t+1)!} > 2R_t(r,h,z_0)$$

Now, $|\xi-z_0| = |z_t'+k(z_t-z_t')-z_0|$

$\leq |z_t'-z_0|+k|z_t-z_t'| = (t-1)\dfrac{r}{h}+k\dfrac{r}{h} = \dfrac{tr}{h}-(1-k)\dfrac{r}{h}$ which implies

$$(2.9) \qquad |\xi-z_0| \leq \frac{tr}{h}.$$

From (2.8) and (2.9) we arrive at a contradiction to the choice of $R_t(r,h,z_0)$. Hence, (i) is proved.

(ii) From (i) we have,

$R_h(r,h,z_0) \leq 2R_{h-1}(r,h,z_0) \leq 2^2R_{h-2}(r,h,z_0) \leq \ldots \leq 2^hR_0(r,h,z_0)$

that is,

$$(2.10) \qquad \max_{\substack{j\leq N_L \\ |z-z_0|=r}}\left\{L(j+2)\frac{\left|f^{(j)}(z)\right|}{J!}\right\} \leq 2^h \max_{j\leq N_L}\left\{L(j+2)\frac{\left|f^{(j)}(z_0)\right|}{J!}\right\}$$

Putting $h = 1$ in (2.10) we get the desired result.

Theorem 2.1. An entire function f is of L-bounded index if and only if for each $r > 0$ there exist an integer $N = N(r)$ and a constant $M = M(r) > 0$ such that for each $z \in C$ there exists an integer $l = l(z)$ with $0 \leq l \leq N$ and $M_l(f,z,r) \leq ML(l+2)\left|f^{(l)}(z)\right|$.

Proof. First we assume that f is of L-bounded index with index N_L. For $r > 0$ we choose $N = N(r) = N_L$ and $M = M(r) = \dfrac{2^h N_L!(N_L+2)}{2L(2)}$ where h is a positive integer such that $\dfrac{r}{h} \leq \dfrac{1}{4(N_L+1)(N_L+3)}$. For $z_0 \in C$ we choose $l = l(z_0)$ to be the L-index of f at z_0. Obviously, $l \leq N_L = N$. From (2.10) of lemma (2.1) we have,

$$(2.11) \qquad \max_{\substack{j \leq N_L \\ |z-z_0|=r}} \left\{ L(j+2)\frac{\left|f^{(j)}(z)\right|}{j!} \right\} \leq 2^h \max_{j \leq N_L} \left\{ L(j+2)\frac{\left|f^{(j)}(z_0)\right|}{j!} \right\} = 2^h L(l+2)\frac{\left|f^{(l)}(z_0)\right|}{l!}.$$

Now, $M_l(f,z_0,r) = \max\limits_{|z-z_0|=r} \left\{\left|f^{(l)}(z)\right|\right\} = \dfrac{l!}{L(l+2)} \max\limits_{|z-z_0|=r} \left\{ L(l+2)\dfrac{\left|f^{(l)}(z)\right|}{l!} \right\}$

$$\leq \frac{N!(l+2)}{(l+2)L(l+2)} \max_{\substack{|z-z_0|=r \\ j \leq N_L}} \left\{ L(l+2)\frac{\left|f^{(l)}(z)\right|}{l!} \right\} \leq \frac{N!(N+2)}{2L(2)} 2^h L(l+2)\frac{\left|f^{(l)}(z_0)\right|}{l!}$$

(Using (2.11) and the fact that $rL(r)$ is monotone increasing)

Hence, $M_l(f,z_0,r) \leq ML(l+2)\left|f^{(l)}(z_0)\right|$.

Since z_0 is arbitrary, the result follows.

To prove the converse part we choose $r = 2$. Then there exist an integer $N = N(2)$ and a constant $M = M(2) > 0$ such that for each $z_0 \in C$ there exists an integer $l_0 = l_0(z_0) \leq N$ satisfying

$$(2.12) \qquad M_{l_0}(f,z_0,2) \leq ML(l_0+2)\left|f^{(l_0)}(z_0)\right|$$

Let m be the least positive integer such that

$$(2.3) \qquad N!L(2)M2^{-(m+1)} < 1$$

Also we can find a positive integer N_1 such that

$$(2.14) \qquad \frac{(n-l_0)!}{n!} \leq \frac{1}{n+2} \text{ for all } n \geq N_1.$$

Let $N_2 = \max\{m + N, N_1\}$. By Cauchy's inequality we have for any entire function g,

$$|g^{(k)}(z)| \leq \frac{k!}{R^k} \max_{|\xi - z| = R} |g(\xi)| \text{ for } k = 0, 1, 2, \ldots \text{ and any } R > 0. \tag{2.15}$$

Taking $g(z) = f^{(l_0)}(z)$ and $R = 2$ in (2.15) we obtain,

$$\frac{|f^{(l_0+k)}(z)|}{k!} \leq 2^{-k} \max_{|\xi - z| = 2} |f^{(l_0)}(\xi)| = 2^{-k} M_{l_0}(f, z, 2). \tag{2.16}$$

Now, for $n \geq N_2$,

$$L(n+2)\frac{|f^{(n)}(z_0)|}{n!} = \frac{(n-l_0)!}{n!} L(n+2) \left|\frac{f^{(l_0+(n-l_0))}(z_0)}{(n-l_0)!}\right|$$

$$\leq \frac{L(n+2)}{n+2} 2^{-(n-l_0)} M_{l_0}(f, z_0, 2) \qquad \text{[using (2.14) and (2.16)]}$$

$$\leq \frac{L(2)}{2} 2^{l_0-n} M_{l_0}(f, z_0, 2), \text{ since } \frac{L(r)}{r} \text{ is monotone decreasing}$$

$$\leq \frac{L(2)}{2} 2^{N-n} ML(l_0+2) |f^{(l_0)}(z_0)| \qquad \text{[by 2.12)]}$$

$$\leq \frac{L(2)}{2} 2^{m} ML(l_0+2) |f^{(l_0)}(z_0)|$$

$$\leq \frac{L(l_0+2)}{N!} |f^{(l_0)}(z_0)| \qquad \text{[using (2.13)]} \leq L(l_0+2)\frac{|f^{(l_0)}(z_0)|}{l_0!}$$

Hence f is of L-bounded index at z_0 and its L-index at z_0 does not exceed N_2 which is independent of z_0. Since z_0 is arbitrary, f is of L-bounded index.

Theorem 2.2. If $f \in$ LBI, then $g \in$ LBI, where $g(z) = f(az+b)$, for any $a, b \in C$.

Proof. If $a = 0$, then g becomes a constant function and hence $g \in$ LBI. So we assume that $a \neq 0$.

Case (i) : Suppose $b = 0$. Then $g(z) = f(az)$. Since $f \in$ LBI, by the theorem (2.1) we have, for each $r > 0$ there exist an integer $N = N(r)$ and a constant $M = M(r) > 0$ such that for each $z \in C$ there exists an integer $l = l(z)$ with $0 \leq l \leq N$ and

$$M_l(f, z, r) \leq ML(l+2)|f^{(l)}(z)| \tag{2.17}$$

Let $z = az_0$ and $r = |a||z_0| = |a|r_0$. Then,

$$M_l(g, z_0, r_0) = \max_{|\xi - z_0| = r_0} \left\{ \left| g^{(l)}(\xi) \right| \right\} = \max_{|\xi - z_0| = r_0} \left\{ \left| a^l f^{(l)}(a\xi) \right| \right\}$$

$$= \left| a^l \right| \max_{\left|\frac{\alpha}{a} - z_0\right| = r_0} \left\{ \left| f^{(l)}(\alpha) \right| \right\}, \text{ where } \alpha = a\xi$$

$$= \left| a^l \right| \max_{|\alpha - az_0| = |a| r_0} \left\{ \left| f^{(l)}(\alpha) \right| \right\} = \left| a^l \right| \max_{|\alpha - z| = r} \left\{ \left| f^{(l)}(\alpha) \right| \right\}$$

$$= \left| a^l \right| M_l(f, z, r) \le \left| a^l \right| ML(l+2) \left| f^{(l)}(z) \right| = ML(l+2) \left| g^{(l)}(z_0) \right|$$

Since z_0 is arbitrary, by theorem (2.1), we have $g(z) = f(az) \in$ LBI for any $a \in C$.

Case (ii) : Let $b \neq 0$. We observe that the L-index of $f(z + b)$ at the point z_0 is equal to the L-index of $f(z)$ at the point $z_0 + b$. Hence $f(az + b) \in$ LBI for any $a, b \in C$.

§ **3** : Let $f, g \in$ LBI with L-indices M_L and N_L respectively. Let $P = M_L + N_L$ and we define the constants $A_0, A_1, \ldots, A_P$ by $A_P = 1$ and $A_n = \frac{1}{2P}(A_{n+1})^2$ for $0 \le n < P$. Also, let $f_k(z) = \frac{f^{(k)}(z)}{k!}$.

Lemma 3.1. Let $f, g \in$ LBI and $h(z) = f(z)g(z)$. Then for each $z_0 \in C$, there exists an integer $k = k(z_0) \le P$ such that

$$L(k+2)\left| h_k(z_o) \right| \ge M' A_k L(i+2) L(j+2) \left| f_i(z_0) g_j(z_0) \right|$$

where i, j are respective L-indices of f and g at z_0 and M' is a suitable constant.

Proof. Let k be the smallest integer n for which

$$(3.1) \qquad L(l+2)L(n-l+2)\left| f_l(z_0) g_{n-l}(z_0) \right| \ge A_n L(i+2)(j+2) \left| f_i(z_0) g_j(z_0) \right|$$

for some l where $0 \le l \le n$.

If we take $n = i + j$ and $l = i$ we see that the inequality (3.1) is satisfied. Hence such a k exists and clearly $k \le P$. For this k we have,

$$(3.2) \qquad L(l+2)L(k-l+2)\left| f_l(z_0) g_{k-l}(z_0) \right| \ge A_k L(i+2) L(j+2) \left| f_i(z_0) g_j(z_0) \right|$$

for some l where $0 \le l \le k$.

Also from the definitions of i and j we have

(3.3) $L(i+2)|f_i(z_0)| \geq L(l+2)|f_l(z_0)|$

(3.4) $L(j+2)|g_j(z_o)| \geq L(k\text{-}l+2)|g_{k-l}(z_o)|$

From (3.2) and (3.4), we have

$L(l+2)L(j+2)|f_l(z_0)g_j(z_0)| \geq A_k L(i+2)L(j+2)|f_i(z_0)g_j(z_0)|$. That is,

(3.5) $L(l+2)|f_l(z_0)| \geq A_k L(i+2)|f_i(z_0)|$

Similarly,

(3.6) $L(k-l+2)|g_{k-l}(z_0)| \geq A_k L(j+2)|g_{j(z_0)}|$

Now, by our choice of k, we must have,

(3.7) $L(u+2)L(v+2)|f_u(z_0)g_v(z_0)| < A_{u+v} L(i+2)L(j+2)|f_i(z_0)g_j(z_0)|$

for all u, v with $u + v < k$.

In particular, taking $u = l$ in (3.7) we get for any v with $0 \leq v < k-l$

$$L(l+2)L(v+2)|f_l(z_0)g_v(z_0)| < A_{l+v} L(i+2)L(j+2)|f_i(z_0)g_j(z_0)|$$

$$\leq A_{k-l} L(i+2)L(j+2)|f_i(z_0)g_j(z_0)|$$

$$= \frac{1}{2P} A_k^2 L(i+2)L(j+2)|f_i(z_0)g_j(z_0)|$$

Using (3.5) we obtain

$$A_k L(i+2)L(v+2)|f_i(z_0)g_v(z_0)| < \frac{1}{2P} A_k^2 L(i+2)L(j+2)|f_i(z_0)g_j(z_0)|$$

for $0 \leq v < k - l$. That is,

(3.8) $L(v+2)|g_v(z_0)| < \frac{1}{2p} A_k L(j+2)|g_j(z_0)|$ for $0 \leq v < k - l$.

Similarly, we have

(3.9) $L(u+2)|f_u(z_0)| < \frac{1}{2p} A_k L(i+2)|f_i(z_0)|$ for $0 \leq u < l$.

Now,

$$L(k+2)\left|h_k(z_0)\right| = L(k+2)\left|\sum_{r=0}^{k} f_r(z_0)g_{k-r}(z_0)\right|$$

$$\geq L(k+2)\left[\left|f_l(z_0)g_{k-l}(z_0)\right| - \left|\sum_{\imath=1}^{k} f_r(z_0)g_{k-r}(z_0)\right|\right]$$

$$= L(k+2)\left[\begin{array}{l}\dfrac{L(l+2)L(k-l+2)}{L(l+2)L(k-l+2)}\left|f_l(z_0)g_{k-l}(z_0)\right| - \displaystyle\sum_{r=0}^{l-1}\left\{\dfrac{L(r+2)}{L(r+2)}\left|f_r(z_0)\right|\dfrac{L(k-r+2)}{L(k-r+2)}\left|g_{k-r}(z_0)\right|\right\} \\ -\displaystyle\sum_{r=l+1}^{k}\left\{\dfrac{L(r+2)}{L(r+2)}\left|f_r(z_0)\right|\dfrac{L(k-r+2)}{L(k-r+2)}\left|g_{k-r}(g_0)\right|\right\}\end{array}\right]$$

$$\geq L(k+2)\left[\begin{array}{l}\dfrac{A_k L(i+2)L(j+2)}{L(l+2)L(k-l+2)}\left|f_i(z_0)g_{k-j}(z_0)\right| - \displaystyle\sum_{r=0}^{l-1}\left\{\dfrac{L(r+2)}{L(r+2)}\left|f_r(z_0)\right|\dfrac{L(j+2)}{L(k-r+2)}\left|g_j(z_0)\right|\right\} \\ -\displaystyle\sum_{r=0}^{k-l-1}\left\{\dfrac{L(i+2)}{L(k-r+2)}\left|f_i(z_0)\right|\dfrac{L(r+2)}{L(r+2)}\left|g_r(z_0)\right|\right\}\end{array}\right]$$

$$\geq L(k+2)\left[\begin{array}{l}\dfrac{A_k L(i+2)L(j+2)}{L(l+2)L(k-l+2)}\left|f_i(z_0)g_j(z_0)\right| - \displaystyle\sum_{r=0}^{l-1}\left\{\dfrac{A_k L(i+2)}{2PL(r+2)}\left|f_r(z_0)\right|\dfrac{L(j+2)}{L(k-r+2)}\left|g_j(z_0)\right|\right\} \\ -\displaystyle\sum_{r=0}^{k-l-1}\left\{\dfrac{L(i+2)}{L(k-r+2)}\left|f_i(z_0)\right|\dfrac{A_k L(j+2)}{2PL(r+2)}\left|g_j(z_0)\right|\right\}\end{array}\right]$$

$$= A_K L(i+2)L(j+2)\left|f_i(z_0)g_j(z_0)\right|\times$$

$$\left[\frac{L(k+2)}{L(l+2)L(k-l+2)} - \frac{L(k+2)}{2P}\sum_{r=0}^{l-1}\frac{1}{L(r+2)L(k-r+2)} - \frac{L(k+2)}{2P}\sum_{r=0}^{k-l-1}\frac{1}{L(r+2)L(k-r+2)}\right]$$

$$= M'A_K L(i+2)L(j+2)\left|f_i(z_0)g_j(z_0)\right|$$

where M' is the constant represented by the expression within the bracket.

Theorem 3.1. Let f, $g \in$ LBI. If $f(az)$, $g(az)$ are entire functions of L-bounded index for some a with $|a| > 1$, then $h(z) = f(z)g(z)$ is of L-bounded index.

Proof. Without loss of generality we may assume that a is real and $a > 1$. Let M_L and N_L be the L-indices of f and g respectively. Let $P = M_L + N_L$, the constants A_0, A_1, ..., A_p are defined as before. Also, let M_a and N_a be the respective L-indices of $u(z) = f(az)$ and $v(z) = g(az)$. Since $a > 1$, we have $a^{-n} = O(n^m)$ for any non-negative integer m. Thereore there exists an interger $N_1 > 0$ such that.

$$(3.10)\qquad \left(\frac{1}{2^3 L(2)}\sum(t+2)t!(n-t)!(n-t+2)\right)a^{-n} \le M'A_0 a^{-M_a-N_a} \text{ for any } n \ge N_1.$$

Again we can find an integer N_2 such that

$$(3.11)\qquad \frac{1}{n!} \le \frac{1}{n+2} \text{ for all } n \ge N_2.$$

Let $n_0 = \max\{N_1, N_2\}$

Again let r and s be the L-indices of $u(z) = f(az)$ and $v(z) = g(az)$ at z_0/a respectively. Then clearly

$$(3.12)\qquad r \le M_a \text{ and } s \le N_a.$$

Now, $u(z) = f(az) \Rightarrow u_r(z) = df_r(az) \Rightarrow u_s\left(\frac{z_0}{r}\right) = a^r f_r(z_0)$

$$\Rightarrow a^r L(r+2)\left|f_r(z_0)\right| = \mathrm{L(r+2)}\left|u_r\left(\frac{z_0}{a}\right)\right|$$

$$\ge L(l+2)\left|u_l\left(\frac{z_0}{a}\right)\right| \text{ for } l = 0, 1, 2, \ldots$$

$$= a^l L(l+2)\left|f_l(z_0)\right|$$

Hence,

$$(3.13)\qquad L(l+2)\left|f_l(z_0)\right| \le a^{r-l} L(r+2)\left|f_r(z_0)\right| \text{ for } l = 0, 1, 2, \ldots$$

Similarly $v(z) = g(az)$ implies that

$$(3.14)\qquad L(l+2)\left|g_l(z_0)\right| \le a^{s-l} L(s+2)\left|g_s(z_0)\right| \text{ for } l = 0, 1, 2, \ldots$$

Thus, for $n \ge n_0$,

$$L(n+2)\frac{\left|h^{(n)}(z_0)\right|}{n!} \le \frac{L(i+2)}{(n+2)}\sum_{t=0}^{n}\left|f^{(t)}(z_0)g^{(n-t)}(z_0)\right| \qquad [\text{ by } (3.11)]$$

$$\leq \frac{L(2)}{2}\sum_{t=0}^{n}\left|f^{(t)}(z_0)\right|\left|g^{(n-t)}(z_0)\right| \qquad \left[\because \ \frac{L(r)}{r} \text{ is monotone decreasing}\right]$$

$$= \frac{L(2)}{2}\sum_{t=0}^{n}\left[\begin{array}{l}\left\{L(t+2)\dfrac{\left|f^{(t)}(z_0)\right|}{t!}\dfrac{t!(t+2)}{(t+2)L(t+2)}\right\} \\ \left\{L(n-t+2)\dfrac{\left|g^{(n-t)}(z_0)\right|}{(n-t)!}\dfrac{(n-t)!(n-t+2)}{(n-t+2)L(n-t+2)}\right\}\end{array}\right]$$

$$\leq \frac{1}{2^3 L(2)}\sum_{t=0}^{n}\left[L(t+2)\left|f_t(z_0)\right|L(n-t+2)\left|g_{n-t}(z_0)\right|t!(t+2)(n-t)!(n-t+2)\right]$$

$$\leq \frac{1}{2^3 L(2)}\sum_{t=0}^{n}\left[a^{r-t}L(r+2)\left|f_r(z_0)\right|a^{s-n+t}L(s+2)\left|g_s(z_0)\right|t!(t+2)(n-t)!(n-t+2)\right]$$

[by using (3.13) and (3.14)]

$$= a^{r+s-n}L(r+2)L(s+2)\left|f_r(z_0)\right|\left|g_s(z_0)\right|\frac{1}{2^3 L(2)}\sum_{t=0}^{n}[t!(t+2)(n-t)!(n-t+2)]$$

$$\leq a^{M_a+N_a+n}L(i+2)L(j+2)\left|f_i(z_0)\right|\left|g_j(z_0)\right|\frac{1}{2^3 L(2)}\sum_{t=0}^{n}[t!(t+2)(n-t)!(n-t+2)]$$

$$\leq M'A_0L(i+2)L(j+2)\left|f_i(z_0)(g_j)(z_0)\right| \qquad [\text{ by using (3.10) }]$$

where i and j are the respective L-indices of f and g at z_0 respectively.

Thus we have

$L(n+2)\left|h_n(z_0)\right| \leq M'A_0L(i+2)L(j+2)\left|f_i(z_0)g_j(z_0)\right| \ \forall n \geq n_0$.

Again by lemma 3.1, we observe that there exists an integer $k \leq P$ such that.

$$\begin{aligned}\mathrm{L(k+2)}\left|h_k(z_0)\right| &\geq M'A_kL(i+2)L(j+2)\left|f_i(z_0)g_j(z_0)\right| \\ &\geq M'A_0L(i+2)L(j+2)\left|f_i(z_0)g_j(z_0)\right| \\ &\geq L(n+2)\left|h_n(z_0)\right| \text{ for all } n \geq n_0.\end{aligned}$$

Thus we see that the L-index of h does not exceed max $\{P, n_0\}$. This completes the proof.

Corollary 3.1. If f, $g \in$ LBI, then $h \in$ LBI, where $h(z) = f(z)g(z)$.

Proof. It can be immediately proved by combining theorems (2.2) and (3.1).

REFERENCES

1. Lepson, B. "Differential equations of finite order, Hyperdirichlet series and entire functions of bounded index", *Proc. Symps. Pure Math.*, vol. II, Amer. Math. Soc., Providence, R. I. (1968) 298-307.
2. Fricke, G. H. "A characterization of functions of bounded index", *Ind. J. Math.*, 14(3) (1972) 207-212.
3. Fricke, G. H. and Powell, R. E. "Some product theorems on functions of bounded index", Indian J. Pure Appl. Math., 3 (1971) 795-800.
4. Lakshminarasimhan, T. V. "A note on entire functions of bounded index", *J. ind. Math. Soc.*, 38(1974) 43-49.
5. Singh S. K. and Barker, G. P. "Slowly changing functions and their applications", *Indian J. Math.*, 19 (1977) 1-6.

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ON SOME PROPERTIES OF HILL'S EQUATION WITH THREE TERMS

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ABSTRACT : One of the main sources of periodic differential equations is the type of problem in which we have to find a solution of a partial differential equation (for example the wave equation) where the solution has to be such as to satisfy given boundary conditions at certain special surfaces, in particular spheroids or ellipsoids. This involves introduction of new coordinates and then separation of the partial differential equation into two or three ordinary differential equations in the new variable.

In this paper, we transform the three dimensional wave equation

$$\frac{\delta^2 w}{\delta x^2} + \frac{\delta^2 w}{\delta y^2} + \frac{\delta^2 w}{\delta z^2} = -\chi^2 w \text{ by paraboloidal coordinates into}$$

$$\frac{d^2 w}{dz^2} + (\lambda + \mu \cos z \, dz - \frac{c^2 \chi^2}{8} \cos 4z)\, w = 0,$$

which is a form of Hill's equation with three terms.

This equation has many properties in common with Mathieu's equation which is of the form $\frac{d^2 w}{dz^2} + (a - 2q \cos 2z)\, w = 0$. This paper discusses these properties. The question naturally arises whether the above special form of Hill's equation with three terms which has so many properties in common with Mathieu's equation has also the feature that two basically-periodic solutions cannot co-exist. It has been shown in this paper that co-existence does not occur also with the above form of Hill's equation with three terms as in Mathieu's equation. An orthogonality relation and an integral relationship between basically periodic solutions of Hill's equation with three terms are also discussed.

Key words and phrases : Periodic differential equation, paraboloidal coordinates, basically periodic functions.

1. INTRODUCTION

The main source of periodic differential equations is the type of problem in which we have to find a solution of a partial differential equation, the wave equation, say, where the solution has to be such as to satisfy given boundary conditions at certain special surfaces, in particular spheroids, ellipsoids or paraboloids. This involves introduction of new co-ordinates and then transforming the partial differential equation with the help of these co-ordinates into two or three ordinary periodic differential equations in the new variables.

In this paper, the three dimensional wave equation is separated into three periodic differential equations with the help of paraboloidal co-ordinates. [For other examples illustrating this method see 1, §1.2] These equations are again transformed into Hill's equation with three terms which is of the form $\frac{d^2w}{dz^2} + (\lambda + \mu \cos z\, dz - \frac{c^2\chi^2}{8} \cos 4z)\, w = 0$ (1.1)

Hill's equation is an equation of the form $\frac{d^2w}{dz^2} + J(z)\, w = 0$ (1.2)

where $J(z) = \theta_0 + \sum_1^\infty 2\theta_r \cos 2rz$, the series converging in an infinite strip of the z-plane which includes the real axis. For a discussion on Hill's equation see [1, ch.7].

If $\theta_r = 0$ for $r \geq 2$, then (1.2) becomes an equation known as Mathieu's equation which is of the form $\frac{d^2w}{dz^2} + (a - 2q \cos dz)\, w = 0$ and when $\theta_r = 0$ for $r \geq 3$, (1.2) becomes Hill's equation with three terms. Mathieu's equation and Hill's equation with three terms have many properties in common. Some of these properties are discussed in this paper. The question naturally arises whether Hill's equation with three terms which has so many properties in common with Mathieu's equation has also the feature that two basically periodic solutions cannot co-exist.

A basically periodic function is a function f(z) which satisfies $f(z + \pi) = \pm f(z)$. [1] gives a discussion of basically periodic function. With the help of a conjecture by Magnus and Winkler in [3], it can be shown that co-existence does not occur in the form of the Hill's equation deduced in this paper i.e., the equation (1.1).

In §2, we give an outline of the method of deduction of the special from of Hill's equation with three terms. §3 deals with the properties common to (1.1) and Mathieu's eauation. The problem of co-existence of basically periodic solutions is also dealt in this section. An orthogonality relation between basically periodic solutions of (1.1) and an integral relationship between two solutions of (1.1) are discussed in §4.

2. AN OUTLINE OF DEDUCTION OF HILL'S EQUATION WITH THREE TERMS

The three dimensional wave equation $\frac{\delta^2 w}{\delta x^2} + \frac{\delta^2 w}{\delta y^2} + \frac{\delta^2 w}{\delta z^2} = -\chi^2 w$ (2.1) is transformed by using paraboloidal co-ordinates α, β, γ given by

$$\begin{aligned} x &= \frac{c}{4}(\cosh 2\alpha + \cos 2\beta - \cosh 2\gamma) \\ y &= c \cosh \alpha \cos \beta \sin \gamma \\ z &= c \sinh \alpha \sin \beta \cosh \gamma \end{aligned} \tag{2.2}$$

$$\text{to } F_1 \frac{\delta^2 w}{\delta x^2} + F_2 \frac{\delta^2 w}{\delta y^2} + F_3 \frac{\delta^2 w}{\delta z^2} + \frac{1}{4} c^2 \chi^2 F_1 F_2 F_3 w = 0 \tag{2.3}$$

where $F_1 = \cos 2\beta + \cosh 2\gamma$, $F_2 = \cosh 2\gamma + \cosh 2\alpha$, $F_3 = \cosh 2\alpha - \cos 2\beta$

Assuming a separated solution of (2.1) in the form $w = A(\alpha)\, B(\beta)\, C(\gamma)$, we obtain the equation

$$\begin{aligned} &A'' + (-p + q \cosh 2\alpha + \frac{1}{4} c^2\chi^2 \cosh^2 2\alpha)\, A = 0 \\ &B'' + (p - q \cos 2\beta - \frac{1}{4} c^2\chi^2 \cos^2 2\beta)\, B = 0 \\ &C'' + (-p - q \cosh 2\gamma + \frac{1}{4} c^2\chi^2 \cosh^2 2\gamma)\, C = 0 \end{aligned} \tag{2.4}$$

p, q being separation constants.

Putting $\alpha = iz$, $\beta = z$, $\gamma = -iz + \frac{\pi i}{2}$, each equation (2.4) can be reduced to the form

$$\frac{d^2 w}{dz^2} + (\alpha + \mu \cos 2z - \frac{c^2\chi^2}{8} \cos 4z) w = 0 \tag{2.5}$$

which is a special form of Hill's equation with three terms. The method of separation of co-ordinates is explained in [1, 1.2].

3. PROPERTIES OF (2.5) AND CO-EXISTENCE PROBLEM

In this section we state without proof some properties which hold for Mathieu's equation as well as Hill's equation with three terms (2.5). The proofs of these theorems are similar to the proofs of corresponding theorems for Mathieu's equation, see [1, 2.1].

Theorem 1. If $w_1(z)$ is a solution of (2.5) then $w_1(n\pi \pm z)$, where n is an integer is also its solution.

Theorem 2. (2.5) always has one even and one odd solution.

Corollary. (2.5) always has one solution which is even and one which is odd about any point $z = \frac{n\pi}{2}$ (n integral) i.e., $w(\frac{n\pi}{2} + z) = \pm w(\frac{n\pi}{2} - z)$

Theorem 3. (2.5) always possesses two solutions $w_1(z)$, $w_2(z)$ such that

(i) $w_1(z)$ is even and $w_2(z)$ is odd.

(ii) $w_1(o) = w_2'(o) = 1$, $w_1'(o) = w_2(o) = 0$

(iii) $w_1(z \pm \pi) = w_1(\pi)\, w_1(z) \pm w_1'(\pi)\, w_2(z)$

(iv) $w_2(z \pm \pi) = \pm w_2(\pi)\, w_1(z) + w_2'(\pi)\, w_2(z)$

(v) $w_1(z)\, w_2'(z) - w_2(z)\, w_1'(z) = 1$

(vi) $w_1(\pi) = w_2'(\pi)$.

As mentioned in §1, the question naturally arises whether Hill's equation with three terms which has many properties in common with Mathieu's equation has also the feature that two basically periodic solutions cannot co-exist. As explained in §1, a basically periodic function is a function f(z) which satisfies $f(z + \pi) = \pm f(z)$.

In 1922, Ince proved a theorem of considerable importance to the effect that if one solution of Mathieu's equation is basically periodic, the other has neither of these periods. The theorem proved by Ince is

Theorem 4. Except in the trivial case $q = 0$, Mathieu's equation never possesses two basically periodic solutions for the same values of a and q. For proof of this theorem see [1].

In 1943, Klotter and Katowski [2] showed that in the case of Hill's equation with three terms, there are certain values of θ_0, θ_1, θ_2 for which two basically periodic solutions, one even and one odd co-exist. On the other hand in 1958, Magnus and Winkler [3] proved that co-existence does not occur if $\theta_2 < 0$.

In eqn (2.5), θ_2 i.e., the coefficient of cos 4z is $-\frac{c^2\chi^2}{8}$ which is negative. Hence with the help of the result proved by Magnus and Winkler, we conclude that two basically periodic solutions cannot co-exist, in (2.5) for the same value of λ, μ and $\frac{c^2\chi^2}{8}$.

4. AN ORTHOGONALITY RELATION AND AN INTEGRAL RELATION

We now prove an orthogonality relation between basically periodic solution in the following theorem :

Theorem 1. Let $w_1(z)$ and $w_2(z)$ be basically periodic solution of (2.5) with the same value of μ and $c^2\chi^2$ but different values λ_1 λ_2 of λ.

Then $\int_0^{2\pi} w_1(z)\ w_2(z)\ dz = 0.$

Proof. From (2.5)

$$w_1'' + (\lambda_1 + \mu \cos 2z - \frac{c^2\chi^2}{8}\cos 4z)\ w_1 = 0 \tag{4.1}$$

$$w_2'' + (\lambda_2 + \mu \cos 2z - \frac{c^2\chi^2}{8}\cos 4z)\ w_2 = 0 \tag{4.2}$$

Multiply (4.1) and (4.2) by w_2 and w_1 respectively, subtract and integrate between the limits 0 and 2π.

$$\text{Then } \int_0^{2\pi}(w_1''w_2 - w_2''w_1)\,dz + (\lambda_1 - \lambda_2)\int_0^{2\pi} w_1 w_2\,dz = 0$$

$$\therefore \int_0^{2\pi} w_1(z)\,w_2(z)dz = -(\lambda_1 - \lambda_2)^{-1}\int_8^{2\pi}(w_1''w_2 - w_2''w_1)\,dz$$

$$= -(\lambda_1 - \lambda_2)^{-1}\left\{\left[w_1'w_2 - w_2'w_1\right]_0^{2\pi} - \int_0^{2\pi} w_1'w_2'\,dz + \int_0^{2\pi} w_2'w_1'\,dz\right\}$$

$$= -(\lambda_1 - \lambda_2)^{-1}\left\{\left[w_1'w_2 - w_2'w_1\right]_0^{2\pi}\right.$$

Since w_1 and w_2 are basically periodic, both they and their derivatives will certainly have 2π as a period and the terms in bracket vanishe, since $\lambda_1 - \lambda_2 \neq 0$, the left-hand side is equal to zero.

We now establish an integral relationship between solution of (2.5).

Theorem 2. Let (a) w(z) be solution of (2.5)

(b) $G(z, z')$ be a solution of the equation

$$\frac{\delta^2 G}{\delta z^2} - \frac{\delta^2 G}{\delta z'^2} = [(-\mu \cos 2z + \mu \cos 2z') + \frac{c^2\chi^2}{8} (\cos 4z - \cos 4z')]\, G$$

such that $G(z, z')$ is analytic when z, z' are in regions R, R' respectively of the complex z, z' planes.

(c) c be a path in the z' plane lying wholly within the region R' and such that

(i) $\left[G(z,z')\,\dfrac{dw(z')}{dz'} - w(z')\dfrac{\delta G(z,z')}{\delta z'}\right]$ has the same value at the two ends of c.

(ii) the integral $\int_c G(z,z')\, w(z')\, dz'$ exists for all z in R and if singular, converges uniformly in z for all z in R then

$w(z) = \int_c G(z,z')\, w(z')\, dz'$ satisfies (2.5) for z in R for the same λ, μ and $c^2\chi^2$ as w(z).

Proof. By condition c(ii) we may differentiate w with respect to z under the integral sign. This gives

$$\frac{d^2 w}{dz^2} = \int_C \frac{\delta^2 G}{\delta z^2} w(z')\, dz'$$

So $\dfrac{d^2 w}{dz^2} + (\lambda + \mu \cos 2z - \dfrac{c^2\chi^2}{8} \cos 4z)\, w$

$$= \int_{c'} \left\{ \frac{\delta^2 G}{\delta z^2} w(z') + (\lambda + \mu \cos 2z - \frac{c^2\chi^2}{8} \cos 4z) G\, w(z') \right\} dz'$$

$$= \int_{c'} \left\{ \begin{array}{c} \dfrac{\delta^2 G}{\delta z'^2} w(z') + (\mu \cos 2z' - \mu \cos 2z) w(z') G + \dfrac{c^2\chi^2}{8} (\cos 4z - \cos 4z') w(z') G \\ + (\lambda + \mu \cos 2z - \dfrac{c^2\chi^2}{8} \cos 4z) G\, w(z') \end{array} \right\} dz'$$

$$= \int_{c'} \left\{ \frac{\delta^2 G}{\delta z'^2} + \lambda G + \mu \cos 2z' G - \frac{c^2\chi^2}{8} \cos 4z' G \right\} \mathrm{w}(z')\ \mathrm{d}z' \tag{1}$$

Now on integrating $\frac{\delta^2 G}{\delta z'^2}\ \mathrm{w}(z')$ by parts twice, we get

$$\int_c \frac{\delta^2 G}{\delta z'^2} \mathrm{w}(z')\ \mathrm{d}z' = \left[w(z') \frac{\delta G}{\delta z'} \right]_c - \int_c \frac{dw(z')}{dz'}\ \frac{\delta G}{\delta z'} \mathrm{d}z'$$

$$= \left[w(z') \frac{\delta G}{\delta z'} \right]_c - \left[G \frac{dw(z')}{dz'} \right]_c + \int_c \frac{d^2 w(z')}{dz'^2} G \mathrm{d}z'$$

$$= \int_c \frac{d^2 w(z')}{dz'^2} G \mathrm{d}z' \text{ by c(i)}$$

substituting in (1)

$$\frac{d^2 w}{dz^2} + (\lambda + \mu \cos 2z - \frac{c^2\chi^2}{8} \cos 4z)\ \mathrm{w}$$

$$= \int_c \frac{d^2 w(z')}{dz'^2} G \mathrm{d}z' + (\lambda + \mu \cos 2z' - \frac{c^2\chi^2}{8} \cos 4z')\ \mathrm{G}\ \mathrm{w}(z')\ \mathrm{d}z'$$

$$= \int_c \left\{ \frac{d^2 w(z')}{dz'^2} + (\lambda + \mu \cos 2z' - \frac{c^2\chi^2}{8} \cos 4z')\, w(z') \right\} \mathrm{G}\ \mathrm{d}z'$$

$= 0$ as $\mathrm{w}(z')$ is a solution of (2.5)

Hence w(z) is a solution of (2.5).

REFERENCES

1. Arscott, F. M., Periodic differential equations. Pergamon Press Ltd., Headingtcn Hill Hall, Oxford.
2. Klotter and Katowski, iibcr die Stabilitat der Losungen Hillscher Differentialgleichungen, z.f. Angew. Mathu. Mech 23 1943, 149-155.

3. Magnus and Winkler, The coexistence problem for Hill's equation. Res. Rep. BR—26, N. Y. University, 1958.
4. Maclahan, Theory and application of Mathieu functions. Oxford University Press, 1947.

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CONCERNING MINIMALITY OF BITOPOLOGICAL SPACES

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ABSTRACT : The Purpose of this paper is to find characterizations of certain minimal bitopological spaces. The sole motivation for such a study is to establish that certain restriction in the corresponding formulations given by Raghavan and Reilly [5] can indeed be dispensed with. Incidentally, it is found that minimal pairwise Hausdorff spaces are associated in some sense to the semiregularizations of the spaces.

Raghavan and Reilly [5] proved some characterizations for minimality of bitopological spaces with regard to certain bitopological separation axioms like pairwise Hausdorffness, pairwise Urysohn property etc. The intent of this article is to establish equivalent formulations of such minimal spaces, which exhibit that certain restriction in the corresponding formulations given in [5] can be dispensed with.

INTRODUCTION

Before we explain our motivation in more clear terms, let us clarify as prerequisities certain concepts and terminologies to be used in the sequel. By (X,Q_1,Q_2) (or (X,P_1,P_2)) we shall mean a bitopological space [3] X endowed with two topologies Q_1 and Q_2 (respectively, P_1 and P_2). For any subset A of (X,Q_1,Q_2), Q_i-clA and Q_i-int A respectively denote the closure and interior of A in (X,Q_i), for i = 1, 2. A subset A of (X,Q_1,Q_2) is said to be ij-regular open [6] if $A = Q_i\text{-int}\,(Q_j\text{-cl}A)$, where (and hereafter also, in any such statement involving both i and j) i, j = 1, 2 and $i \neq j$. A space (X,Q_1,Q_2) is called ij-semiregular [6] if Q_i has a base of ij-regular open sets ; the space X is called pairwise semiregular if it is 12-and 21-semiregular. For a given space (X,Q_1,Q_2), the collection of all ij-regular open sets forms an open basis for a topology Q_i^S on X coarser than Q_i [6]. The space (X, Q_1^S, Q_2^S) which is pairwise semiregular, is called the semiregularization of (X,Q_1,Q_2). Listed below are some other definitions that will be required in course of the deliberation.

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Definition 1. A space (X,Q_1,Q_2) is said to be

(a) pairwise Hausdorff [3] (pairwise Urysohn [7]) if for any two distinct points x,y of X, there exist a Q_1-open neighbourhood U of x and a Q_2-open neighbourhood V of y such that $U \cap V = \Phi$ (respectively, $Q_2\text{–cl}U \cap Q_1\text{–cl}V = \Phi$) ;

(b) ij-regular [3] if for each Q_i-open set V of X and each $x \varepsilon V$, there is a Q_i-open set U containing x such that $Q_j\text{-cl}U \subseteq V$;

(c) pairwise regular [3] if X is 12-regular as well as 21-regular.

Definition 2. [5] Let R denote any one of the properties : pairwise Hausdorff, pairwise Urysohn, pairwise regular. Then a space (X,Q_1,Q_2) with property R is called minimal R if for any space (X,P_1,P_2) possessing the property R and such that $P_i \subseteq Q_i$ for i = 1, 2, one has $P_i = Q_i$ for i = 1 and 2.

In [5] it is proved that a pairwise Hausdorff space (X,Q_1,Q_2) is minimal pairwise Hausdorff iff whenever $\mathcal{T}_1$ is a Q_1-open filterbase with a unique Q_2-adherent point p and $\mathcal{T}_2$ is a Q_2-open filterbase with the same point p as the unique Q_1-adherent point, then $\mathcal{T}_1$ is Q_1-convergent to p and $\mathcal{T}_2$ is Q_2-convergent to p. In practice, the information derived from the theorem is somewhat difficult to apply in a minimal pairwise Hausdorff space, because one has to ensure the same point p as the unique Q_1-(respectively Q_2-) adherent point of two filterbases of which one is Q_2-open and the other is Q_1-open respectively. Similar restriction regarding sharing of the same point p is also assumed in the characterizations of minimal pairwise Urysohn and minimal pairwise regular spaces in [5]. We propose to establish here equivalent criteria for such minimal spaces, where the above stated restriction can be dispensed with. We first take up the case of minimal pairwise Hausdorffness as follows.

Theorem 3. A pairwise Hausdorff space (X,Q_1,Q_2) is minimal pairwise Hausdorff iff the statements S(1,2) and S(2,1) are true, where S(i,j) stands for the statement : "every Q_i-open filterbase on X with a unique O_j-adherent point p_i is Q_i-convergent to p_i".

Proof. Let (X,Q_1,Q_2) be minimal pairwise Hausdorff and if possible. at least one of S(1,2) and S(2,1) be false. Let S(1,2) be false, then there exists a Q_1-open filterbase $\mathcal{T}$ with a unique Q_2-adherent point q such that for some $V_q \varepsilon Q_1$ with $q \varepsilon V_q$, no member of $\mathcal{T}$ is contained in v_q. Let $P_1 = \{U : U \varepsilon Q_1 \text{ and } q \not\varepsilon U\} \cup \{V \cup F : q \varepsilon V \varepsilon Q_1$ and F is a Q_1-open set which contains some member of $\mathcal{T}\}$. Then P_1 is a topology on X such that $P_1 \subseteq Q_1$. Clearly, $V_q \notin P_1$ so that $P_1 \neq Q_1$. Let $P_2 = Q_2$. We show that (X,P_1,P_2) is a pairwise Hausdorff space. Let $x,y \varepsilon X$ such that $x \neq y$ and both are different from q. By pairwise Hausdorff property of (X,Q_1,Q_2), we take $A_x \varepsilon Q_i$ and $B_y \in Q_j$ such that $x \varepsilon A_x$, $q \notin A_x$, $y \in B_y$, $q \in B_y$ and $A_x \cap B_y = \Phi$; then $A_x \in P_i$ and $B_y \in P_j$. Now let x be a point distinct from q. There is $F \in \mathcal{T}$ such that $x \notin Q_2\text{–cl}F$. Choose $C_x \in Q_2$ and $D_q \in Q_1$ such that $x \in C_x$, $q \in D_q$ and $C_x \cap D_q = \Phi$. Let $V_x = C_x \cap (X \setminus Q_2\text{–cl}F)$ and $W_q = D_q \cup F$; then

$x \varepsilon V_x$, $q \varepsilon W_q$ and $V_x \cap W_q = \Phi$, also $W_q \varepsilon P_1$ and $V_x \varepsilon P_2$. If $x \varepsilon U_x \varepsilon Q_1$ and $q \in U_q \varepsilon Q_2$ with $U_x \cap U_q = \Phi$, then $U_x \varepsilon P_1$ and $V_q \varepsilon P_2$. Hence (X,P_1,P_2) is pairwise Hausdorff which is strictly weaker than (X,Q_1,Q_2) which is a contradiction. Thus S(1,2) and S(2,1) are ture.

Conversely, let (X,Q_1,Q_2) be a pairwise Hausdorff space which is not minimal pairwise Hausdorff. Then there is a pairwise Hausdorff space (X,P_1,P_2) which is strictly coarser than (X,Q_1,Q_2). Let $P_1 \neq Q_1$, where we necessarily have $P_i \subseteq Q_i$ for $i = 1,2$. Let $U \varepsilon Q_1$ such that $U \not\in P_1$. Then there exists a point $p \varepsilon U$ such that U is not a P_1-neighbourhood of p. Let $\mathcal{T} = \{F : p \varepsilon F \varepsilon P_1\}$. As (X,P_1,P_2) is pairwise Hausdorff, $\{p\} = \cap \{P_2\text{–cl}F : F \varepsilon \mathcal{T}\}$. Since $Q_2\text{–cl}F \subseteq P_2\text{–cl}F$ and $p \varepsilon F \varepsilon Q_1$ for all $F \varepsilon \mathcal{T}$, $\mathcal{T}$ is a Q_1-open filterbase with a unique Q_2-adherent point p. But U contains no member of $\mathcal{T}$ and then $\mathcal{T}$ is not Q_1-convergent to p. Thus S(1,2) is false. Hence the theorem is completely proved.

Having furnished an equivalent and simplified verson of Raghavan and Reilly's criterion for minimal pairwise Hausdorff spaces, we now like to ascertain, incidentally, that such minimal spaces are associated in some way to the semiregularizations of the spaces. For this we need to recall certain definitions and results as follows :

For any $A \subseteq X$, the ij-θ-closure of A, to be denoted by ij-θ-clA, is defined to be the set consisting of precisely the points x of X such that for every Q_i-open neighbourhood (henceforth nbd, for short) U of x, $(Q_j\text{-cl}U) \cap A \neq \phi$ [1,2]. It is known that

Lemma 4. For a subset A of (X,Q_1,Q_2),

(a) [2] $A \varepsilon Q_j \Rightarrow Q_i\text{-cl}A = ij\text{-}\theta\text{-cl}A$

(b) [4] $ij\text{-}\theta\text{-cl}A = \cap\{Q_i\text{-cl}V : A \subseteq V \in Q_j\}$.

A point x in a space (X,Q_1,Q_2) is called an ij-θ-adherent point of a filterbase $\mathcal{T}$ on X, or $\mathcal{T}$ is said to ij-θ-adherent at x [2] if $x \varepsilon \cap \{ij\text{-}\theta\text{-cl}F : F \varepsilon \mathcal{T}\}$. The set of all ij-θ-adherent points of $\mathcal{T}$ is called the ij-θ-adherence of $\mathcal{T}$, denoted by ij-θ-ad$\mathcal{T}$. The filterbase $\mathcal{T}$ on X is said to ij-θ-converge to a point x of X [2], written as $\mathcal{T} \xrightarrow{ij-\theta} x$, if for each Q_i-open nbd U of x, there exists $F \varepsilon \mathcal{T}$ such that $F \subseteq Q_j\text{–cl}U$. We shall denote, as usual, by Q-ad$\mathcal{T}$ the set of all adherent points of a filterbase $\mathcal{T}$ on a topological space (X,Q). With the concepts, explained so far, we now have :

Theorem 5. In a space (X,Q_1,Q_2), every Q_j-open filterbase with a unique Q_i-adherent point Q_j^s-converges to the point iff every filterbase with a unique ij-θ-adherent point ji-θ-converges to the point.

Proof. First suppose that every Q_j-open filterbase in X with a unique Q_i-adherent point is Q_j^s-convergent to the point. Let $\mathcal{T}$ be a filterbase on X such that ij-θ-adf$\mathcal{T} = \{x\}$. Let $\mathcal{G} = \{V \varepsilon Q_j : F \subseteq V$, for some $F \varepsilon \mathcal{T}\}$. The $\mathcal{G}$ is a Q_j-open filterbase on X and by Lemma

4(b) we easily get Q_i-ad$\mathcal{T} = \{x\}$. Then by hypothesis, $\mathcal{G}$ is Q_j^s-convergent to x, Now, if $x \varepsilon V \varepsilon Q_j$ then since $x \varepsilon O_j$-int Q_i-cl $V \varepsilon Q_j^s$, there is $U \varepsilon \mathcal{G}$ such that $U \subseteq Q_j$-int Q_i-clV. Thus if $F \varepsilon \mathcal{T}$ such that $F \subseteq U$, then $F \subseteq Q_i$-clV, i.e., $\mathcal{T}$ is ji-θ-convergent to x. Conversely, let $\mathcal{G}$ be a Q_j-open filterbase such that $\cap \{Q_i\text{-cl } U : U \varepsilon \mathcal{G}\} = \{x\}$. Then in view of Lemma 4(a), it follows that ij-θ-ad$\mathcal{G} = \{x\}$ and hence $\mathcal{G}$ is ji-θ-convergent to x. Let $x \varepsilon W_X \varepsilon Q_j^s$ with $W_X = Q_j$-int Q_i-clW_X. Then there is $U \varepsilon \mathcal{G}$ such that $U \subseteq Q_i$-cl W_X. Since $U \varepsilon Q_j$, $U \subseteq Q_j$-int Q_i-cl$W_X = W_X$. Hence $\mathcal{G}$ is Q_j^s-convergent to p.

Theorem 6. In a pairwise Hausdorff space (X,Q_1,Q_2) the following statements are equivalent :

(a) Every Q_i-open filterbase with a unique O_j-adherent point, Q_i^s-Converges to the point, and $Q_i = Q_i^s$.

(b) Every Q_i-open filterbase with a unique Q_j-adherent point is Q_i-convergent to the point.

Proof. '(a) ⇒ (b)' is obvious, since $Q_i^s = Q_i$. For establishing '(b) ⇒ (a)' it suffices to prove that $Q_i \subseteq Q_i^s$. Let $U \varepsilon Q_i$ and $x \varepsilon U$. As X is pairwise Hausdorff, it is easy to see that $\mathcal{B}(X) = \{V : x \varepsilon V \varepsilon Q_i \text{ and } V = Q_i\text{-int } O_j\text{-cl}V\}$ is a Q_i-open filterbase with the unique Q_j-adherent point x. So for some $V \varepsilon \mathcal{B}(X)$, $V \subseteq U$. Since V is an ij-regular open set containing x, we have $U \varepsilon Q_i^s$. Hence the theorem is proved.

Combining the results deduced so far, we obtain the following theorem which also relates minimality of a pairwise Hausdorff space with the pairwise semiregularity of the space.

Theorem 7. For a pairwise Hausdorff space (X,Q_1,Q_2), the following statements are equivalent :

(a) (X,Q_1,Q_2) is pairwise semiregular and every filterbase with a unique ij-θ-adherent point p_i is ji-θ-convergent to p_i.

(b) Every Q_j-open filterbase with a unique Q_i-adherent point p_i is Q_j-convergent to p_i.

(c) X is minimal pairwise Hausdorff.

(In statements (a) and (b), we mean, as usual, ij = 1,2 and i ≠ j).

We now turn our attention to minimal pairwise Urysohn and minimal pairwise regular spaces :

Definition 8. [5] A Q_i-open filterbase $\mathcal{T}$ on a space (X,Q_1,Q_2) is said to be

(a) Q_i-Urysohn with respect to Q_j if for each point $p \notin Q_j$-ad$\mathcal{T}$, there is a Q_j-open set U containing p and a set $F \varepsilon \mathcal{T}$ such that Q_i-clU $\cap$ Q_j-clF $= \Phi$;

(b) Q_i-regular with respect to Q_j if for each $F \varepsilon \mathcal{T}$, there exists a $G \varepsilon \mathcal{T}$ such that Q_j-clG $\subseteq$ F.

Theorem 9. A pairwise G-space (X,Q_1,Q_2) is minimal pairwise G iff the statements T(1,2) and T(2,1) are true, where 'G' stands for 'Urysohn' or 'regular', and T(i,j) stands for the statement "every Q_i-open filterbase on X, which is Q_i-G with respect to Q_j and which has a unique Q_j-adherent point p_i is Q_i-convergent to p_i".

Proof. We prove only the case of pairwise Urysohn property, the other case of pairwise regularity being quite analogous.

Let (X,Q_1,Q_2) be minimal pairwise Urysohn and if possible, let at least one of T(1,2) and T(2,1), say T(1,2), be false. Then there exists a Q_1-open filterbase $\mathcal{T}$ on X, which is Q_1-Urysohn with respect to Q_2, and with a unique Q_2-adherent point q, such that for some $V_q \varepsilon Q_1$ with $q \varepsilon V_q$, no member of $\mathcal{T}$ is contained in V_q. Let, as in the proof of Theorem 3, $P_1 = \{U \varepsilon Q_1 : q \notin U\} \cup \{V \cup F : q \varepsilon v \varepsilon Q_1$ and F is a Q_1-open set which contains some member of $\mathcal{T}\}$, and $P_2 = Q_2$. Then P_1 is clearly a topology on X such that $P_1 \subseteq Q_1$. Clearly $V_q \notin P_1$ so that $P_1 \neq Q_1$. To arrive at a contradiction, it suffices to show that (X,P_1,P_2) is a pairwise Urysohn space.

Let $x \varepsilon X$ such that $x \neq q$. Now, since $\mathcal{T}$ is Q_1-Urysohn with respect to Q_2, there exists a Q_2-open nbd U of x such that Q_1-clU $\cap$ Q_2-clF $= \Phi$ for some $F \varepsilon \mathcal{T}$. Again, (X,Q_1,Q_2) being pairwise Urysohn, there exist $V \varepsilon Q_1$ and $W \varepsilon Q_2$ such that $q \varepsilon V$, $x \varepsilon W$ and Q_2-clV $\cup$ Q_1-clW $= \Phi$. Then $W \cap U \varepsilon P_2$. We claim that Q_1-cl$(W \cap U) = P_1$-cl $(W \cap U)$. In fact, Q_1-cl $(W \cap U) \subseteq P_1$-cl $(W \cap U)$ is clear. Now, let $y \notin Q_1$-cl$(W \cap U)$, i.e., there exists a $V_y \varepsilon Q_1$ such that $y \varepsilon V_y$ and $V_y \cap (W \cap U) = \Phi$. If $q \notin V_y$ then $V_y \varepsilon P_1$ which implies that $y \notin P_1$-cl $(W \cap U)$. If $q \varepsilon V_y$, then we can assume $q \neq y$ (indeed, $q \notin P_1$-cl$(W \cap U)$. For, $q \varepsilon P_1$-cl$(W \cap U) \Rightarrow [V \cup (X \setminus Q_1$–clU$)] \cap (W \cap U) \neq \Phi \Rightarrow V \cap W \cap U \neq \Phi$, a contradiction). By pairwise Urysohn property of (X,Q_1,Q_2), there exists $W_y \varepsilon Q_1$ with $y \varepsilon W_y$ such that $q \notin W_y$. We see that $y \varepsilon V_y \cap W_y \varepsilon P_1$ for which $(V_y \cap W_y) \cap (W \cap U) = \Phi$. Hence $y \notin P_1$-cl $(W \cap U)$, and our claim is justified. Now, since $q \varepsilon V \cup F$ and Q_2-cl $(V \cup F) = P_2$-cl $(V \cup F)$, we have P_1-cl $(W \cap U) \cap P_2$-cl $(V \cup F) = \Phi$. Further, if $x \varepsilon U \varepsilon Q_1$ and $q \varepsilon V \varepsilon Q_2$ and Q_2-clU $\cap$ Q_1-clV $= \Phi$, then $U \varepsilon P_1$ and $V \varepsilon P_2$. We show that P_2-cl U $\cap$ P_1-clV $= \Phi$ for which we only prove that P_1-clV $\subseteq Q_1$-clV. If not, let $z \notin Q_1$-clV, then there exists $W \varepsilon Q_1$ with $z \varepsilon W$ such that $W \cap V = \Phi$. If $q \notin W$ then $W \varepsilon P_1$ with $z \varepsilon W$ so that $z \notin P_1$-clV. If $q \varepsilon W$, then by pairwise Urysohn property of (X,Q_1,Q_2), there is $W_z \varepsilon Q_1$ with $z \varepsilon W_z$ such that $q \notin W_z$, then $z \varepsilon W \cap W_z \varepsilon Q_1$ such that $(W \cap W_z) \cap V = \Phi$. Hence P_1-clV $\subseteq$ Q_1-clV.

If x and y are distinct points of X, both different from z, then clearly they can be separated by suitable sets. Hence (X,P_1,P_2) is pairwise Urysohn with $P_1 \subsetneq Q_1$ and we arrive at the desired contradiction.

Conversely, let (X,P_1,P_2) be pairwise Urysohn such that $P_i \subseteq Q_i$ for i = 1,2, but $P_1 \neq Q_1$ or $P_2 \neq Q_2$. For definiteness, suppose $P_1 \neq Q_1$ so that for some $U \varepsilon Q_1$, $U \not\varepsilon P_1$. Then there exists a point $q \varepsilon U$ such that U is not a P_1-open nbd of q. Let $\mathcal{T} = \{F \in P_1 : q \varepsilon F\}$. Clearly, $\mathcal{T}$ is a Q_1-open filterbase. Let $x \not\varepsilon Q_2$-ad$\mathcal{T}$. As $x \neq q$ and (x,P_1,P_2) is pairwise Urysohn, there are a P_2-open nbd U of x and a P_1-open nbd V of q (i.e., $V \varepsilon \mathcal{T}$) such that P_1-clU $\cap$ P_2-clV $= \Phi$ and hence Q_1-clU $\cap$ Q_2-clV $= \Phi$.

Thus $\mathcal{T}$ is Q_1-Urysohn with respect to Q_2. Also, $\mathcal{T}$ has a unique Q_2-adherent q. In fact, let $p \varepsilon X$ such that $p \neq q$. By pairwise Urysohn property of (X,P_1,P_2), there exist a P_2-open nbd W of p and a P_1-open nbd V of q such that P_1-clW $\cap$ P_2-clV $\neq \Phi$. Then $V \varepsilon \mathcal{T}$ and W is a Q_2-open nbd of p with $W \cap V = \Phi$. Thus p cannot be a Q_2-adherent point of $\mathcal{T}$. But U contains no member of $\mathcal{T}$ and thus $\mathcal{T}$ is not Q_1-convergent to q. Hence T(1,2) is false. This completes the proof.

REFERENCES

1. G. K. Banerjee, On pairwise almost strongly θ-continuous mappings, Bull. Cal. Math. Soc. 79 (1987), 314-320.
2. Ch. G. Kariofillis, On pairwise almost compactness, Ann. Soc. Sci. Bruxelles 100 (1986), 129-137.
3. J. C. Kelley, Bitopological spaces, Proc. London Math. Soc. 13 (1963), 71-89.
4. M. N. Mukherjee and G. K. Banerjee, Bitopological QHC-spaces, Indian Jour. Pure Appl. Math. 21 (7) (1990), 639-648.
5. T. G. Raghavan and I. L. Reilly, on minimal bitopological spaces, Kyungpook Math. Jour. 17 (1977), 57-65.
6. A. R. Singal and S. P. Arya, On pairwise almost regular spaces, Glasnik Mat. 6 (26) (1971), 335-343.
7. M. K. Singal and A. R. Singal, Some more separation axioms in bitopological spaces, Ann. Soc. Sci. Bruxellers 84 (1970), 207-230.

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ON A TYPE OF KENMOTSU MANIFOLD

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ABSTRACT : The object of the present paper is to study a Kenmotsu manifold which is Ricci-generalized pseudosymmetric. It is proved that a Ricci-generalized pseudosymmetric Kenmotsu manifold is either locally isometric with the Hyperbolic space $H^n(1)$ or an Einstein manifold.

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0. INTRODUCTION

Let $M^n(\phi,\xi,\eta,g)$ $(n = 2m+1)$ be an almost contact Riemannian manifold with a contact form η, the associated vector field ξ, (1-1) tensor field ϕ and the associated Riemannian metric g. In 1971 K. Kenmotsu [3] studied a class of contact Riemannian manifold which satisfy some special conditions. We call it Kenmotsu manifold.

A Riemannian manifold (M^n,g) $(n \geq 3)$ is said to be Ricci-generalized pseudosymmetric [2] if and only if the relation

$$R.R = L\ Q\ (S,R) \tag{0.1}$$

holds on the set $U = \{x \in M^n : Q(S,R) \neq 0 \text{ at } x\}$, where L is a constant on U, R is the curvature tensor of type (1,3) and S is the Ricci tensor of type (0,2). R.R is defined by

$$\begin{aligned}(R(X,Y).R)(U,V)W &= R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W \\ &\quad - R(U,V)R(X,Y)W \end{aligned} \tag{0.2}$$

where R(X,Y) denotes the derivation of the tensor algebra at each point of the tangent space and Q(S,R) is defined by

$$\begin{aligned}Q(S,R) &= (X\wedge_s Y).R)(U,V)W \\ &= (X\wedge_s Y)R(U,V)W - R((X\wedge_s Y)U,V)W \\ &\quad - R(U,(X\wedge_s Y)V)W - R(U,V)(X\wedge_s Y)W \end{aligned} \tag{0.3}$$

for all $X,Y,U,V,W \in \chi(M^n)$, $\chi(M^n)$ being the Lie algebra for all differentiable vector fields on M where the endomorphism $X\wedge_s Y$ is defined by

$$(X\wedge_s Y)Z = S(Y,Z)X - S(X,Z)Y. \tag{0.4}$$

It is clear that any semi symmetric (R.R = 0) as well as any Ricci flat (S = 0) manifold is Ricci-generalized pseudosymmetric. In the present paper we prove that a Ricci-generalized pseudosymmetric manifold is either locally isometric with the Hyperbolic space $H^n(1)$ or an Einstein manifold.

1. KENMOTSU MANIFOLD

Let (M^n,ϕ,ξ,η,g) be an n dimensional (where n = 2m+1) almost contact Riemannian manifold where ϕ is a (1,1) tensor field, ξ is the structure vector field, η is a 1-form and g is a Riemannian metric. It is well known that (ϕ,ξ,η,g) structure satisfy [1].

$$\phi\xi = 0 \tag{1.1}$$

$$\eta(\xi) = 1 \tag{1.2}$$

$$\eta(\phi X) = 0 \tag{1.3}$$

$$\phi^2 X = -X + \eta(X)\xi \tag{1.4}$$

$$g(X,\xi) = \eta(X) \tag{1.5}$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\,\eta(Y) \tag{1.6}$$

for every vector fields X,Y.

If more over,

$$(\nabla_x\phi)Y = -g(X,\phi Y)\xi - \eta(Y)\phi X \tag{1.7}$$

$$\nabla_x\xi = X - \eta(X)\xi \tag{1.8}$$

where ∇ denotes the Riemannian connection of g, then (M^n,ϕ,ξ,η,g) is called a Kenmotsu manifold.

We have [3]

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X \tag{1.9}$$

$$S(X,\xi) = -(n-1)\eta(X) \tag{1.10}$$

where S denotes the Ricci-tensor. From (1.9) it easily follows that

$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X \tag{1.11}$$

$$R(X,\xi)\xi = \eta(X)\xi - X \tag{1.12}$$

2. RICCI-GENERALIZED PSEUDOSYMMETRIC KENMOTSU MANIFOLD

Let (M^n,g) $(n \geq 3)$ be a Ricci-generalized pseudosymmetric Kenmotsu manifold. Then we have the relation (0.1) which can be written by virtue of (0.3)

$$(R(X,Y).R)(U,V)W = L[((X\wedge_s Y).R)(U,V)W] \tag{2.1}$$

for all $X,Y,\ldots\ldots\ldots W \in \chi\,(M^n)$. From (2.1), it follows that

$$(R(X,\xi).R)(U,V)W = L[((X\wedge_s \xi).R)(U,V)W] \tag{2.2}$$

By definition we have

$$\begin{aligned}(R(X,\xi).R)(U,V)W = {} & R(X,\xi)R(U,V)W-R(R(X,\xi)U,V)W\\ & -R(U,R(X,\xi)V)W-R(U,V)R(X,\xi)W.\end{aligned} \tag{2.3}$$

Applying (1.11) we obtain from (2.3)

$$\begin{aligned}(R(X,\xi).R)(U,V)W = {} & {}'R(U,V,W,X)\xi-\eta(R(U,V)W)X+\eta(U)R(X,V)W\\ & -g(X,U)R(\xi,V)W+\eta(V)R(U,X)W-g(X,V)R(U,\xi)W\\ & +\eta(W)R(U,V)X-g(X,W)R(U,V)\xi\end{aligned} \tag{2.4}$$

where ${}'R(U,V,W,X)= g(R(U,V)W,X)$.

Also from definition we have

$$\begin{aligned}((X\wedge_s \xi).R(U,V)W = {} & (X\wedge_s \xi)R(U,V)W-R((X\wedge_s \xi)U,V)W\\ & -R(U,(X\wedge_s \xi)V)W-R(U,V)(X\wedge_s \xi)W.\end{aligned} \tag{2.5}$$

In view of the definition (0.4), it follows from (2.5) that

$$\begin{aligned}((X\wedge_s \xi).R)(U,V)W = {} & S(\xi,R(U,V)W)X-S(X,R(U,V)W)\xi-R(S(\xi,U)X-S(X,U)\xi,V)W\\ & -R(U,S(\xi,V)X-S(X,V)\xi)W-R(U,V)(S(\xi,W)X-S(X,W)\xi).\end{aligned} \tag{2.6}$$

By virtue of (1.10) we obtain from (2.6)

$$\begin{aligned}((X\wedge_s \xi).R)(U,V)W = {} & -(n-1)[\eta(R(U,V)W)X-\eta(U)R(X,V)W\\ & -\eta(V)R(U,X)W-\eta(W)R(U,V)X]-S(X,R(U,V)W)\xi\\ & +S(X,U)R(\xi,V)W+S(X,V)R(U,\xi)W+S(X,W)R(U,V)\xi\end{aligned} \tag{2.7}$$

Using (2.4) and (2.7) in (2.2) we get

$$\begin{aligned}& {}'R(U,V,W,X)\xi-\eta(R(U,V)W)X+\eta(U)R(X,V)W-g(X,U)R(\xi,V)W\\ & +\eta(V)R(U,X)W-g(X,V)R(U,\xi)W+\eta(W)R(U,V)X-g(X,W)R(U,V)\xi\\ & =-(n-1)L[\eta(R(U,V)W)X+\eta(U)(R(X,V)W)+\eta(V)R(U,X)W\\ & +\eta(W)R(U,V)X]-L[S(X,R(U,V)W)\xi-S(X,U)R(\xi,V)W-S(X,V)R(U,\xi)W\\ & -S(X,W)R(U,V)\xi].\end{aligned} \tag{2.8}$$

Taking the inner product on both sides of (2.8) by ξ we obtain by virtue of (1.2), (1.5), (1.10) and $\eta(R(U,V)\xi) = 0$,

$$
\begin{aligned}
&{}'R(U,V,W,X)-\eta(R(U,V)W)\eta(X)+\eta(U)\eta(R(X,V)W)-g(X,U)\eta(R(\xi,V)W)\\
&+\eta(V)\eta(R(U,X)W)-g(X,V)\eta(R(U,\xi)W)+\eta(W)\eta(R(U,V)X)\\
&=-(n-1)L[\eta(R(U,V)W)\eta(X)-\eta(U)\eta(R(X,V)W)-\eta(V)\eta(R(U,X)W)\\
&-\eta(W)\eta(R(U,V)X)]-L[S(X,R(U,V)W)-S(X,U)\eta(R(\xi,V)W)\\
&-S(X,V)\eta(R(U,\xi)W)].
\end{aligned}
\tag{2.9}
$$

Putting $W = \xi$ in (2.9) and noting that $\eta(R(U,V)\xi) = 0$ for all U,V, we obtain

$$L[(n-1)\eta(R(U,V)X-S(X,R(U,V)\xi)] = 0,$$

from which it follows that

either $L = 0$,

$$\text{or, } (n-1)\eta(R(U,V)X-S(X,R(U,V)\xi) = 0. \tag{2.10}$$

If $L = 0$, then from (2.1) we have

$$R.R = 0. \tag{2.11}$$

which implies that the manifold is semi symmetric and hence by the proposition 7 of [3] is a space of constant negative curvature-1. Consequently the space is locally isometric with the Hyperbolic space $H^n(1)$, [4].

Again, replacing U by ξ in (2.10) and applying (1.2) and (1.10) we obtain

$$S(X,V) = -(n-1)g(X,V) \text{ for all } X,V. \tag{2.12}$$

This implies that the manifold is an Einstein manifold. Contracting (2.12) we get

$$r = -n(n-1), \text{ where } r \text{ is the scalar curvature.} \tag{2.13}$$

This leads to the following :

Theorem 1. A Ricci-generalized pseudosymmetric Kenmotsu manifold (M^n,g) $(n \geq 3)$ is either locally isometric with the Hyperbolic space $H^n(1)$ or an Einstein manifold.

Now for a Kenmotsu manifold (M^n,g) $(n \geq 3)$ satisfying the condition $(X\wedge_S Y).R = 0$ we have the relation (2.10) and hence (2.12). Thus we have the following :

Theorem 2. A Kenmotsu manifold (M^n,g) $(n \geq 3)$ satisfying the condition $(X\wedge_S Y).R = 0$ for all X,Y is an Einstein manifold.

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REFERENCES

1. D. E. Blair : Contact manifolds in Riemannian geometry, Lecture notes in Mathematics, 509, Springer-Verlag, Berlin, 1976.
2. R. Deszcz : On pseudosymmetric spaces, Bull. Belg. Math Soc. Ser A. 44 (1992), 1-34.
3. K. Kenmotsu : A class of contact Riemannian manifolds, Tohoku Math Journ. 24 (1972) 93-103.
4. B.O'Neill : Semi-Riemannian Geometry with application to Relativity, Academic Press p. 228.

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